



Research article

On the time-optimal control problem for a pseudo-parabolic equation with variable thermal conductivity in a bounded domain

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Abstract: This paper studied a time-optimal control problem for a pseudo-parabolic equation in a bounded multidimensional domain. The equation involved a uniformly elliptic operator and a small parameter characterizing memory effects. The control acted through a distributed source term, where the spatial part was fixed and the time-dependent part was a bounded measurable function. The goal was to steer the system from a zero initial state to a prescribed spatially averaged temperature in minimal time and maintain it thereafter. Using spectral decomposition, we derived an explicit solution representation and reduced the control problem to a Volterra integral equation. Under natural positivity assumptions on the Fourier coefficients, we established that the kernel was continuous, positive, and monotone. These properties yielded an explicit upper bound for the minimal time in terms of the first eigenvalue, the relaxation parameter, and the system constraints. A comparison with the parabolic case showed that the pseudo-parabolic term led to longer minimal times, reflecting the memory effects inherent in such models.

Keywords: pseudo-parabolic equation; thermal conductivity; Volterra integral equation; admissible control; minimal time; memory effect

Mathematics Subject Classification: 35K70, 35K35

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. We introduce the linear elliptic operator

$$\mathcal{A}u := - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right),$$

where the coefficients $a_{ij}(x)$ satisfy

$$a_{ij}(x) = a_{ji}(x), \quad a_{ij} \in C^1(\overline{\Omega}),$$

and the uniform ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \beta |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega,$$

with a constant $\beta > 0$. From a physical point of view, the operator $\mathcal{A}$ characterizes the diffusion mechanism of heat within the medium. The matrix of coefficients $\{a_{ij}(x)\}$ represents the thermal conductivity tensor, which allows the model to account for spatial heterogeneity and anisotropic heat transfer effects. Such a formulation is appropriate for describing heat propagation in complex materials, including composites and heterogeneous solids.

In this paper, we consider the pseudo-parabolic equation

$$(I + \gamma \mathcal{A})u_t + \mathcal{A}u(x, t) = h(x) \mu(t), \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

with homogeneous Dirichlet boundary condition

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0, \quad (1.2)$$

together with the initial condition

$$u(x, 0) = 0, \quad x \in \overline{\Omega}. \quad (1.3)$$

Here, $\mu(t)$ is the control function, $h(x)$ is a prescribed spatial distribution function, and $\gamma > 0$ is a fixed positive constant. The function $h(x)$ characterizes the spatial distribution of the heat source, while $\mu(t)$ represents the time-dependent control input. Thus, the source term $h(x)\mu(t)$ describes a controlled heat source that can be adjusted over time. The parameter γ characterizes the relaxation time or memory effect in the medium. When $\gamma = 0$, equation (1.1) reduces to the classical parabolic heat equation. Thus, the presence of the term $\gamma \mathcal{A}u_t$ introduces a regularization effect and accounts for more complex thermal phenomena in which the heat flux depends not only on the present temperature gradient but also on its history.

Equation (1.1) represents the law of conservation of thermal energy in the domain Ω . The term $(I + \gamma \mathcal{A})u_t$ captures both the rate of change of temperature and its interaction with the diffusion mechanism, a characteristic feature of pseudo-parabolic equations. This type of equation arises naturally in models describing relaxation or memory effects in heat conduction, where the heat flux depends not only on the instantaneous temperature gradient but also on its past history. They also appear in the theory of incompressible fluids with memory, describing the evolution of velocity fields in materials that retain information about past deformations [1].

Control problems associated with parabolic equations have been extensively studied over the past decades due to their fundamental importance in modeling heat conduction and diffusion processes. One of the earliest contributions to this area was made by Friedman [2], who investigated control problems for parabolic equations from a variational perspective. Foundational advances in the controllability theory of linear parabolic equations were initiated by Fattorini and Russell [3], where key concepts related to exact and approximate controllability were established. Subsequent advances further developed control

problems in infinite-dimensional settings. In particular, Egorov [4] extended Pontryagin's maximum principle to evolution equations in Banach spaces and proved a bang–bang principle under suitable assumptions. A comprehensive account of optimal control theory for distributed parameter systems, including parabolic equations, can be found in the classical monograph by Lions [5].

Further developments addressed various control problems for heat equations and related parabolic models. The boundary control problem for the heat conduction equation in a multidimensional parallelepiped was studied in [6], where control function was proved using the Laplace control method. Time-optimal control for fourth-order parabolic equations was investigated in [7] in a three-dimensional domain. In the case of bounded n -dimensional domains, sharp estimates for the minimal time required to achieve a prescribed average temperature were obtained in [8]. In parallel, substantial progress has been made in the analysis of nonlinear and degenerate parabolic equations, including problems involving logarithmic nonlinearities, reaction-diffusion systems, and nonlocal terms; see, for instance, [9–12]. These works focus primarily on qualitative properties of solutions such as existence, blow-up, and long-time dynamics. Recent developments have also addressed controllability issues from a spectral and observability viewpoint. In [13], spectral inequalities for the Laplace-Beltrami operator were derived, leading to observability and null-controllability results for the spherical heat equation. Numerical aspects of boundary control and discretization schemes for heat equations were analyzed in [14].

Despite these advances, control problems for pseudo-parabolic equations remain comparatively less explored. While the well-posedness theory for pseudo-parabolic initial-boundary value problems has been established in [15], and certain controllability issues have been considered in [16–18], a systematic analysis of time-optimal control in such systems is still largely absent. The presence of the additional memory term significantly alters the dynamical structure of the equation and raises new analytical challenges that are not present in classical parabolic models.

The purpose of the present paper is to investigate a time-optimal control problem for the pseudo-parabolic equation (1.1) in a bounded multidimensional domain. The equation contains a uniformly elliptic operator and an additional term with a small parameter characterizing memory effects in the medium. The control is exercised through a distributed source term, where the spatial part is fixed and the time-dependent part is a bounded measurable function. The goal is to steer the system from zero initial state to a prescribed spatially averaged temperature level in minimal time and maintain it thereafter. To address this problem, we first introduce the concept of a generalized solution for the initial-boundary value problem and establish its existence and uniqueness using spectral decomposition. The solution is represented as a Fourier series in terms of the eigenfunctions of the elliptic operator, which allows us to reduce the control problem to a Volterra integral equation of the first kind for the unknown control function. The kernel of this integral equation is expressed through the eigenvalues and Fourier coefficients of the given data. Under natural positivity assumptions on the Fourier coefficients of the spatial source distribution and the weight function, we prove that the kernel is positive, continuous, and strictly decreasing. These properties are essential for the subsequent analysis. Using a lower bound for the kernel, we derive an explicit upper estimate for the minimal time required to reach the desired average temperature. This estimate is expressed in terms of the first eigenvalue of the elliptic operator, the relaxation parameter, and the system constraints. The main novelties of this paper are threefold. First, we provide the first explicit upper bound for the minimal time in a time-optimal control problem for a pseudo-parabolic equation. Second, we reduce the control problem to a Volterra integral equation of the first kind and prove that its kernel is positive, continuous, and strictly decreasing, which allows

a clear analysis of the controllability. Third, we obtain a quantitative comparison $T_\gamma = (1 + \gamma\lambda_1)T_0$, showing exactly how the memory parameter γ slows down the controllability relative to the classical parabolic case. Finally, we compare the obtained estimate with the classical parabolic case, showing that the presence of the pseudo-parabolic term leads to longer minimal times. This result quantifies the effect of memory phenomena on the speed of controllability and provides a rigorous justification for the intuitive understanding that relaxation mechanisms slow down the system's response.

2. Problem statement

In this section, we give a precise statement of the control problem and explain what it means from a physical perspective. The idea comes from thermal processes where we regulate the overall temperature of the medium by adjusting an internal heat source distributed throughout the domain.

Let $M > 0$ be a given constant. A function $\mu \in L^2(\mathbb{R}^+)$ is called an *admissible control* if it satisfies the pointwise bound

$$|\mu(t)| \leq M, \quad \text{for almost every } t \geq 0.$$

We define the class of weight functions as:

$$\mathcal{R}(\Omega) := \left\{ \rho \in L^2(\Omega) \mid \rho(x) \geq 0 \text{ a.e. in } \Omega, \quad \int_{\Omega} \rho(x) \, dx = 1, \quad \rho_k \geq 0 \text{ for all } k \in \mathbb{N} \right\},$$

where ρ_k are the Fourier coefficients of function ρ .

The main objective of this work is to investigate a time-optimal control problem associated with the pseudo-parabolic system (1.1)–(1.3).

Time-optimal problem. *For a given constant $\theta > 0$ find the minimal time $T_{\min} \geq 0$, such that for $t \geq T_{\min}$ the solution $u(x, t)$ of the initial-boundary value problem (1.1)–(1.3) with some admissible control $\mu(t)$ exists, and satisfies the integral condition*

$$\frac{1}{|\Omega|} \int_{\Omega} \rho(x) u(x, t) \, dx = \theta, \quad t \in [T_{\min}, T_1], \quad (2.1)$$

for some $T_1 > T_{\min}$.

This formulation represents a physically meaningful requirement that the spatially averaged temperature in the domain Ω , weighted by $\rho(x)$, attains and preserves a prescribed level θ over the time interval $[T_{\min}, T_1]$. In contrast to pointwise temperature constraints, this condition characterizes a global thermal performance criterion, which naturally arises in heat conduction processes governed by diffusion. Such an averaged formulation is particularly relevant in practical applications where only the overall thermal state of the system is important, for example, in thermal processing of composite materials, climate control of enclosed spaces, or heating of biological tissues. From a control-theoretic viewpoint, the integral constraint allows for greater flexibility in the temperature distribution while still guaranteeing the required total thermal effect, thereby promoting physically realistic and energy-efficient control strategies.

The condition $\rho_k \geq 0$ for all $k \in \mathbb{N}$ is a technical assumption that simplifies the analysis by ensuring the kernel $K(t)$ remains positive. It is automatically satisfied for the first eigenfunction ϕ_1 (which can be chosen strictly positive), but for higher eigenfunctions it imposes a genuine restriction. For example, in a one-dimensional interval with eigenfunctions $\phi_k(x) = \sqrt{2} \sin(k\pi x)$, a nonnegative function $\rho(x)$ may

have negative Fourier coefficients for some k . Relaxing this assumption while maintaining positivity of the kernel is an interesting direction for future research.

We note that the time-optimal boundary control problem for the standard parabolic equation with the Laplace operator under Robin boundary conditions in n -dimensional domains was previously investigated in [8], where sharp estimates for the minimal time and admissible controls were derived.

To the best of our knowledge, the time-optimal control problem for the pseudo-parabolic equation with a general uniformly elliptic operator $\mathcal{A}$ has not been studied before. Our work extends the results of [8] to the pseudo-parabolic case and, for the first time, addresses the optimal time control problem for this class of equations.

We now consider the spectral problem associated with the elliptic operator $\mathcal{A}$, which plays a fundamental role in the analysis of the pseudo-parabolic system (1.1)–(1.3). Consider the spectral problem

$$\mathcal{A}\varphi(x) = \lambda\varphi(x), \quad x \in \Omega,$$

with the homogeneous Dirichlet boundary condition

$$\varphi(x) = 0, \quad x \in \partial\Omega.$$

This problem is understood in the weak sense: we seek $\lambda \in \mathbb{R}$ and $\varphi \in H_0^1(\Omega)$, $\varphi \neq 0$, satisfying

$$\int_{\Omega} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx = \lambda \int_{\Omega} \varphi \psi dx, \quad \forall \psi \in H_0^1(\Omega).$$

Under the stated assumptions on the coefficients $a_{ij}(x)$, the operator $\mathcal{A}$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ is self-adjoint, positive definite, and has a compact resolvent in $L^2(\Omega)$. Consequently, the spectrum of $\mathcal{A}$ consists of a sequence of real eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots, \quad \lambda_k \rightarrow +\infty \text{ as } k \rightarrow \infty,$$

each of finite multiplicity. The corresponding eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ form a complete orthonormal system in $L^2(\Omega)$. The first eigenvalue λ_1 is simple, and the corresponding eigenfunction φ_1 can be chosen strictly positive almost everywhere in Ω . This property follows from the strong maximum principle for second-order elliptic operators and the Krein-Rutman theorem for positive compact operators [19].

The following result provides an explicit estimate for the minimal time required to achieve the prescribed average temperature via a controlled heat source.

For a given function $h \in L^2(\Omega)$, we denote its Fourier coefficients by

$$h_k = \int_{\Omega} h(x) \varphi_k(x) dx, \quad k \in \mathbb{N}.$$

We assume that the function $h(x)$ is chosen such that its Fourier coefficients are nonnegative:

$$h_k \geq 0 \quad \text{for all } k \in \mathbb{N}.$$

For the weight function $\rho \in \mathcal{R}(\Omega)$, we have $\rho_k \geq 0$ for all $k \in \mathbb{N}$ by definition of the class $\mathcal{R}(\Omega)$. In particular, for the first eigenfunction φ_1 , the Fourier coefficients

$$\rho_1 = \int_{\Omega} \rho(x) \varphi_1(x) dx, \quad h_1 = \int_{\Omega} h(x) \varphi_1(x) dx$$

are well-defined. Since the first eigenfunction φ_1 can be chosen strictly positive almost everywhere in Ω (by the strong maximum principle) and the functions ρ and h are nonnegative, it follows that $\rho_1 > 0$ and $h_1 > 0$.

Theorem 1. Assume that

$$0 < \theta < \frac{\rho_1 h_1 M}{\lambda_1 |\Omega|},$$

where λ_1 is the first eigenvalue of $\mathcal{A}$, M is the maximal control amplitude, and $|\Omega|$ is the volume of Ω . Set

$$T_\gamma = -\frac{1 + \gamma\lambda_1}{\lambda_1} \ln \left(1 - \frac{\theta\lambda_1|\Omega|}{\rho_1 h_1 M} \right).$$

Then, a solution $T_{\min}$ of the time-optimal control problem exists, and the estimate $T_{\min} \leq T_\gamma$ is valid.

Physically, the condition

$$0 < \theta < \frac{\rho_1 h_1 M}{\lambda_1 |\Omega|}$$

ensures that the desired average temperature is attainable given the maximal amplitude M of the admissible control $\mu(t)$. Since the control acts through the distributed source term $h(x)\mu(t)$, its effectiveness depends on the spatial correlation between the source distribution $h(x)$ and the weight function $\rho(x)$, captured by the product $\rho_1 h_1$ of their first Fourier coefficients. The presence of λ_1 in the denominator reflects the fact that the diffusion operator $\mathcal{A}$ resists temperature accumulation: higher diffusion rates (larger λ_1) require stronger control effort to maintain a given average temperature. The parameter γ does not appear explicitly in the condition but influences the minimal time estimate through the factor $(1 + \gamma\lambda_1)/\lambda_1$ in T_γ , indicating that the relaxation effect ($\gamma > 0$) slows down the system's response. Thus, the inequality guarantees that the pseudo-parabolic system can be driven to the target thermal state within a finite time using admissible controls, respecting both the control constraints and the diffusion properties of the medium.

3. Volterra integral equation for control function

In this section, we demonstrate how the control problem governed by the pseudo-parabolic equation can be reduced to an equivalent integral representation. This reduction enables us to apply spectral decomposition techniques in order to investigate the existence and properties of the solution.

Let B be a Banach space and $T > 0$ a fixed constant. We denote by $C([0, T]; B)$ the Banach space of all continuous mappings $u: [0, T] \rightarrow B$ endowed with the norm

$$\|u\|_T := \sup_{t \in [0, T]} \|u(t)\|_B.$$

By $H_0^1(\Omega)$, we denote the Sobolev space of functions with zero trace on $\partial\Omega$. We define the generalized solution as follows.

Definition 1. Assume that $h \in L^2(\Omega)$ is given, and let $\mu \in L^2(0, T)$. A function $u(x, t)$ is called a generalized solution of the initial-boundary problem (1.1)–(1.3) if:

- 1) $u \in C([0, T]; H_0^1(\Omega))$ for any $T > 0$;
- 2) $u(x, 0) = 0$ for almost every $x \in \Omega$;

3) For every $\chi \in H_0^1(\Omega)$ and almost every $t > 0$,

$$\int_{\Omega} u_t \chi \, dx + \gamma \int_{\Omega} \sum_{i,j=1}^n a_{ij} \frac{\partial u_t}{\partial x_i} \frac{\partial \chi}{\partial x_j} \, dx + \int_{\Omega} \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \chi}{\partial x_j} \, dx = \mu(t) \int_{\Omega} h \chi \, dx.$$

The class $C([0, T]; H_0^1(\Omega))$ considered in Definition 1 is a subset of the class $H_0^1(\Omega_T)$ studied in the monograph [20] for problems with homogeneous boundary conditions (see Chapter III, Theorem 3.2, pp. 173–176), where $\Omega_T = \Omega \times (0, T)$. Consequently, the generalized solution defined above is also a generalized solution in the sense of [20]. However, solutions in $C([0, T]; H_0^1(\Omega))$ are continuous in $t \in [0, T]$ with respect to the $L^2(\Omega)$ -norm, whereas solutions in $H_0^1(\Omega_T)$ are only guaranteed to have a trace for almost every $t \in [0, T]$.

Lemma 1. Let $h \in L^2(\Omega)$ and $\mu \in L^2(0, T)$. Then, the function

$$u(x, t) = \sum_{k=1}^{\infty} \frac{h_k}{1 + \gamma \lambda_k} \left(\int_0^t \mu(s) e^{-\frac{\lambda_k}{1+\gamma \lambda_k}(t-s)} \, ds \right) \varphi_k(x), \quad (3.1)$$

is the unique generalized solution of problem (1.1)–(1.3) in the sense of Definition 1.

Proof. We need to verify that the function defined by (3.1) satisfies all conditions of Definition 1.

First, we show that the solution (3.1) belongs to the class $u \in C([0, T]; H_0^1(\Omega))$. Note that each term of the series belongs to $H_0^1(\Omega)$ since $\varphi_k \in H_0^1(\Omega)$. To show that the series converges in $H_0^1(\Omega)$, we estimate the L^2 -norm of its gradient. Using $\|\nabla \varphi_k\|_{L^2(\Omega)}^2 = \lambda_k$ and the orthonormality of $\{\varphi_k\}$ in $L^2(\Omega)$, we have

$$\|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} \lambda_k \left| \frac{h_k}{1 + \gamma \lambda_k} \int_0^t \mu(s) e^{-\frac{\lambda_k}{1+\gamma \lambda_k}(t-s)} \, ds \right|^2.$$

Applying the Cauchy-Schwarz inequality, we may write

$$\left| \int_0^t \mu(s) e^{-\frac{\lambda_k}{1+\gamma \lambda_k}(t-s)} \, ds \right|^2 \leq \left(\int_0^t e^{-\frac{2\lambda_k}{1+\gamma \lambda_k}(t-s)} \, ds \right) \left(\int_0^t |\mu(s)|^2 \, ds \right).$$

Since

$$\int_0^t e^{-\frac{2\lambda_k}{1+\gamma \lambda_k}(t-s)} \, ds = \frac{1 + \gamma \lambda_k}{2\lambda_k} \left(1 - e^{-\frac{2\lambda_k}{1+\gamma \lambda_k}t} \right) \leq \frac{1 + \gamma \lambda_k}{2\lambda_k},$$

we obtain

$$\begin{aligned} \|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 &\leq \|\mu\|_{L^2(0,T)}^2 \sum_{k=1}^{\infty} \frac{\lambda_k |h_k|^2}{(1 + \gamma \lambda_k)^2} \cdot \frac{1 + \gamma \lambda_k}{2\lambda_k} \\ &= \frac{1}{2} \|\mu\|_{L^2(0,T)}^2 \sum_{k=1}^{\infty} \frac{|h_k|^2}{1 + \gamma \lambda_k} \\ &\leq \frac{1}{2} \|\mu\|_{L^2(0,T)}^2 \sum_{k=1}^{\infty} |h_k|^2 = \frac{1}{2} \|\mu\|_{L^2(0,T)}^2 \|h\|_{L^2(\Omega)}^2 < \infty. \end{aligned}$$

Thus, $\nabla u(\cdot, t) \in L^2(\Omega)$ for each t , which implies $u(\cdot, t) \in H_0^1(\Omega)$. The continuity in t follows from the continuity of the exponential factors and the uniform convergence of the series. Hence, $u \in C([0, T]; H_0^1(\Omega))$.

Next, we verify that u satisfies the integral identity. For any fixed $\chi \in H_0^1(\Omega)$, we expand it in the Fourier series:

$$\chi(x) = \sum_{k=1}^{\infty} \chi_k \varphi_k(x) \quad \text{with} \quad \chi_k = \int_{\Omega} \chi(x) \varphi_k(x) \, dx.$$

Substituting the series representation of u and using the orthonormality of $\{\varphi_k\}$, we compute each term of the identity.

Note that

$$u_t(x, t) = \sum_{k=1}^{\infty} \frac{h_k}{1 + \gamma \lambda_k} \left(\mu(t) - \frac{\lambda_k}{1 + \gamma \lambda_k} \int_0^t \mu(s) e^{-\frac{\lambda_k}{1 + \gamma \lambda_k}(t-s)} \, ds \right) \varphi_k(x).$$

Now, using $\mathcal{A}\varphi_k = \lambda_k \varphi_k$ and the orthonormality, we obtain:

$$\begin{aligned} \int_{\Omega} u_t \chi \, dx &= \sum_{k=1}^{\infty} \frac{h_k}{1 + \gamma \lambda_k} \left(\mu(t) - \frac{\lambda_k}{1 + \gamma \lambda_k} \int_0^t \mu(s) e^{-\frac{\lambda_k}{1 + \gamma \lambda_k}(t-s)} \, ds \right) \chi_k, \\ \gamma \int_{\Omega} \sum_{i,j=1}^n a_{ij} \frac{\partial u_t}{\partial x_i} \frac{\partial \chi}{\partial x_j} \, dx &= \gamma \sum_{k=1}^{\infty} \lambda_k \frac{h_k}{1 + \gamma \lambda_k} \left(\mu(t) - \frac{\lambda_k}{1 + \gamma \lambda_k} \int_0^t \mu(s) e^{-\frac{\lambda_k}{1 + \gamma \lambda_k}(t-s)} \, ds \right) \chi_k, \\ \int_{\Omega} \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \chi}{\partial x_j} \, dx &= \sum_{k=1}^{\infty} \lambda_k \frac{h_k}{1 + \gamma \lambda_k} \left(\int_0^t \mu(s) e^{-\frac{\lambda_k}{1 + \gamma \lambda_k}(t-s)} \, ds \right) \chi_k. \end{aligned}$$

Summing these three expressions, the terms containing the integral cancel each other:

$$\begin{aligned} &\int_{\Omega} u_t \chi \, dx + \gamma \int_{\Omega} \sum_{i,j=1}^n a_{ij} \frac{\partial u_t}{\partial x_i} \frac{\partial \chi}{\partial x_j} \, dx + \int_{\Omega} \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \chi}{\partial x_j} \, dx \\ &= \sum_{k=1}^{\infty} \frac{h_k}{1 + \gamma \lambda_k} \mu(t) \chi_k + \gamma \sum_{k=1}^{\infty} \lambda_k \frac{h_k}{1 + \gamma \lambda_k} \mu(t) \chi_k \\ &= \mu(t) \sum_{k=1}^{\infty} h_k \chi_k = \mu(t) \int_{\Omega} h(x) \chi(x) \, dx. \end{aligned}$$

Thus, the integral identity holds for every $\chi \in H_0^1(\Omega)$ and almost every $t > 0$. $\square$

By substituting the expression (3.1) into the integral condition (2.1), we arrive at the following relation:

$$\begin{aligned} f(t) &= \frac{1}{|\Omega|} \int_{\Omega} \rho(x) u(x, t) \, dx \\ &= \frac{1}{|\Omega|} \sum_{k=1}^{\infty} \frac{h_k}{1 + \gamma \lambda_k} \left(\int_0^t \mu(s) e^{-\frac{\lambda_k}{1 + \gamma \lambda_k}(t-s)} \, ds \right) \int_{\Omega} \rho(x) \varphi_k(x) \, dx \\ &= \frac{1}{|\Omega|} \sum_{k=1}^{\infty} \frac{h_k \rho_k}{1 + \gamma \lambda_k} \int_0^t \mu(s) e^{-\frac{\lambda_k}{1 + \gamma \lambda_k}(t-s)} \, ds, \end{aligned}$$

where function $f(t) = \theta$ for $t \in [T_{\min}, T_1]$. Define the kernel function

$$K(t) := \sum_{k=1}^{\infty} \frac{h_k \rho_k}{1 + \gamma \lambda_k} e^{-\frac{\lambda_k}{1 + \gamma \lambda_k} t}, \quad t \geq 0. \quad (3.2)$$

Consequently, we obtain the following Volterra integral equation of the first kind for the admissible control $\mu(t)$:

$$\frac{1}{|\Omega|} \int_0^t K(t-\tau) \mu(\tau) d\tau = f(t), \quad t > 0, \quad (3.3)$$

where $f(t) = \theta$ for $t \in [T_{\min}, T_1]$, and $|\Omega|$ denotes the volume of the domain Ω .

The integral equation (3.3) plays a central role in determining the control function $\mu(t)$ that drives the system to the desired thermal state. The solvability of this equation depends on the properties of the kernel $K(t)$. Therefore, we first establish the continuity of the kernel, which is essential for the existence of a solution.

Lemma 2. *Let $h \in L^2(\Omega)$ and $\rho \in \mathcal{R}(\Omega)$. Then, the kernel function $K(t)$, defined by (3.2), is continuous for all $t \geq 0$.*

Proof. The kernel function is given by the series

$$K(t) = \sum_{k=1}^{\infty} \frac{h_k \rho_k}{1 + \gamma \lambda_k} e^{-\frac{\lambda_k}{1+\gamma \lambda_k} t}, \quad t \geq 0.$$

Note that, for all $k \in \mathbb{N}$ and $t \geq 0$, we have the estimate

$$\left| \frac{h_k \rho_k}{1 + \gamma \lambda_k} e^{-\frac{\lambda_k}{1+\gamma \lambda_k} t} \right| \leq \frac{|h_k \rho_k|}{1 + \gamma \lambda_k} \leq |h_k \rho_k|,$$

since $1 + \gamma \lambda_k \geq 1$ for $k \in \mathbb{N}$. We show that this series converges uniformly on $[0, \infty)$. Since $\rho \in \mathcal{R}(\Omega)$, its Fourier coefficients satisfy $\rho_k \geq 0$ for all $k \in \mathbb{N}$. Moreover, by the Cauchy-Schwarz inequality and the fact that $\{\varphi_k\}$ is an orthonormal basis in $L^2(\Omega)$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} |h_k \rho_k| &\leq \left(\sum_{k=1}^{\infty} |h_k|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |\rho_k|^2 \right)^{1/2} \\ &= \|h\|_{L^2(\Omega)} \|\rho\|_{L^2(\Omega)} < \infty. \end{aligned}$$

By the Weierstrass M-test, the series converges uniformly on $[0, \infty)$. Each term of the series is a continuous function of t (as an exponential function). Uniform convergence preserves continuity, therefore $K(t)$ is continuous on $[0, \infty)$. This completes the proof. $\square$

Corollary 1. *Under the same assumptions as in Lemma 2, the kernel function $K(t)$ is positive and strictly decreasing for all $t > 0$. Moreover,*

$$K'(t) < 0 \quad \text{for} \quad t > 0.$$

Proof. Since $h_k \geq 0$, $\rho_k \geq 0$, and $\lambda_k > 0$, each term of the series is nonnegative. Moreover, at least one coefficient $h_k \rho_k$ is positive (otherwise $h \equiv 0$ or $\rho \equiv 0$, which is excluded by the problem setting). Therefore, for any $t \geq 0$,

$$K(t) = \sum_{k=1}^{\infty} \frac{h_k \rho_k}{1 + \gamma \lambda_k} e^{-\frac{\lambda_k}{1+\gamma \lambda_k} t} > 0.$$

To ensure that termwise differentiation is valid, we note that for any $\delta > 0$ and $t \geq \delta$,

$$\left| \frac{d}{dt} \left(\frac{h_k \rho_k}{1 + \gamma \lambda_k} e^{-\frac{\lambda_k}{1 + \gamma \lambda_k} t} \right) \right| = \frac{|h_k \rho_k| \lambda_k}{(1 + \gamma \lambda_k)^2} e^{-\frac{\lambda_k}{1 + \gamma \lambda_k} t} \leq |h_k \rho_k| \cdot \frac{\lambda_k}{(1 + \gamma \lambda_k)^2} e^{-\frac{\lambda_1}{1 + \gamma \lambda_1} \delta}.$$

Since $\frac{\lambda_k}{(1 + \gamma \lambda_k)^2} \leq \frac{1}{4\gamma}$ and $\sum_{k=1}^{\infty} |h_k \rho_k| < \infty$, the series of derivatives converges uniformly on $[\delta, \infty)$ by the Weierstrass M-test. Hence, termwise differentiation is valid for all $t > 0$.

Differentiating term by term, we obtain

$$\begin{aligned} K'(t) &= - \sum_{k=1}^{\infty} \frac{h_k \rho_k}{1 + \gamma \lambda_k} \cdot \frac{\lambda_k}{1 + \gamma \lambda_k} e^{-\frac{\lambda_k}{1 + \gamma \lambda_k} t} \\ &= - \sum_{k=1}^{\infty} \frac{h_k \rho_k \lambda_k}{(1 + \gamma \lambda_k)^2} e^{-\frac{\lambda_k}{1 + \gamma \lambda_k} t}. \end{aligned}$$

Since all terms in the series are nonnegative, and at least one term is positive, we have $K'(t) < 0$ for all $t > 0$. Thus, $K(t)$ is strictly decreasing. $\square$

Moreover, from the series representation (3.2) and the properties of $K(t)$ established in Corollary 1, we have

$$\begin{aligned} \int_0^{\infty} |K'(s)| ds &= - \int_0^{\infty} K'(s) ds \\ &= K(0), \end{aligned}$$

since $K(t)$ is positive, strictly decreasing, and $\lim_{t \rightarrow \infty} K(t) = 0$.

4. Estimate of the minimal time

In this section, we derive an explicit upper bound for the minimal time required to reach a prescribed averaged temperature level in the domain. The estimate is obtained by means of a lower bound for the kernel of the corresponding Volterra integral equation, which plays a key role in the time-optimal control analysis.

Lemma 3. *The following estimate holds:*

$$K(t) \geq \frac{h_1 \rho_1}{1 + \gamma \lambda_1} e^{-\frac{\lambda_1}{1 + \gamma \lambda_1} t}, \quad t > 0,$$

where λ_1 is the first eigenvalue of the operator $\mathcal{A}$, and h_1, ρ_1 are the Fourier coefficients of h and ρ with respect to the first eigenfunction φ_1 .

Proof. Recall that the kernel function $K(t)$ is defined by the series

$$K(t) = \sum_{k=1}^{\infty} \frac{h_k \rho_k}{1 + \gamma \lambda_k} e^{-\frac{\lambda_k}{1 + \gamma \lambda_k} t}, \quad t \geq 0.$$

Since all terms in the series are nonnegative (by the assumptions $h_k \geq 0, \rho_k \geq 0$), the sum is greater than or equal to any individual term. In particular, taking the first term corresponding to $k = 1$, we obtain

$$K(t) \geq \frac{h_1 \rho_1}{1 + \gamma \lambda_1} e^{-\frac{\lambda_1}{1 + \gamma \lambda_1} t}, \quad t > 0.$$

This completes the proof. $\square$

We introduce the function

$$W(t) := \frac{1}{|\Omega|} \int_0^t K(t - \tau) d\tau = \frac{1}{|\Omega|} \int_0^t K(\tau) d\tau, \quad t > 0, \quad (4.1)$$

where the second equality follows from the change of variables $\eta = t - \tau$. This function $W(t)$ has a natural physical meaning: it describes how the average temperature in Ω accumulates over time when the system is subjected to a unit control input. In other words, it is the total thermal response of the medium up to time t resulting from an impulse of unit strength.

From the previously established properties of $K(t)$, we immediately obtain

$$W(0) = 0, \quad W'(t) = \frac{1}{|\Omega|} K(t) \geq 0,$$

so $W(t)$ is nonnegative, continuous, and strictly increasing for $t > 0$. Furthermore, since $K(t)$ is nonnegative, decreasing, and integrable on $(0, \infty)$, the limit

$$W^* := \frac{1}{|\Omega|} \int_0^\infty K(\tau) d\tau < +\infty \quad (4.2)$$

is finite. Physically, W^* represents the maximum possible accumulated average temperature that can be attained in Ω if a unit control were applied indefinitely. It reflects the system's inherent thermal capacity under the given operator $\mathcal{A}$ and boundary conditions.

As a consequence, for any admissible control $\mu(t)$ bounded by $|\mu(t)| \leq M$, the total accumulated average temperature produced by the control cannot exceed MW^* .

Lemma 4. Assume that the desired average temperature θ satisfies

$$0 < \theta < MW^*,$$

where W^* is defined by (4.2). Then, one can find a time $T_{\min} > 0$ and an admissible control $\mu(t)$ with $|\mu(t)| \leq M$ such that

$$\frac{1}{|\Omega|} \int_0^{T_{\min}} K(T_{\min} - \tau) \mu(\tau) d\tau = \theta. \quad (4.3)$$

Proof. Since the kernel $K(t)$ is positive and strictly decreasing, the convolution $\int_0^t K(t - \tau) \mu(\tau) d\tau$ for a fixed bound $|\mu(t)| \leq M$ is maximized by the constant control $\mu \equiv M$; this follows from a rearrangement (or Chebyshev-type) inequality. Hence, the constant control provides the fastest accumulation of the average temperature and is therefore a natural candidate for estimating the minimal time. Take the simplest possible control, namely, the constant function $\mu(t) \equiv M$. Substituting this into the left-hand side of (4.3) yields

$$\frac{1}{|\Omega|} \int_0^t K(t - \tau) \mu(\tau) d\tau = \frac{M}{|\Omega|} \int_0^t K(t - \tau) d\tau = MW(t),$$

where $W(t)$ is defined in (4.1). As shown above, $W(t)$ is continuous, strictly increasing, and satisfies $W(0) = 0$ and $\lim_{t \rightarrow \infty} W(t) = W^*$. Consequently, the function $MW(t)$ also increases continuously from 0 to MW^* .

The condition $0 < \theta < MW^*$ places θ exactly within the range of $MW(t)$. By the intermediate value theorem, there exists a unique $T_{\min} > 0$ such that

$$MW(T_{\min}) = \theta. \quad (4.4)$$

For this $T_{\min}$ and the control $\mu \equiv M$, equation (4.3) holds. Clearly, this control satisfies the bound $|\mu(t)| \leq M$. Thus, the desired time and control exist. $\square$

The main objective of the present work is to obtain a quantitative estimate for the minimal time $T_{\min}$ required to reach the prescribed averaged temperature level θ .

Lemma 5. Assume that the target average temperature θ satisfies

$$0 < \theta < \frac{\rho_1 h_1 M}{\lambda_1 |\Omega|}, \quad (4.5)$$

where ρ_1 and h_1 are the first Fourier coefficients of ρ and h , λ_1 is the first eigenvalue of $\mathcal{A}$, M is the maximal control amplitude, and $|\Omega|$ denotes the volume of the domain. Then, there exists a minimal time $T_{\min} > 0$ such that $T_{\min} \leq T_\gamma$ and equation (4.3) holds, with T_γ given explicitly by

$$T_\gamma = -\frac{1 + \gamma\lambda_1}{\lambda_1} \ln \left(1 - \frac{\theta \lambda_1 |\Omega|}{\rho_1 h_1 M} \right).$$

Proof. We start from the lower bound for the kernel $K(t)$ obtained in Lemma 3:

$$K(t) \geq \frac{h_1 \rho_1}{1 + \gamma\lambda_1} e^{-\frac{\lambda_1}{1 + \gamma\lambda_1} t}, \quad t > 0.$$

Integrating this estimate gives a corresponding lower bound for $W(t)$:

$$W(t) = \frac{1}{|\Omega|} \int_0^t K(\tau) d\tau \geq \frac{h_1 \rho_1}{\lambda_1 |\Omega|} \left(1 - e^{-\frac{\lambda_1}{1 + \gamma\lambda_1} t} \right).$$

Note that the integration preserves the exponential exponent:

$$\int_0^t e^{-\frac{\lambda_1}{1 + \gamma\lambda_1} \tau} d\tau = \frac{1 + \gamma\lambda_1}{\lambda_1} \left(1 - e^{-\frac{\lambda_1}{1 + \gamma\lambda_1} t} \right).$$

Thus the factor $\frac{1 + \gamma\lambda_1}{\lambda_1}$ appears in the final estimate leading to the expression for T_γ . The minimal time $T_{\min}$ is determined by the condition $MW(T_{\min}) = \theta$. The righthand side of the above inequality is increasing in t and tends to $\frac{h_1 \rho_1 M}{\lambda_1 |\Omega|}$ as $t \rightarrow \infty$. Therefore, under condition (4.5), the equation $MW(t) = \theta$ has a unique solution $T_{\min} > 0$.

Now let T_γ be the time at which the lower bound itself reaches θ , i.e.,

$$\frac{h_1 \rho_1 M}{\lambda_1 |\Omega|} \left(1 - e^{-\frac{\lambda_1}{1 + \gamma\lambda_1} t} \right) = \theta.$$

Solving this equation yields

$$T_\gamma = -\frac{1 + \gamma\lambda_1}{\lambda_1} \ln \left(1 - \frac{\theta \lambda_1 |\Omega|}{\rho_1 h_1 M} \right).$$

Since $W(t)$ is increasing and satisfies $MW(t) \geq \frac{h_1 \rho_1 M}{\lambda_1 |\Omega|} \left(1 - e^{-\frac{\lambda_1}{1 + \gamma\lambda_1} t} \right)$ for all t , at time T_γ we have

$$MW(T_\gamma) \geq \theta = MW(T_{\min}).$$

Monotonicity of W then implies $T_\gamma \geq T_{\min}$. Hence, $T_{\min} \leq T_\gamma$, which is the desired estimate. $\square$

Lemma 6. Let $T_{\min} > 0$ be the minimal time satisfying equality (4.4) together with condition (4.5). Then, there exists a time $T_1 > T_{\min}$ and a measurable real-valued control function $\mu(t)$ such that

$$|\mu(t)| \leq M \quad \text{a.e. on } (0, T_1),$$

and the integral condition (2.1) is fulfilled.

Proof. Recall that the average temperature at time t is given by the convolution

$$\frac{1}{|\Omega|} \int_0^t K(t - \tau) \mu(\tau) d\tau.$$

We already know from Lemma 5 that by using the constant control $\mu(t) = M$ up to time $T_{\min}$, we achieve exactly $MW(T_{\min}) = \theta$. Our task now is to extend the control beyond $T_{\min}$ so that the temperature remains at this same level θ for all t in some interval $[T_{\min}, T_1]$.

To do this, we define a target function $f(t)$ that represents the desired average temperature:

$$f(t) = \begin{cases} MW(t), & 0 \leq t \leq T_{\min}, \\ \theta, & T_{\min} \leq t \leq T_1. \end{cases}$$

Notice that $f(t)$ is continuous at $t = T_{\min}$ because $MW(T_{\min}) = \theta$, and it is smooth on each subinterval. We now look for a control of the form

$$\mu(t) = \begin{cases} M, & 0 \leq t \leq T_{\min}, \\ \mu_1(t), & T_{\min} \leq t \leq T_1, \end{cases}$$

where $\mu_1(t)$ is to be determined. Substituting this into the convolution for $t \geq T_{\min}$ gives

$$\frac{M}{|\Omega|} \int_0^{T_{\min}} K(t - \tau) d\tau + \frac{1}{|\Omega|} \int_{T_{\min}}^t K(t - \tau) \mu_1(\tau) d\tau = \theta, \quad t \in [T_{\min}, T_1].$$

To find $\mu_1(t)$, we differentiate this equation with respect to t . Using the Leibniz rule and the fact that $K(0) > 0$, we obtain a Volterra integral equation of the second kind:

$$K(0)\mu_1(t) + \int_{T_{\min}}^t K'(t - \tau) \mu_1(\tau) d\tau = M(K(t - T_{\min}) - K(t)), \quad t \in [T_{\min}, T_1].$$

Setting $t = T_{\min}$ in the Volterra equation, the integral term vanishes, leaving

$$K(0)\mu_1(T_{\min}) = M(K(0) - K(T_{\min})).$$

Since K is strictly decreasing and positive, we have $K(T_{\min}) < K(0)$ and $K(T_{\min}) > 0$. Therefore,

$$\mu_1(T_{\min}) = M \left(1 - \frac{K(T_{\min})}{K(0)} \right) < M.$$

Because μ_1 is continuous (as the solution of a Volterra equation of the second kind with continuous kernel), there exists $\delta > 0$ such that $|\mu_1(t)| \leq M$ for all $t \in [T_{\min}, T_{\min} + \delta]$. The kernel function $K(t)$ has

several nice properties: it is positive, strictly decreasing, and continuously differentiable on $(0, \infty)$, and $K(0) > 0$. Moreover, $K'(t)$ is integrable. For such kernels, the standard theory of Volterra equations guarantees that for any $T_1 > T_{\min}$, there exists a unique continuous solution $\mu_1(t)$ on $[T_{\min}, T_1]$.

Let us examine this solution at the left endpoint $t = T_{\min}$. Setting $t = T_{\min}$ in the equation, the integral term vanishes, leaving

$$K(0)\mu_1(T_{\min}) = M(K(0) - K(T_{\min})),$$

which simplifies to

$$\mu_1(T_{\min}) = M \left(1 - \frac{K(T_{\min})}{K(0)} \right).$$

Since K is strictly decreasing, we have $K(T_{\min}) < K(0)$, and therefore $\mu_1(T_{\min}) < M$. By continuity of μ_1 , there exists some $\delta > 0$ such that $|\mu_1(t)| \leq M$ for all $t \in [T_{\min}, T_{\min} + \delta]$. We may then choose $T_1 = T_{\min} + \delta$. With this choice, the control $\mu(t)$ defined above satisfies $|\mu(t)| \leq M$ almost everywhere on $(0, T_1)$. Moreover, by construction, it fulfills

$$\frac{1}{|\Omega|} \int_0^t K(t - \tau)\mu(\tau) d\tau = \theta \quad \text{for all } t \in [T_{\min}, T_1],$$

which is exactly the integral condition (2.1). This completes the proof. $\square$

Remark 1. The control $\mu(t)$ has a jump discontinuity at $t = T_{\min}$, corresponding to switching from maximal heating to a regulated regime. This does not violate admissibility, since $\mu \in L^\infty(0, T_1)$, and the associated weak solution remains well defined.

The proof of Theorem 1 follows now easily from Lemmas 5 and 6.

5. Physical interpretation of the parameter γ

The estimate $T_{\min} \leq T_\gamma$ obtained in Lemma 5 reveals an interesting physical phenomenon. Recall that T_γ depends on the parameter γ through the factor $\frac{1+\gamma\lambda_1}{\lambda_1}$:

$$T_\gamma = -\frac{1 + \gamma\lambda_1}{\lambda_1} \ln \left(1 - \frac{\theta \lambda_1 |\Omega|}{\rho_1 h_1 M} \right).$$

When $\gamma = 0$, equation (1.1) reduces to the classical parabolic heat equation

$$u_t + \mathcal{A}u = h(x)\mu(t),$$

and the corresponding minimal time estimate becomes

$$T_0 = -\frac{1}{\lambda_1} \ln \left(1 - \frac{\theta \lambda_1 |\Omega|}{\rho_1 h_1 M} \right).$$

Comparing the two expressions, we see that

$$T_\gamma = (1 + \gamma\lambda_1) T_0.$$

Since $\gamma > 0$ and $\lambda_1 > 0$, we have $1 + \gamma\lambda_1 > 1$, and consequently

$$T_\gamma > T_0.$$

This inequality carries a clear physical message: in a pseudo-parabolic medium, reaching a prescribed average temperature level requires more time than in a purely parabolic one. The additional time factor $1 + \gamma\lambda_1$ reflects the “relaxation” or “memory” effect inherent in pseudo-parabolic equations. The term $\gamma\mathcal{A}u_t$ introduces inertia into the system and resists rapid changes in the temperature gradient, effectively slowing down the diffusion process. As a result, the control needs to act over a longer period to accumulate the same thermal effect.

In other words, the parameter γ quantifies how strongly the medium “remembers” past temperature gradients. Larger values of γ mean stronger memory effects and, accordingly, longer minimal times to achieve the desired thermal state. This observation aligns perfectly with the physical interpretation of pseudo-parabolic equations as models for heat conduction in complex materials with internal relaxation mechanisms. We note a conceptual similarity with the Cattaneo-Vernotte model of heat conduction, where a relaxation time leads to finite speed of heat propagation. In both models, memory effects result in a slower thermal response compared to the classical Fourier law, although the mathematical structure differs: the Cattaneo-Vernotte model is hyperbolic, while our pseudo-parabolic model remains parabolic in nature.

Author contributions

Farrukh Dekhkonov: Writing-original draft, writing-review & editing, methodology, formal analysis, investigation; Zhuang Han: Writing-review & editing, methodology, formal analysis, validation; Malika Nematjonova: Writing-review & editing, methodology, validation; Weipeng Wu: Writing-review & editing, methodology.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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References

1. B. D. Coleman, W. Noll, n approximation theorem for functionals, with applications in continuum mechanics, *Arch. Ration. Mech. Anal.*, **6** (1960), 355–370. <https://doi.org/10.1007/BF00276168>
2. A. Friedman, Optimal control for parabolic equations, *J. Math. Anal. Appl.*, **18** (1967), 479–491. [https://doi.org/10.1016/0022-247X\(67\)90040-6](https://doi.org/10.1016/0022-247X(67)90040-6)
3. H. O. Fattorini, D. L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, *Arch. Ration. Mech. Anal.*, **43** (1971), 272–292. <https://doi.org/10.1007/BF00250466>

4. Yu. V. Egorov, The optimal control in Banach space, *Uspekhi Mat. Nauk*, **18** (1963), 211–213.
5. J. L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Dunod Gauthier-Villars, Paris, 1968.
6. F. Dekhkonov, W. Li, W. Wu, Boundary control problem for a heat equation in a multidimensional parallelepiped, *Discrete Contin. Dyn. Syst. Ser. S*, **29** (2026), 53–70. <https://doi.org/10.3934/dcdss.2025112>
7. F. Dekhkonov, W. Li, W. Wu, On the time-optimal control problem for a fourth order parabolic equation in the three-dimensional space, *Commun. Anal. Mech.*, **17** (2025), 413–428. <https://doi.org/10.3934/cam.2025017>
8. S. Albeverio, Sh. A. Alimov, On one time-optimal control problem associated with the heat exchange process, *Appl. Math. Optim.*, **57** (2008), 58–68. <https://doi.org/10.1007/s00245-007-9008-7>
9. H. Xu, Existence and blow-up of solutions for finitely degenerate semilinear parabolic equations with singular potentials, *Commun. Anal. Mech.*, **15** (2023), 132–161. <https://doi.org/10.3934/cam.2023008>
10. X. Wu, Y. Zhao, X. Yang, On a singular parabolic p -Laplacian equation with logarithmic nonlinearity, *Commun. Anal. Mech.*, **16** (2024), 528–553. <https://doi.org/10.3934/cam.2024025>
11. Y. Chen, V. Radulescu, R. Xu, High energy blowup and blowup time for a class of semilinear parabolic equations with singular potential on manifolds with conical singularities, *Commun. Math. Sci.*, **21** (2023), 25–63. <https://dx.doi.org/10.4310/CMS.2023.v21.n1.a2>
12. R. Xu, W. Lian, Y. Niu, Global well-posedness of coupled parabolic systems, *Sci. China Math.*, **63** (2020), 321–356. <https://doi.org/10.1007/s11425-017-9280-x>
13. A. Dicke, I. Veselić, Spherical Logvinenko-Sereda-Kovrijkine type inequality and null-controllability of the heat equation on the sphere, *Arch. Math.*, **123** (2024), 543–556. <https://doi.org/10.1007/s00013-024-02051-4>
14. B. Allal, G. Fragnelli, J. Salhi, Null controllability for degenerate parabolic equations with a nonlocal space term, *Discrete Contin. Dyn. Syst. Ser. S*, **17** (2024), 1821–1856. <https://doi.org/10.3934/dcdss.2022164>
15. B. D. Coleman, R. J. Duffin, V. J. Mizel, Instability, uniqueness, and nonexistence theorems for the equation on a strip, *Arch. Ration. Mech. Anal.*, **19** (1965), 100–116. <https://doi.org/10.1007/BF00282277>
16. W. Lian, J. Wang, R. Xu, Global existence and blow up of solutions for pseudo-parabolic equation with singular potential, *J. Differential Equations*, **269** (2020), 4914–4959. <https://doi.org/10.1016/j.jde.2020.03.047>
17. J. Cheng, Q. Wang, Global existence and finite-time blowup for a mixed pseudo-parabolic $r(x)$ -Laplacian equation, *Adv. Nonlinear Anal.*, **13** (2024), 1–26. <https://doi.org/10.1515/anona-2023-0133>
18. F. Dekhkonov, On one boundary control problem for a pseudo-parabolic equation in a two-dimensional domain, *Commun. Anal. Mech.*, **17** (2025), 1–14. <https://doi.org/10.3934/cam.2025001>

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19. H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011. <https://doi.org/10.1007/978-0-387-70914-7>
 20. O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, 1968. <https://doi.org/10.1090/mmono/023>



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