



Research article

A new analytical technique of the fully implicit Crank–Nicolson discontinuous Galerkin method for the Ginzburg–Landau Model

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Abstract: In this paper, a fully implicit Crank–Nicolson discontinuous Galerkin method is proposed for solving the Ginzburg–Landau equation. By leveraging a novel analytical technique, we rigorously establish the unique solvability of the constructed numerical scheme as well as its unconditionally optimal error estimates under both the L^2 -norm and the energy norm. The core of the proof hinges on the L^2 -norm boundedness of the numerical solution and the refined estimation of the cubic nonlinear term. Finally, four numerical examples are presented to validate the theoretical findings.

Keywords: Crank–Nicolson; discontinuous Galerkin method; Ginzburg–Landau equation; unconditionally optimal error estimates; cubic nonlinear term

Mathematics Subject Classification: 65M60, 65N15

1. Introduction

The Ginzburg–Landau equation is a class of important complex-valued nonlinear partial differential equations, first proposed by physicists Vitaly Ginzburg and Lev Landau in the 1950s to model low-temperature superconductivity, this model finds significant applications in fields including condensed matter physics, nonlinear optics, and nonequilibrium hydrodynamic systems [1]. Under different application contexts, the Ginzburg–Landau equation exhibits distinct formulations such as the nonlinear Ginzburg–Landau equation with a Caputo derivative, the Ginzburg–Landau–Schrödinger equation, the Kuramoto–Tsuzuki complex equation with strong nonlinear effects, the dynamical Ginzburg–Landau equation under the Coulomb gauge, and the time-dependent Ginzburg–Landau equation under the temporal gauge.

In this paper, we are devoted to the discontinuous Galerkin approximation of the following two-dimensional Ginzburg–Landau equation with cubic nonlinearity in a bounded convex polygonal domain $\Omega \subset \mathbb{R}^2$:

$$u_t - (\nu + i\alpha)\Delta u + (\kappa + i\beta)|u|^2 u - \gamma u = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T], \quad (1.1)$$

subject to the homogeneous Dirichlet boundary condition

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T], \quad (1.2)$$

and the initial condition

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}. \quad (1.3)$$

Here, $u(\mathbf{x}, t)$ is a complex-valued order parameter that describes the macroscopic state of the physical system, such as superconductivity, nonlinear optics, or nonequilibrium hydrodynamics. Specifically, the parameter $\nu > 0$ is the dissipation coefficient, α is the linear dispersion coefficient, $\kappa > 0$ measures the cubic nonlinear interaction, β is the nonlinear dispersion coefficient, and γ is the linear growth–decay coefficient, with relevant physical background available in [1]. Additionally, $\partial\Omega$ represents the boundary of Ω , $\mathbf{x} = (x, y)$; Δ is the Laplace operator; $i = \sqrt{-1}$ denotes the imaginary unit; and $u^0(\mathbf{x})$ is a sufficiently smooth complex-valued function with a zero boundary trace.

Due to the significance of this equation, numerous scholars have proposed various effective numerical methods for solving this model, including the finite difference method (FDM), the finite element method (FEM), the meshless method, and the discontinuous polygonal method. More precisely, Zhang and Yang [2] proposed a linearized three-layer finite difference scheme for the nonlinear variable-coefficient Kuramoto–Tsuzuki complex equation. The scheme employs standard central finite differences for spatial discretization, combines central averaging with central differences for temporal discretization, and adopts a semi-implicit linearization approach to handle the nonlinear terms. Li et al. [3] developed a linearized Galerkin finite element method for solving the two-dimensional and three-dimensional Kuramoto–Tsuzuki equation. By classifying and refining the treatment of the nonlinear terms, the authors derived the optimal L^2 -norm error estimate for the numerical solution. Li et al. [4] studied a fast element-free Galerkin (EFG) method for solving the nonlinear complex Ginzburg–Landau equation. Using the space–time error-splitting technique, they achieved optimal order convergence. Recently, Guan and Cao [5] proposed a discontinuous Galerkin method on polygonal meshes, implemented via the weighted θ scheme. They proved boundedness and optimal convergence under various norms. From the aforementioned literature review, it can be observed that most numerical studies employ semi-implicit schemes wherein the nonlinear terms are treated explicitly. Although such schemes afford computational convenience, they compromise certain inherent properties of the original equation to a limited extent. Motivated by this consideration, a fully implicit numerical scheme is proposed in this paper, which combines the discontinuous Galerkin method for spatial discretization with the Crank–Nicolson scheme for temporal discretization. To the best of our knowledge, such a fully implicit Crank–Nicolson discontinuous Galerkin scheme for the Ginzburg–Landau equation has not been reported in the existing literature. Compared with existing semi-implicit schemes, our fully implicit method has two major merits: unconditional convergence without restriction on the space–time mesh ratio and superior preservation of the original model’s physical properties. However, full implicitness necessitates solving a nonlinear algebraic system at each time level, leading to higher computational overhead and more intricate theoretical analysis.

As is well-established in the field, two distinct classes of numerical schemes are commonly employed for nonlinear partial differential equations: fully implicit schemes and linearized schemes. For linearized schemes, the unconditionally optimal error estimates are usually derived using the space–time error splitting technique proposed by Li and Sun [6]. The core idea is to first introduce a time-discrete system

and then use the inverse inequality to prove the boundedness of the numerical solution in the infinity norm. More precisely,

$$\begin{aligned}\|u_h^n\|_\infty &\leq \|R_h U^n\|_\infty + \|u_h^n - R_h U^n\|_\infty \\ &\leq C\|U^n\|_2 + Ch^{-\frac{d}{2}}\|u_h^n - R_h U^n\| \\ &\leq C\|U^n\|_2 + Ch^{2-\frac{d}{2}} \\ &\leq C,\end{aligned}$$

where U^n is the numerical solution of time-discrete system at the n th level, R_h is a suitable elliptic projection operator, and h and τ denote the spatial and temporal mesh sizes, respectively. Another method employed in the literature is to prove the boundedness of the numerical solution in H^1 -norm by means of mathematical induction [7]. However, those methods are only applicable to linearized numerical schemes, and they fail for fully implicit algorithms. For fully implicit schemes, there are generally two analytical methods, one of which is the duality argument technique proposed by Feng et al. [8]. The key points of its proof lie in the Schaefer's fixed-point theorem and the discrete Ladyzhenskaya inequality. This method has been successfully applied to solving other nonlinear partial differential equations, including the Schrödinger–Helmholtz equation [9] and the coupled Schrödinger equation [10]. Another approach involves the cut-off function technique [11]: The core idea lies in truncating the nonlinear term to satisfy the global Lipschitz condition. Both of the aforementioned techniques merely establish the existence of a solution to the fully discrete numerical scheme and the uniqueness of a numerical solution in a neighborhood of an exact solution. However, they fail to address the global unique solvability of the scheme.

To establish the global unique solvability of the proposed fully implicit numerical scheme, we develop a novel analytical approach for its proof, drawing on the core concepts of the two techniques outlined earlier. This constitutes the primary contribution of this paper. First, we establish the uniform boundedness of the numerical solution in the L^2 -norm. Subsequently, by utilizing this result and conducting a refined estimate of the cubic nonlinear term, we derive an estimation formula for the numerical solution and Ritz projection of the true solution. Finally, we analyze the time step τ and spatial step h under two cases ($\tau \leq h$ and $\tau > h$), obtain the boundedness of the numerical solution in the energy norm, and further establish the unique solvability and optimal convergence rate of the fully discrete numerical scheme. We conjecture that the analytical technique proposed in this paper can be extended to other types of nonlinear partial differential equations.

The structure of this paper is organized as follows. In Section 2, we introduce the notations, lemmas, and the numerical scheme under investigation. Section 3 is dedicated to the theoretical analysis of the numerical scheme, encompassing the global existence and uniqueness of the numerical solution, as well as its convergence analysis. In Section 4, four numerical examples are presented to validate the theoretical results. Finally, Section 5 provides a summary of the entire paper and an outlook for future research.

2. The fully implicit Crank–Nicolson discontinuous Galerkin scheme

In this section, we first introduce key notations and preliminary lemmas employed throughout the paper upon which we formulate the fully implicit Crank–Nicolson discontinuous Galerkin method.

Throughout the paper, we adopt the standard Sobolev space notation commonly used in the discontinuous finite element literature. For any integer $m \geq 1$, let $|\cdot|_{H^m(\Omega)}$ and $\|\cdot\|_{H^m(\Omega)}$ denote the seminorm and norm of the Sobolev space $H^m(\Omega)$, respectively. The norm of the complex-valued Banach space $L^p(\Omega)$ is denoted by $\|\cdot\|_p$, where $1 \leq p \leq \infty$. When $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with respect to the following inner product and norm:

$$(u, v) = \int_{\Omega} uv^* d\mathbf{x}, \quad \|u\| = \sqrt{(u, u)}, \quad u, v \in L^2(\Omega),$$

where v^* denotes the conjugate of v . When no ambiguity arises, this paper adopts similar notations for the inner product and norm of vector functions. Finally, for a strongly measurable function $v : (0, T) \rightarrow X$, where X is a complex Banach space, we introduce the Bochner space defined as

$$L^p(0, T; X) = \left\{ v : \int_0^T \|v\|_X^p dt < \infty \right\}$$

with the associated norm

$$\|v\|_{L^p(0, T; X)} = \left(\int_0^T \|v\|_X^p dt \right)^{\frac{1}{p}}.$$

When $p = \infty$, the norm and integral defined above are implicitly subject to the usual modification. Throughout this paper, C denotes a generic positive constant independent of the mesh parameters h and τ , whose value may vary in different places.

Let $\mathcal{K}_h$ denote a sequence of quasi-uniform triangulations of Ω , where h denotes the mesh size (defined as the diameter of the largest element in $\mathcal{K}_h$). For simplicity, we assume $\mathcal{K}_h$ is conforming: The intersection of any two distinct elements in the triangulation is either the empty set, a single vertex, or a complete edge. We denote the set of all edges in the partition $\mathcal{K}_h$ by $\mathcal{E}_h$. Moreover, interior edges are collected in the set $\mathcal{E}_h^i$ and boundary edges in $\mathcal{E}_h^b$. For every element $K \in \mathcal{K}_h$, h_K is the diameter of K . Similarly, define h_E as the length of each edge E . We orient each edge E with a fixed normal vector $\mathbf{n}_E$. When E is on the boundary of the region Ω , take $\mathbf{n}_E$ as the unit outward normal vector of the boundary $\partial\Omega$. Let v be a scalar-valued function defined on Ω , and suppose v is sufficiently smooth. This smoothness ensures that v has a possibly two-valued trace on every $E \in \mathcal{E}_h^i$. Then, for all $E \in \mathcal{E}_h^i$ shared by two adjacent elements K_1 and K_2 , the average and jump are defined as

$$\{v\} = \frac{1}{2}(v|_{K_1} + v|_{K_2}), \quad [v] = v|_{K_1} - v|_{K_2},$$

respectively, where the outer normal vector of K_1 on the edge E is consistent with $\mathbf{n}_E$. The concepts of average and jump can be extended to the boundary edge $E \in \mathcal{E}_h^b$

$$\{v\} = [v] = v|_K.$$

Next, for a fixed positive integer k , we give the complex-valued discontinuous Galerkin approximation subspace V_h^k defined as

$$V_h^k = \{v_h \in L^2(\Omega) : v_h|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{K}_h\},$$

where $\mathbb{P}_k(K)$ denotes the space of polynomials of total degree less than or equal to k . In the finite-dimensional space V_h^k , the discontinuous Galerkin norm is defined as follows:

$$\|v_h\|_{\text{DG}} := \left(\sum_{K \in \mathcal{K}_h} \int_K |\nabla v_h|^2 d\mathbf{x} + \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \int_E [v_h]^2 ds \right)^{1/2}, \quad (2.1)$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{C}^2$. Based on the arguments in [12], it can be concluded that the following Sobolev embedding inequality holds:

$$\|v_h\|_p \leq C \|v_h\|_{\text{DG}}, \quad \forall v_h \in V_h^k, \quad 1 \leq p < \infty. \quad (2.2)$$

We further present the discrete Ladyzhenskaya inequality [13], which is essential to the derivation process of the optimal error estimate:

$$\|v_h\|_4 \leq C \|v_h\|_{\text{DG}}^{1/2} \|v_h\|^{1/2}, \quad \forall v_h \in V_h^k. \quad (2.3)$$

We also recall the following global inverse inequality [5, 14]:

$$\|v_h\|_{\text{DG}} \leq Ch^{-1} \|v_h\|, \quad \forall v_h \in V_h^k. \quad (2.4)$$

Moreover, let us define the discrete bilinear form corresponding to the continuous bilinear form $(\nabla u, \nabla v)$:

$$\begin{aligned} a_h(v_h, w_h) := & \int_{\Omega} \nabla_h v_h \cdot \nabla_h w_h^* d\mathbf{x} - \sum_{E \in \mathcal{E}_h} \int_E \{\nabla_h v_h\} \cdot \mathbf{n}_E [w_h^*] ds - \sum_{E \in \mathcal{E}_h} \int_E [v_h] \{\nabla_h w_h^*\} \cdot \mathbf{n}_E ds \\ & + \sum_{E \in \mathcal{E}_h} \frac{\lambda}{h_E} \int_E [v_h] [w_h^*] ds, \quad \forall v_h, w_h \in V_h^k, \end{aligned}$$

where $\nabla \cdot$ is the piecewise-defined gradient operator, and the parameter λ , known as the penalty term, is a sufficiently large non-negative real number in this context. It is a well-established fact that the discrete bilinear form introduced earlier satisfies the coercivity and continuity properties outlined below:

$$C_1 \|v_h\|_{\text{DG}}^2 \leq a_h(v_h, v_h) =: \|v_h\|_{a_h}^2 \leq C_2 \|v_h\|_{\text{DG}}^2, \quad \forall v_h \in V_h^k. \quad (2.5)$$

On this basis, we define the elliptic projection operator $R_h : H^{k+1}(\Omega) \rightarrow V_h^k$ satisfying

$$a_h(R_h u, v_h) = -(\Delta u, v_h), \quad \forall v_h \in V_h^k. \quad (2.6)$$

It is well-known that the following boundedness and approximation properties hold for all $u \in H^{k+1}(\Omega)$:

$$\|u - R_h u\|_{\infty} + \|R_h u\|_{\infty} \leq C \|u\|_{k+1}, \quad (2.7)$$

$$\|R_h u - u\| + h \|R_h u - u\|_{\text{DG}} \leq Ch^{k+1} \|u\|_{k+1}. \quad (2.8)$$

Let $\{t_n : n = 0, 1, 2, \dots, N\}$ be the uniform partition of the interval $[0, T]$ with the time step $\tau = T/N$, and denote $t_{n-1/2} := (n - 1/2)\tau$. For a series of functions $\{\phi^n : n = 0, 1, 2, \dots, N\}$ defined on the domain Ω , we write

$$D_{\tau} \phi^n := \frac{\phi^n - \phi^{n-1}}{\tau}, \quad \hat{\phi}^n := \frac{\phi^n + \phi^{n-1}}{2}, \quad n = 1, 2, \dots, N-1, N.$$

Using some of the notations given in the preceding text, the fully implicit Crank–Nicolson discontinuous Galerkin algorithm is to seek $u_h^n \in V_h^k$ such that for $n = 1, 2, \dots, N-1, N$,

$$(D_\tau u_h^n, v_h) + (\nu + i\alpha) a_h(\hat{u}_h^n, v_h) + (\kappa + i\beta)(|\hat{u}_h^n|^2 \hat{u}_h^n, v_h) - \gamma(\hat{u}_h^n, v_h) = 0, \quad \forall v_h \in V_h^k, \quad (2.9)$$

where u_h^0 is taken as the Ritz projection $R_h u^0$ of u^0 .

Lemma 2.1. (Brouwer fixed-point theorem [15]) Let $(H, (\cdot, \cdot))$ be a finite-dimensional inner product space with $\|\cdot\|$ the associated norm, and let $\Pi : H \rightarrow H$ be continuous. Assume, moreover, that there exists an $a > 0$ such that for any $z \in H$ and $\|z\| = a$, it holds that

$$\operatorname{Re}(\Pi(z), z) \geq 0.$$

Then, there is a $z' \in H$ satisfying $\|z'\| \leq a$ such that $\Pi(z') = 0$.

Lemma 2.2. (Discrete Grönwall inequality [16]) Let $A \geq 0, B > 0$, $\{\eta^i\}_{i=1}^N$ and $\{\xi^i\}_{i=1}^N$ be two sequences of non-negative real numbers satisfying

$$\eta^n + \tau \sum_{i=1}^n \xi^i \leq A + B\tau \sum_{i=1}^n \eta^i, \quad 1 \leq n \leq N.$$

Then, the following holds when $\tau \leq 1/(2B)$:

$$\eta^n + \tau \sum_{i=1}^n \xi^i \leq A \exp(2Bn\tau), \quad 1 \leq n \leq N.$$

Lemma 2.3. (Discrete Grönwall inequality [17]) Suppose $\{F^n\}_{n=0}^N$ is a non-negative sequence, and C and G are two non-negative constants satisfying

$$F^n \leq (1 + C\tau)F^{n-1} + \tau G, \quad n = 1, 2, \dots, N.$$

Then,

$$F^n \leq e^{Cn\tau} \left(F^0 + \frac{G}{C} \right), \quad n = 1, 2, \dots, N. \quad (2.10)$$

Lemma 2.4. (Transfer inequality [18]) For a normed linear space $\mathcal{V}$ with the norm $\|\cdot\|_{\mathcal{V}}$ and for elements $v^0, v^1, \dots, v^N \in \mathcal{V}$, the result below holds:

$$\|v^n\|_{\mathcal{V}} \leq 2 \sum_{m=1}^n \|\hat{v}^m\|_{\mathcal{V}} + \|v^0\|_{\mathcal{V}}, \quad 1 \leq n \leq N. \quad (2.11)$$

3. Numerical analysis of the fully implicit discontinuous Galerkin scheme

In this section, we initiate the theoretical analysis of the proposed numerical scheme. To this end, we first establish the existence of the numerical solution, followed by a rigorous proof of its L^2 -norm boundedness. Based on this result, we further derive the boundedness of the numerical solution under the energy norm as well as the optimal order error estimates in both the L^2 -norm and the energy norm. In the subsequent analysis, we assume that the solution to (1.1)–(1.3) uniquely exists and satisfies

$$\|u^0\|_{H^{k+1}} + \|u\|_{L^\infty(0,T;H^{k+1})} + \|u_t\|_{L^2(0,T;H^{k+1})} + \|u_{tt}\|_{L^2(0,T;H^2)} + \|u_{ttt}\|_{L^2(0,T;L^2)} \leq C.$$

3.1. Existence of the discontinuous Galerkin solution

Theorem 3.1. When $\tau\gamma < 2$, the fully discrete discontinuous Galerkin algorithm (2.9) has a solution.

Proof. Assume that the numerical solution u_h^{n-1} at the $(n-1)$ -th layer has been obtained. Then, (2.9) can be interpreted as a nonlinear system in terms of the average solution $\hat{u}_h^n$:

$$\frac{2}{\tau} (\hat{u}_h^n - u_h^{n-1}, v_h) + (\nu + i\alpha) a_h(\hat{u}_h^n, v_h) + (\kappa + i\beta) (|\hat{u}_h^n|^2 \hat{u}_h^n, v_h) - \gamma(\hat{u}_h^n, v_h) = 0, \quad \forall v_h \in V_h^k. \quad (3.1)$$

Obviously, if we can prove the existence of a solution to Eq (3.1), then we can easily know that $u_h^n = 2\hat{u}_h^n - u_h^{n-1}$ is a solution of (2.9). For this purpose, define the operator $\Pi : V_h^k \rightarrow V_h^k$ by

$$(\Pi(w), v_h) = \frac{2}{\tau} (w - u_h^{n-1}, v_h) + (\nu + i\alpha) a_h(w, v_h) + (\kappa + i\beta) (|w|^2 w, v_h) - \gamma(w, v_h), \quad \forall v_h \in V_h^k. \quad (3.2)$$

Next, we will prove that this operator is continuous. In fact, given $w_1, w_2 \in V_h^k$, it can be deduced from (3.2) that

$$\begin{aligned} (\Pi(w_1) - \Pi(w_2), v_h) &= \frac{2}{\tau} (w_1 - w_2, v_h) + (\nu + i\alpha) a_h(w_1 - w_2, v_h) + (\kappa + i\beta) (|w_1|^2 w_1 - |w_2|^2 w_2, v_h) \\ &\quad - \gamma(w_1 - w_2, v_h), \quad \forall v_h \in V_h^k. \end{aligned} \quad (3.3)$$

By taking $v_h = \Pi(w_1) - \Pi(w_2)$ in (3.3) and employing the Cauchy–Schwarz inequality, we find that

$$\begin{aligned} \|\Pi(w_1) - \Pi(w_2)\|^2 &\leq C \|w_1 - w_2\|_{a_h} \|\Pi(w_1) - \Pi(w_2)\|_{a_h} + C \left\| |w_1|^2 w_1 - |w_2|^2 w_2 \right\| \|\Pi(w_1) - \Pi(w_2)\| \\ &\quad + \left(C + \frac{2}{\tau} \right) \|w_1 - w_2\| \|\Pi(w_1) - \Pi(w_2)\| \\ &\leq C \|w_1 - w_2\|_{\text{DG}} \|\Pi(w_1) - \Pi(w_2)\|_{\text{DG}} + C \left\| |w_1|^2 w_1 - |w_2|^2 w_2 \right\| \|\Pi(w_1) - \Pi(w_2)\| \\ &\quad + \left(C + \frac{2}{\tau} \right) \|w_1 - w_2\| \|\Pi(w_1) - \Pi(w_2)\| \\ &\leq \frac{C}{h^2} \|w_1 - w_2\| \|\Pi(w_1) - \Pi(w_2)\| + \left(C + \frac{2}{\tau} \right) \|w_1 - w_2\| \|\Pi(w_1) - \Pi(w_2)\| \\ &\quad + C \left(\|w_1\|_\infty^2 + \|w_1\|_\infty \|w_2\|_\infty + \|w_2\|_\infty^2 \right) \|w_1 - w_2\| \|\Pi(w_1) - \Pi(w_2)\|, \end{aligned} \quad (3.4)$$

where we have also used (2.5), (2.4), and the following elementary inequality:

$$\left| |z_1|^2 z_1 - |z_2|^2 z_2 \right| \leq \left(|z_1|^2 + |z_1| |z_2| + |z_2|^2 \right) |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C}. \quad (3.5)$$

By dividing both sides of Ineq (3.4) by $\|\Pi(w_1) - \Pi(w_2)\|$ simultaneously, it holds that

$$\|\Pi(w_1) - \Pi(w_2)\| \leq C \left(\frac{1}{h^2} + 1 + \frac{2}{\tau} + \|w_1\|_\infty^2 + \|w_1\|_\infty \|w_2\|_\infty + \|w_2\|_\infty^2 \right) \|w_1 - w_2\|. \quad (3.6)$$

From (3.6), we can easily see that the operator Π is continuous. Furthermore, choosing $v_h = w$ in (3.2), taking the real part of both sides, and using the Cauchy–Schwarz inequality gives that

$$\text{Re}(\Pi(w), w) = \frac{2}{\tau} \left(\|w\|^2 - \text{Re}(u_h^{n-1}, w) \right) + \nu \|w\|_{a_h} + \kappa \|w\|_4^4 - \gamma \|w\|^2$$

$$\begin{aligned}
&\geq \frac{2}{\tau} \left(\|w\|^2 - \|u_h^{n-1}\| \|w\| \right) - \gamma \|w\|^2 \\
&\geq \frac{2}{\tau} \left[\left(1 - \frac{1}{2} \gamma \tau \right) \|w\| - \|u_h^{n-1}\| \right] \cdot \|w\|.
\end{aligned} \tag{3.7}$$

Obviously, when $\tau\gamma < 2$, and $\|w\| = \|u_h^{n-1}\|/(1 - 1/2\tau\gamma) + 1$, it follows that

$$\operatorname{Re}(\Pi(w), w) \geq 0.$$

By the Brouwer fixed-point theorem described in Lemma 2.1, we can conclude that there exists a w' satisfying $\|w'\| \leq \|u_h^{n-1}\|/(1 - 1/2\tau\gamma) + 1$ such that

$$\Pi(w') = 0.$$

Therefore, there is a solution to (2.9). All this completes the proof. $\square$

3.2. Boundedness of the discontinuous Galerkin solution

Theorem 3.2. Suppose u_h^n is the solution of the fully discrete numerical scheme (2.9). Then, when $\tau\gamma \leq 1$, we have

$$\|u_h^n\| \leq e^{2T \max\{0, \gamma\}} \|u_h^0\|, \quad 1 \leq n \leq N.$$

Proof. Taking $v_h = \hat{u}_h^n$ in (2.9), we have

$$(D_\tau u_h^n, \hat{u}_h^n) + (\nu + i\alpha) \|\hat{u}_h^n\|_{a_h}^2 + (\kappa + i\beta) \|\hat{u}_h^n\|_4^4 - \gamma \|\hat{u}_h^n\|^2 = 0. \tag{3.8}$$

Noticing the fact that

$$\operatorname{Re}(D_\tau u_h^n, \hat{u}_h^n) = \frac{1}{2\tau} \left(\|u_h^n\|^2 - \|u_h^{n-1}\|^2 \right),$$

we take the real part of both sides of Eq (3.8) to obtain

$$\|u_h^n\|^2 - \|u_h^{n-1}\|^2 \leq 2\tau\gamma \|\hat{u}_h^n\|^2, \quad 1 \leq n \leq N. \tag{3.9}$$

Therefore, when $\gamma \leq 0$, it arrives that

$$\|u_h^n\| \leq \|u_h^0\|, \quad 1 \leq n \leq N. \tag{3.10}$$

When $\gamma > 0$, it follows from (3.9) that

$$\|u_h^n\|^2 - \|u_h^{n-1}\|^2 \leq 2\tau\gamma \|\hat{u}_h^n\|^2 \leq \frac{\tau\gamma}{2} \left(\|u_h^n\| + \|u_h^{n-1}\| \right)^2.$$

A simple transformation gives

$$\left(1 - \frac{\tau\gamma}{2} \right) \|u_h^n\| \leq \left(1 + \frac{\tau\gamma}{2} \right) \|u_h^{n-1}\|, \quad 1 \leq n \leq N.$$

When $\tau\gamma \leq 1$, we have

$$\|u_h^n\| \leq \frac{1 + \frac{\tau\gamma}{2}}{1 - \frac{\tau\gamma}{2}} \|u_h^{n-1}\| = \left(1 + \frac{\tau\gamma}{1 - \frac{\tau\gamma}{2}}\right) \|u_h^{n-1}\| \leq (1 + 2\tau\gamma) \|u_h^{n-1}\|, \quad 1 \leq n \leq N.$$

By virtue of the discrete Grönwall inequality given in Lemma 2.3, it holds that

$$\|u_h^n\| \leq e^{2\gamma n\tau} \|u_h^0\| \leq e^{2\gamma T} \|u_h^0\|, \quad 1 \leq n \leq N. \quad (3.11)$$

It can be concluded that the theorem holds true by combining Ineqs (3.10) and (3.11). $\square$

Remark 3.1. By utilizing the definition of the initial approximate solution u_h^0 and the approximation property (2.8) of the Ritz projection, we can easily obtain that

$$\|u_h^0\| = \|R_h u^0\| \leq \|R_h u^0 - u^0\| + \|u^0\| \leq C. \quad (3.12)$$

Using the conclusions of Theorem 3.2 and (3.12), it is known that there exists a positive constant C such that the following equation holds:

$$\|u_h^n\| \leq C, \quad 0 \leq n \leq N.$$

That is, the numerical solution $\|u_h^n\|$ is bounded in the L^2 -norm. Furthermore, with the help of the estimate (2.8) of the Ritz projection R_h , we know that

$$\|\eta_h^n\| = \|R_h u^n - u_h^n\| \leq \|R_h u^n\| + \|u_h^n\| \leq C, \quad 0 \leq n \leq N. \quad (3.13)$$

To prove the boundedness of the numerical solution under the energy norm, we present the following crucial lemma.

Lemma 3.1. Let u_h^n be the solution of the Eq (1.1)–(1.3) and $u(t_n)$ be the solution of the discrete problem (2.9). Denote

$$\xi^n = u^n - R_h u^n, \quad \eta_h^n = R_h u^n - u_h^n, \quad 0 \leq n \leq N.$$

Then, there exists a positive constant τ_1 such that when $\tau \leq \tau_1$, we have

$$\|\eta_h^n\| + \tau \|\eta_h^n\|_{DG} \leq C \left(\tau^2 + h^{k+1} \right), \quad 0 \leq n \leq N. \quad (3.14)$$

Proof. By virtue of the strong consistency of the interior penalty discontinuous Galerkin scheme (2.9), it can be concluded that the exact solution satisfies the following equation at time $t_{n-1/2}$:

$$(D_\tau u^n, v_h) + (\nu + i\alpha) a_h(\hat{u}^n, v_h) + (\kappa + i\beta) (|\hat{u}^n|^2 \hat{u}^n, v_h) - \gamma(\hat{u}^n, v_h) = (R^n, v_h), \quad \forall v_h \in V_h^k, \quad (3.15)$$

where

$$\begin{aligned} R^n &= D_\tau u^n - u_t^{n-\frac{1}{2}} + (\nu + i\alpha) \left(\Delta u^{n-\frac{1}{2}} - \Delta \hat{u}^n \right) \\ &\quad + (\kappa + i\beta) \left(|\hat{u}^n|^2 \hat{u}^n - |u^{n-\frac{1}{2}}|^2 u^{n-\frac{1}{2}} \right) - \gamma \left(\hat{u}^n - u^{n-\frac{1}{2}} \right), \end{aligned}$$

and

$$u_t^{n-\frac{1}{2}} = u_t(t_{n-\frac{1}{2}}), \quad u^{n-\frac{1}{2}} = u(t_{n-\frac{1}{2}}).$$

Subtracting (3.15) from (2.9) and utilizing the fact $u^n = \xi^n + \eta_h^n$, we have the system of error equations

$$\begin{aligned} & (D_\tau \eta_h^n, v_h) + (\nu + i\alpha) a_h(\hat{\eta}_h^n, v_h) + (\kappa + i\beta) \left(|\hat{u}^n|^2 \hat{u}^n - |\hat{u}_h^n|^2 \hat{u}_h^n, v_h \right) - \gamma(\hat{\eta}_h^n, v_h) \\ & = (R^n, v_h) - (D_\tau \xi^n, v_h) + \gamma(\hat{\xi}^n, v_h), \quad \forall v_h \in V_h^k. \end{aligned} \quad (3.16)$$

Choosing $v_h = \hat{\eta}_h^n$ in (3.16) and taking the real part of the resulting equation, it arrives at

$$\begin{aligned} \frac{1}{2\tau} \left(\|u_h^n\|^2 - \|u_h^{n-1}\|^2 \right) + \nu \|\hat{\eta}_h^n\|_{a_h}^2 & = -\operatorname{Re} \left\{ (\kappa + i\beta) \left(|\hat{u}^n|^2 \hat{u}^n - |\hat{u}_h^n|^2 \hat{u}_h^n, \hat{\eta}_h^n \right) \right\} + \gamma \|\hat{\eta}_h^n\|^2 \\ & \quad + \operatorname{Re}(R^n, \hat{\eta}_h^n) - \operatorname{Re}(D_\tau \xi^n, \hat{\eta}_h^n) + \gamma \operatorname{Re}(\hat{\xi}^n, \hat{\eta}_h^n). \end{aligned} \quad (3.17)$$

Next, we estimate each term on the right-hand side of Eq (3.17). Noticing that

$$\begin{aligned} |\hat{u}^n|^2 \hat{u}^n - |\hat{u}_h^n|^2 \hat{u}_h^n & = |\hat{u}^n|^2 (\hat{\xi}^n + R_h \hat{u}^n) - |\hat{u}_h^n|^2 (R_h \hat{u}^n - \hat{\eta}_h^n) \\ & = |\hat{u}^n|^2 \hat{\xi}^n + R_h \hat{u}^n \left(|\hat{u}^n|^2 - |\hat{u}_h^n|^2 \right) + |\hat{u}_h^n|^2 \hat{\eta}_h^n \\ & = |\hat{u}^n|^2 \hat{\xi}^n + R_h \hat{u}^n \left(\hat{u}^n (\hat{u}^n - \hat{u}_h^n)^* + (\hat{u}^n - \hat{u}_h^n) (\hat{u}_h^n)^* \right) + |\hat{u}_h^n|^2 \hat{\eta}_h^n \\ & = |\hat{u}^n|^2 \hat{\xi}^n + R_h \hat{u}^n \left(\hat{u}^n (\hat{\xi}^n + \hat{\eta}_h^n)^* + (\hat{\xi}^n + \hat{\eta}_h^n) (R_h \hat{u}^n - \hat{\eta}_h^n)^* \right) + |\hat{u}_h^n|^2 \hat{\eta}_h^n \\ & = |\hat{u}^n|^2 \hat{\xi}^n + R_h \hat{u}^n \left(\hat{u}^n (\hat{\xi}^n)^* + \hat{u}^n (\hat{\eta}_h^n)^* + \hat{\xi}^n (R_h \hat{u}^n)^* - \hat{\xi}^n (\hat{\eta}_h^n)^* + \hat{\eta}_h^n (R_h \hat{u}^n)^* \right) \\ & \quad - (R_h \hat{u}^n) \hat{\eta}_h^n (\hat{\eta}_h^n)^* + |\hat{u}_h^n|^2 \hat{\eta}_h^n, \end{aligned}$$

we can readily obtain the estimation for the first term on the right-hand side of Eq (3.17). In fact,

$$\begin{aligned} & -\operatorname{Re} \left\{ (\kappa + i\beta) \left(|\hat{u}^n|^2 \hat{u}^n - |\hat{u}_h^n|^2 \hat{u}_h^n, \hat{\eta}_h^n \right) \right\} \\ & \leq C \|\hat{u}^n\|_\infty \|\hat{\xi}^n\| \|\hat{\eta}_h^n\| + \|R_h \hat{u}^n\|_\infty (\|\hat{u}^n\|_\infty \|\hat{\xi}^n\| \|\hat{\eta}_h^n\| + \|\hat{u}^n\|_\infty \|\hat{\eta}_h^n\|^2 \\ & \quad + \|R_h \hat{u}^n\|_\infty \|\hat{\xi}^n\| \|\hat{\eta}_h^n\| + \|\hat{\xi}^n\|_\infty \|\hat{\eta}_h^n\|^2 + \|R_h \hat{u}^n\|_\infty \|\hat{\eta}_h^n\|^2) + \|R_h \hat{u}^n\|_\infty \|\hat{\eta}_h^n\|_4^2 \|\hat{\eta}_h^n\| \\ & \leq C (\|\hat{\xi}^n\|^2 + \|\hat{\eta}_h^n\|^2 + \|\hat{\eta}_h^n\|_4^2) \\ & \leq C \left(\|\hat{\xi}^n\|^2 + \|\hat{\eta}_h^n\|^2 + \|\hat{\eta}_h^n\|_{\text{DG}} \|\hat{\eta}_h^n\| \right), \end{aligned} \quad (3.18)$$

where the Cauchy–Schwarz inequality is applied in the first inequality above, the boundedness (3.13) of R_h in L^2 -norm is used in the second inequality, and the discrete Ladyzhenskaya inequality (2.3) is employed in the third inequality. Next, the Taylor expansion formula with an integral remainder term can be used to obtain

$$\begin{aligned} \operatorname{Re}(R^n, \hat{\eta}_h^n) & \leq C \left(\|D_\tau u^n - u^{n-\frac{1}{2}}\| + \|\Delta u^{n-\frac{1}{2}} - \Delta \hat{u}^n\| \right. \\ & \quad \left. + \left\| |\hat{u}^n|^2 \hat{u}^n - |u^{n-\frac{1}{2}}|^2 u^{n-\frac{1}{2}} \right\| + \|\hat{u}^n - u^{n-\frac{1}{2}}\| \right) \|\hat{\eta}_h^n\| \\ & \leq C(\tau^4 + \|\hat{\eta}_h^n\|^2), \end{aligned} \quad (3.19)$$

where we have used the following estimates:

$$\|D_\tau u^n - u^{n-\frac{1}{2}}\| = \frac{1}{2\tau} \left\| \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (s - t_{n-1})^2 u_{ttt}(s) ds + \int_{t_{n-\frac{1}{2}}}^{t_n} (s - t_n)^2 u_{ttt}(s) ds \right\| \leq C\tau^2,$$

$$\|\Delta u^{n-\frac{1}{2}} - \Delta \hat{u}^n\| \leq C\tau \int_{t_{n-1}}^{t_n} \|\Delta u_{tt}\| ds \leq C\tau^2,$$

$$\begin{aligned} \left\| |\hat{u}^n|^2 \hat{u}^n - |u^{n-\frac{1}{2}}|^2 u^{n-\frac{1}{2}} \right\| &= \left\| |\hat{u}^n|^2 (\hat{u}^n - u^{n-\frac{1}{2}}) + (|\hat{u}^n|^2 - |u^{n-\frac{1}{2}}|^2) u^{n-\frac{1}{2}} \right\| \\ &= \left\| |\hat{u}^n|^2 (\hat{u}^n - u^{n-\frac{1}{2}}) + \operatorname{Re} \left((\hat{u}^n - u^{n-\frac{1}{2}})(\hat{u}^n + u^{n-\frac{1}{2}})^* \right) u^{n-\frac{1}{2}} \right\| \\ &\leq C\tau^2, \end{aligned}$$

and

$$\|u^{n-\frac{1}{2}} - \hat{u}^n\| \leq C\tau \int_{t_{n-1}}^{t_n} \|u_{tt}\| ds \leq C\tau^2.$$

In the last, applying the Cauchy–Schwarz inequality and the approximation property (2.8), it follows that

$$\begin{aligned} -\operatorname{Re}(D_\tau \xi^n, \hat{\eta}_h^n) + \gamma \operatorname{Re}(\hat{\xi}^n, \hat{\eta}_h^n) &\leq C(\|D_\tau \xi^n\| + \|\hat{\xi}^n\|) \|\hat{\eta}_h^n\| \\ &= C \left(\frac{1}{\tau} \left\| \int_{t_{n-1}}^{t_n} (R_h(u_t) - u_t) ds \right\| + \|\hat{\xi}^n\| \right) \|\hat{\eta}_h^n\| \\ &\leq C(h^{2k+2} + \|\hat{\eta}_h^n\|^2). \end{aligned} \quad (3.20)$$

Substituting (3.18)–(3.20) into (3.17) and multiplying both sides of the resulting equation by 2τ yields

$$\begin{aligned} \|\eta_h^n\|^2 - \|\eta_h^{n-1}\|^2 + 2\nu C_1 \tau \|\hat{\eta}_h^n\|_{\text{DG}}^2 &\leq C(\tau^5 + \tau h^{2k+2} + \tau \|\hat{\eta}_h^n\|^2 + \tau \|\hat{\eta}_h^n\|_{\text{DG}} \|\hat{\eta}_h^n\|) \\ &\leq C(\tau^5 + \tau h^{2k+2} + \tau \|\hat{\eta}_h^n\|^2) + \nu C_1 \tau \|\hat{\eta}_h^n\|_{\text{DG}}^2. \end{aligned}$$

Combining like terms, we have

$$\|\eta_h^n\|^2 - \|\eta_h^{n-1}\|^2 + \nu C_1 \tau \|\hat{\eta}_h^n\|_{\text{DG}}^2 \leq C(\tau^5 + \tau h^{2k+2} + \tau \|\hat{\eta}_h^n\|^2), \quad 1 \leq n \leq N. \quad (3.21)$$

Replacing n by j in (3.21) and summing over j from 1 to n , we get

$$\|\eta_h^n\|^2 + \nu C_1 \tau \sum_{j=1}^n \|\hat{\eta}_h^j\|_{\text{DG}}^2 \leq C(\tau^4 + h^{2k+2}) + C\tau \sum_{j=1}^n \|\eta_h^j\|^2, \quad 1 \leq n \leq N,$$

where we have utilized the initial condition $\eta_h^0 = 0$ together with the following bound:

$$\sum_{j=1}^n \|\hat{\eta}_h^j\|^2 \leq C \sum_{j=1}^n \|\eta_h^j\|^2.$$

According to the Grönwall inequality presented in Lemma 2.3, it holds that

$$\|\eta_h^n\|^2 + \nu C_1 \tau \sum_{j=1}^n \|\hat{\eta}_h^j\|_{\text{DG}}^2 \leq C(\tau^4 + h^{2k+2}). \quad (3.22)$$

Employing the transfer inequality (2.11), (3.22), and the Cauchy–Schwarz inequality gives that

$$\tau \|\eta_h^n\|_{\text{DG}} \leq 2\tau \sum_{j=1}^n \|\hat{\eta}_h^j\| \leq C \sqrt{n\tau} \sqrt{\sum_{j=1}^n \nu C_1 \tau \|\hat{\eta}_h^j\|_{\text{DG}}^2} \leq C(\tau^2 + h^{k+1}). \quad (3.23)$$

In combination with (3.22), it follows that the conclusion in (3.14) is true. $\square$

With the above preparations, the boundedness of the numerical solution under the energy norm follows immediately.

Theorem 3.3. *The solution u_h^n of the fully discrete algorithm (2.9) is bounded in the discontinuous Galerkin (DG)-norm; that is, there exists a positive constant C such that*

$$\|u_h^n\|_{DG} \leq C, \quad 0 \leq n \leq N.$$

Proof. The core idea of the argument is to conduct a classified discussion on the spatial parameter h and the temporal parameter τ . When $\tau \leq h$, it follows from the global inverse inequality (2.4) and (3.14) that

$$\|\eta_h^n\|_{DG} \leq Ch^{-1}\|\eta_h^n\| \leq Ch \leq C. \quad (3.24)$$

When $h < \tau$, we can easily obtain from (3.14) that

$$\|\eta_h^n\|_{DG} \leq C\tau^{-1}(\tau^2 + h^{k+1}) \leq C\tau \leq C. \quad (3.25)$$

Through the combination of (3.24) and (3.25) and the fact that $\|R_h u^n\|_{DG} \leq C\|u^n\|_{DG}$, the truth of the theorem can be established. $\square$

Using the boundedness of the numerical solution under the energy norm, the uniqueness of the numerical solution follows immediately.

Theorem 3.4. *The solution of the discrete Galerkin scheme (2.9) is unique for sufficiently small τ .*

Proof. Adopt the background in Lemma 3.1. To prove the uniqueness of the numerical solution, suppose that w_1 and w_2 satisfy

$$\frac{2}{\tau}(w_1 - u_h^{n-1}, v_h) + (\nu + i\alpha)a_h(w_1, v_h) + (\kappa + i\beta)(|w_1|^2 w_1, v_h) - \gamma(w_1, v_h) = 0, \quad \forall v_h \in V_h^k, \quad (3.26)$$

$$\frac{2}{\tau}(w_2 - u_h^{n-1}, v_h) + (\nu + i\alpha)a_h(w_2, v_h) + (\kappa + i\beta)(|w_2|^2 w_2, v_h) - \gamma(w_2, v_h) = 0, \quad \forall v_h \in V_h^k, \quad (3.27)$$

respectively. Subtracting (3.27) from (3.26), taking $v_h = w_1 - w_2$, and considering the real part on both sides of the corresponding equation gives

$$\begin{aligned} & \frac{2}{\tau}\|w_1 - w_2\|^2 + \nu\|w_1 - w_2\|_{a_h}^2 \\ &= -\operatorname{Re}\left((\kappa + i\beta)(|w_1|^2 w_1 - |w_2|^2 w_2, w_1 - w_2)\right) + \gamma\|w_1 - w_2\|^2 \\ &\leq C\left((\|w_1\|_8^2 + \|w_1\|_8\|w_2\|_8 + \|w_2\|_8^2)\|w_1 - w_2\|_4\|w_1 - w_2\| + \|w_1 - w_2\|^2\right) \\ &\leq C\left((\|w_1\|_{DG}^2 + \|w_1\|_{DG}\|w_2\|_{DG} + \|w_2\|_{DG}^2)\|w_1 - w_2\|_{DG}^{\frac{1}{2}}\|w_1 - w_2\|^{\frac{3}{2}} + \|w_1 - w_2\|^2\right) \\ &\leq C\|w_1 - w_2\|^2 + \nu\|w_1 - w_2\|_{a_h}^2, \end{aligned} \quad (3.28)$$

where we have used the elementary inequality (3.5) and the Hölder inequality in the first inequality, followed by the imbedding inequality (2.2) and the discrete Ladyzhenskaya inequality (2.3) in the second equality, and the Young inequality in the third inequality. From (3.28), we can immediately obtain

$$\|w_1 - w_2\| \leq C\tau\|w_1 - w_2\|.$$

Therefore, for a suitably small τ , we have $w_1 = w_2$; that is, the scheme (2.9) is uniquely solvable. $\square$

In the last, we establish the unconditional optimal error estimates of the numerical solution in both the L^2 -norm and the energy norm.

Theorem 3.5. *Let u^n and u_h^n be the solutions of (1.1)–(1.3) and (2.9), respectively. Then, there exist two positive constants τ_0 and h_0 such that when $\tau \leq \tau_0$, $h \leq h_0$, it holds that*

$$\|u^n - u_h^n\| \leq C(\tau^2 + h^{k+1}), \quad \|u^n - u_h^n\|_{DG} \leq C(\tau^2 + h^k). \quad (3.29)$$

Proof. We first present the optimal order error estimate in the L^2 -norm. For this purpose, utilizing the triangle inequality, Lemma 3.1, and the elliptic projection estimate (2.8), we have

$$\|u^n - u_h^n\| \leq \|\xi^n\| + \|\eta_h^n\| \leq C(\tau^2 + h^{k+1}). \quad (3.30)$$

Next, taking $v_h = D_\tau \eta_h^n$ in (3.16) and the real part together with the similar procedure as in Lemma 3.1 leads to

$$\begin{aligned} & \|D_\tau \eta_h^n\|^2 + \frac{\nu}{2\tau} (\|\eta_h^n\|_{a_h}^2 - \|\eta_h^{n-1}\|_{a_h}^2) \\ &= -\operatorname{Re} \left\{ (\kappa + i\beta) \left(|\hat{u}^n|^2 \hat{u}^n - |\hat{u}_h^n|^2 \hat{u}_h^n, D_\tau \eta_h^n \right) \right\} - \gamma (\hat{\eta}_h^n, D_\tau \eta_h^n) \\ & \quad + \operatorname{Re}(R^n, D_\tau \eta_h^n) - \operatorname{Re}(D_\tau \xi^n, D_\tau \eta_h^n) + \gamma \operatorname{Re}(\hat{\xi}^n, D_\tau \eta_h^n) \\ & \leq C(\tau^4 + h^{2k+2}) + \|D_\tau \eta_h^n\|^2 + C\|\eta_h^n\|_{DG}^2. \end{aligned} \quad (3.31)$$

It should be emphasized here that we used the L^2 -norm error estimation (3.30) in the above argument. Multiplying both sides of Ineq (3.31) by 2τ and summing up, we get

$$\|\eta_h^n\|_{DG} \leq C\tau \sum_{m=1}^n \|\eta_h^m\|_{DG}^2 + C(\tau^2 + h^{2k+2}).$$

Applying the discrete Grönwall inequality, we have

$$\|\eta_h^n\|_{DG} \leq C(\tau^2 + h^{k+1}).$$

It follows from the triangle inequality that

$$\|u^n - u_h^n\|_{DG} \leq \|\xi^n\|_{DG} + \|\eta_h^n\|_{DG} \leq C(\tau^2 + h^k).$$

The proof is completed. $\square$

4. Numerical examples

In this section, we present four numerical examples to validate the theoretical findings of this work. All numerical results are implemented using the open-source finite element software FreeFEM. Because the fully discrete scheme proposed in this paper is fully implicit, the Picard iteration is used to solve the nonlinear system derived from the numerical algorithm (2.9) during practical implementation.

Step 1. Initialization: $u_h^0 = R_h u^0$.

Step 2. Picard iteration: Given $u_h^{n-1} \in V_h^k$, $n \geq 1$, find $u_h^{n,l} \in V_h^k$, $l \geq 1$ such that

$$\begin{aligned} & \left(\frac{u_h^{n,l} - u_h^{n-1}}{\tau}, v_h \right) + (\nu + i\alpha) a_h \left(\frac{u_h^{n,l} + u_h^{n-1}}{2}, v_h \right) \\ & + (\kappa + i\beta) \left(\left| \frac{u_h^{n,l-1} + u_h^{n-1}}{2} \right|^2 \frac{u_h^{n,l} + u_h^{n-1}}{2}, v_h \right) - \gamma \left(\frac{u_h^{n,l} + u_h^{n-1}}{2}, v_h \right) = 0, \quad \forall v_h \in V_h^k, \end{aligned}$$

where

$$u_h^{n,0} = u_h^{n-1}.$$

Step 3. Termination criterion for iteration: Given a fixed tolerance $\varepsilon = 10^{-10}$, check whether the stopping condition $\|u_h^{n,l} - u_h^{n,l-1}\| \leq \varepsilon$ is satisfied. If the condition holds, terminate the iteration and set $u_h^n = u_h^{n,l}$. Otherwise, increment the iteration counter as $l - 1 \leftarrow l$ and return to **Step 2**.

Step 4. Time marching: If $n = N$, terminate the time marching process. Otherwise, update the time index by letting $n - 1 \leftarrow n$ and return to **Step 2**.

The convergence of the above fixed-point iteration can be proven via an argument analogous to Theorem 6.1 in [19]. Because the relevant derivation follows standard routine, we omit detailed derivations to save space.

Example 4.1. Consider the following Ginzburg–Landau equation:

$$u_t - (1 + i)\Delta u + (1 + i)|u|^2 u - u = f(x, y, t), \quad (x, y) \in (0, 1) \times (0, 1), \quad t \in (0, T],$$

which has the exact solution

$$u(x, y, t) = \sin(\pi x) \sin(\pi y) e^{it^2},$$

and the source function $f(x, y, t)$ is determined by the above exact solution.

The numerical results of this example are presented in Tables 1–4. To test the convergence orders in the spatial direction, we select $T = 2 \times 10^{-6}$, $N = 20$, and the spatial step sizes $h = 1/5, 1/10, 1/15, 1/20, 1/25$ under different polynomial degrees $k = 1, 2, 3$. In Tables 1 to 3, the errors and optimal convergence accuracies under different norms are presented. As is evident from the tabulated numerical results, it can be observed that the convergence orders of the errors in the L^2 -norm and DG-norm both achieve the theoretically expected accuracies $O(h^{k+1})$ and $O(h^k)$ as derived earlier. To test the convergence orders in the temporal direction, we set $T = 1$, $\tau = h$, and $k = 3$, and the relevant computational results are presented in Table 4. The results suggest that the accuracy in the temporal direction achieves second-order convergence. Furthermore, we demonstrate the unconditional convergence by testing the method with three different temporal resolutions $N = 5, 10, 15$ on progressively finer spatial grids, where $1/h = 5, 10, 20, 40, 80, 160$. The L^2 -norm errors are shown in Figure 1. From the results, we see that as the spatial grid is refined, and the L^2 -norm errors stabilize, indicating that the fully implicit numerical scheme (2.9) we propose converges unconditionally. The above results in the spatial and temporal directions fully demonstrate the effectiveness and robustness of the algorithm proposed in this paper.

Table 1. The spatial convergence orders with $k = 1$ at the final time $T = 2e - 6$.

h	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{DG}$	Order
1/5	5.0867e-02	–	6.7062e-01	–
1/10	1.3085e-02	1.9588	3.3807e-01	0.9882
1/15	5.7763e-03	2.0167	2.2379e-01	1.0175
1/20	3.2100e-03	2.0421	1.6631e-01	1.0318
1/25	2.0268e-03	2.0608	1.3178e-01	1.0428

Table 2. The spatial convergence orders with $k = 2$ at the final time $T = 2e - 6$.

h	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{DG}$	Order
1/5	1.8174e-03	–	8.1516e-02	–
1/10	2.1470e-04	3.0815	2.0340e-02	2.0027
1/15	6.0179e-05	3.1369	8.9062e-03	2.0368
1/20	2.4057e-05	3.1871	4.9354e-03	2.0520
1/25	1.1687e-05	3.2354	3.1162e-03	2.0606

Table 3. The spatial convergence orders with $k = 3$ at the final time $T = 2e - 6$.

h	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{DG}$	Order
1/5	1.6529e-04	–	3.2644e-03	–
1/10	9.9187e-06	4.0587	3.9142e-04	3.0600
1/15	1.9195e-06	4.0505	1.1472e-04	3.0269
1/20	5.9857e-07	4.0507	4.8330e-05	3.0049
1/25	2.4216e-07	4.0555	2.4829e-05	2.9848

Table 4. The temporal convergence orders with $k = 3$ at the final time $T = 1$.

τ	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{DG}$	Order
1/20	1.8177e-05	–	9.8640e-05	–
1/25	1.1607e-05	2.0100	6.0324e-05	2.2037
1/30	8.0512e-06	2.0063	4.0810e-05	2.1435
1/35	5.9112e-06	2.0044	2.9499e-05	2.1056
1/40	4.5238e-06	2.0032	2.2344e-05	2.0804

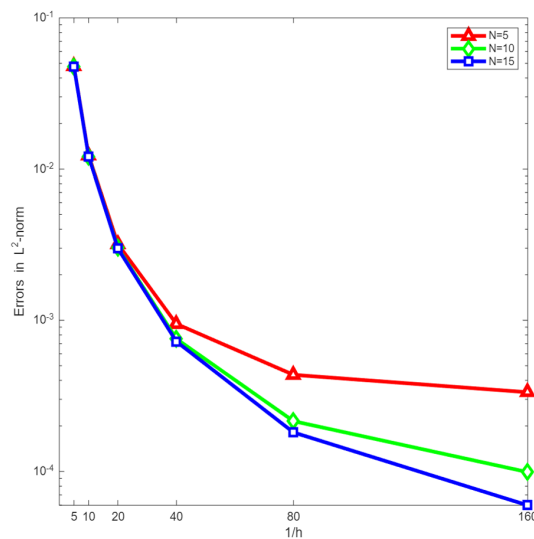


Figure 1. L^2 -norm errors of the linear discontinuous Galerkin method with varying spatial mesh size h for Example 4.1.

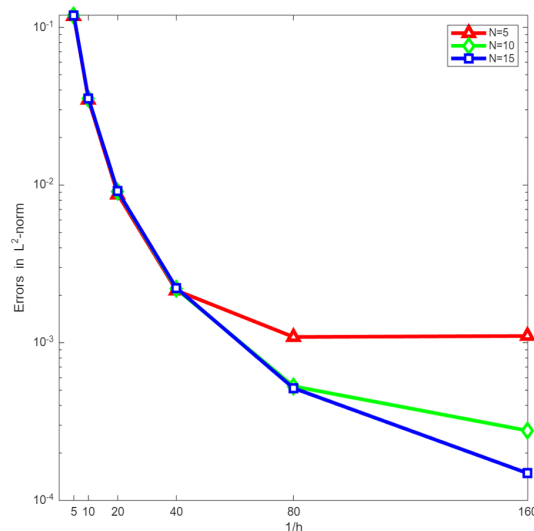


Figure 2. L^2 -norm errors of the linear discontinuous Galerkin method with varying spatial mesh size h for Example 4.2.

Example 4.2. In (1.1)–(1.3), let $\alpha = \beta = \gamma = \nu = \kappa = 1$ and $\Omega = (-1, 1) \times (-1, 1)$. The analytical solution is taken as

$$u(x, y, t) = (1 + x)^4(1 - x)^4(1 + y)^4(1 - y)^4 e^{it}.$$

From the above exact solution, the initial condition and the right-hand side function can be directly derived.

The results of the second numerical example are presented in Tables 5 to 8. They are in excellent agreement with the theoretical arguments. In the test of convergence accuracy in the temporal direction, we fix $h = 1/50$ and vary the time step size, and the numerical results achieve the expected second-order

accuracy. In addition, it can be seen from Figure 2 that the numerical algorithm constructed in this paper has no mesh ratio restriction. The above numerical results further verify the correctness of the theoretical findings.

Table 5. The spatial convergence orders with $k = 1$ at the final time $T = 2e - 6$.

h	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{\text{DG}}$	Order
2/10	3.7771e-02	—	4.9915e-01	—
2/15	1.7320e-02	1.9229	3.3706e-01	0.9684
2/20	9.8114e-03	1.9756	2.5324e-01	0.9939
2/25	6.2780e-03	2.0009	2.0231e-01	1.0063
2/30	4.3464e-03	2.0168	1.6815e-01	1.0143

Table 6. The spatial convergence orders with $k = 2$ at the final time $T = 2e - 6$.

h	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{\text{DG}}$	Order
2/10	1.6974e-03	—	6.8834e-02	—
2/15	4.9250e-04	3.0517	3.0846e-02	1.9797
2/20	2.0308e-04	3.0794	1.7317e-02	2.0069
2/25	1.0153e-04	3.1065	1.1027e-02	2.0228
2/30	5.7362e-05	3.1319	7.6110e-03	2.0332

Table 7. The spatial convergence orders with $k = 3$ at the final time $T = 2e - 6$.

h	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{\text{DG}}$	Order
2/10	1.6237e-04	—	3.5477e-03	—
2/15	3.2170e-05	3.9925	1.0471e-03	3.0095
2/20	1.0154e-05	4.0083	4.4184e-04	2.9993
2/25	4.1470e-06	4.0132	2.2656e-04	2.9933
2/30	1.9939e-06	4.0166	1.3139e-04	2.9883

Table 8. The temporal convergence orders with $k = 3$ at the final time $T = 1$.

τ	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{\text{DG}}$	Order
1/10	2.8739e-04	—	7.1514e-04	—
1/15	1.2805e-04	1.9938	3.2113e-04	1.9745
1/20	7.2071e-05	1.9979	1.8064e-04	2.0000
1/25	4.6139e-05	1.9987	1.1568e-04	1.9971
1/30	3.2046e-05	1.9991	8.0405e-05	1.9952

Example 4.3. This numerical example focuses on a variable-coefficient complex Ginzburg–Landau equation from [4] defined on the square domain $(x, y) \in (0, 2\pi)^2$ over the time interval $t \in (0, T]$:

$$\frac{\partial u}{\partial t} - \frac{i}{2}\Delta u + i|u|^2u + i(1 - \sin^2(x)\sin^2(y))u = 0.$$

The analytical solution is given by

$$u(x, y, t) = \sin(x)\sin(y)\exp(-2it).$$

The numerical results of Example 4.3 are presented in Tables 9–11. As can be observed from the tabulated data, the accuracy of our proposed numerical algorithm agrees well with the preceding theoretical analysis. Moreover, it is worth emphasizing that the theoretical framework derived above remains valid even for the case of variable coefficients.

Table 9. The spatial convergence orders with $k = 1$ at the final time $T = 2e - 6$.

h	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{\text{DG}}$	Order
$2\pi/10$	3.4479e-01	–	1.3523e-00	–
$2\pi/15$	1.5806e-01	1.9237	9.1365e-01	0.9670
$2\pi/20$	8.9747e-02	1.9673	6.8768e-01	0.9876
$2\pi/25$	5.7614e-02	1.9863	5.5052e-01	0.9969
$2\pi/30$	4.0031e-02	1.9971	4.5856e-01	1.0025

Table 10. The spatial convergence orders with $k = 2$ at the final time $T = 2e - 6$.

h	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{\text{DG}}$	Order
$2\pi/10$	1.1988e-02	–	1.6641e-01	–
$2\pi/15$	3.5212e-03	3.0214	7.4813e-02	1.9718
$2\pi/20$	1.4721e-03	3.0315	4.2148e-02	1.9946
$2\pi/25$	7.4679e-04	3.0414	2.6938e-02	2.0061
$2\pi/30$	4.2817e-04	3.0509	1.8661e-02	2.0135

Table 11. The spatial convergence orders with $k = 3$ at the final time $T = 2e - 6$.

h	$\ u^N - u_h^N\ $	Order	$\ u^N - u_h^N\ _{\text{DG}}$	Order
$2\pi/10$	1.0347e-03	–	6.3670e-03	–
$2\pi/15$	2.0049e-04	4.0473	1.8453e-03	3.0545
$2\pi/20$	6.2776e-05	4.0364	7.7041e-04	3.0362
$2\pi/25$	2.5544e-05	4.0295	3.9214e-04	3.0263
$2\pi/30$	1.2262e-05	4.0255	2.2611e-04	3.0200

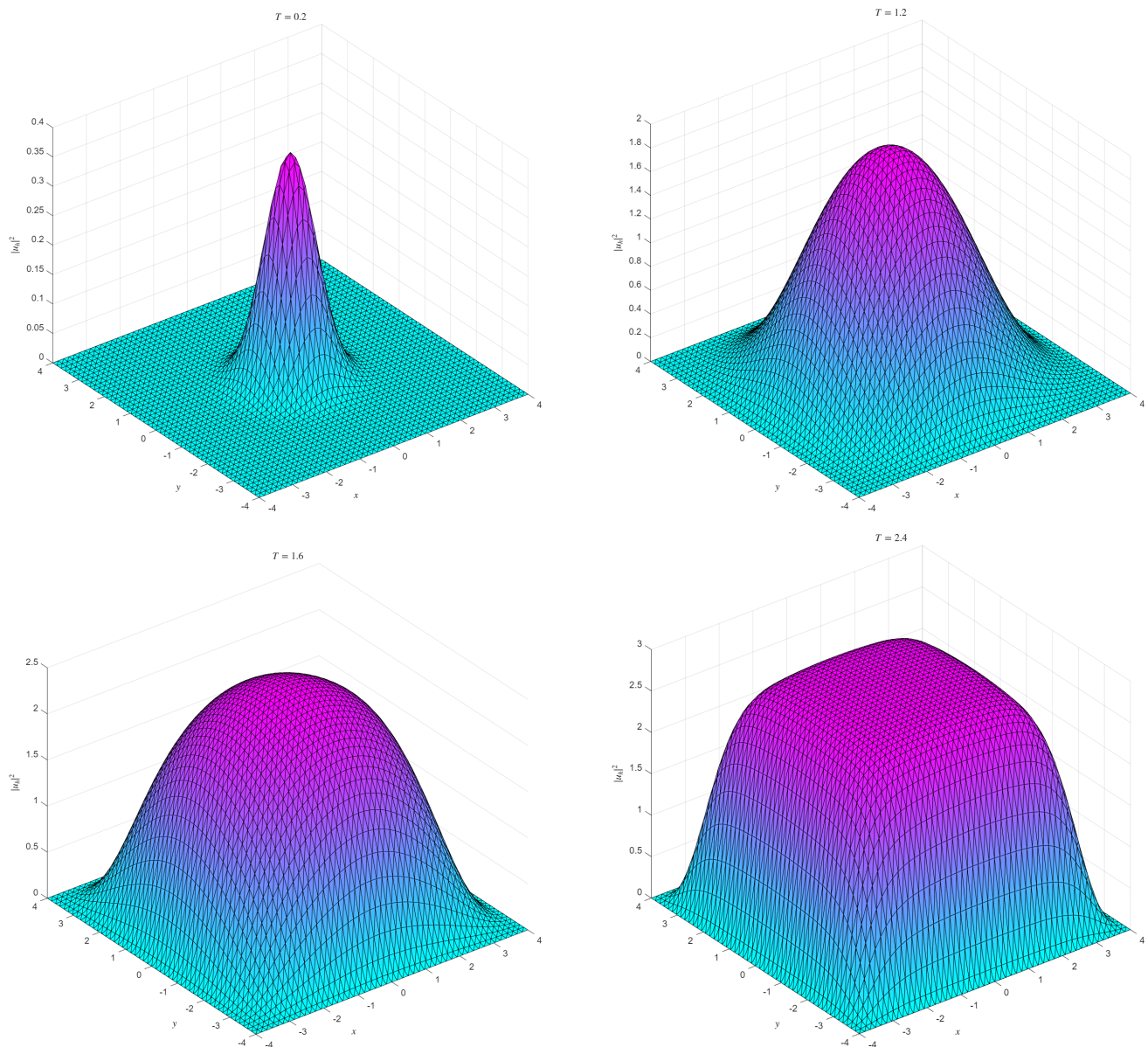


Figure 3. Evolution profiles of the modulus squared $|u_h|^2$ for the Ginzburg–Landau equation starting from Gaussian symmetric initial data, captured at $T = 0.2, 1.2, 1.6$, and 2.4 .

Example 4.4. This example deals with the two-dimensional Ginzburg–Landau equation equipped with the complex-valued Gaussian-type initial condition

$$u(x, y, 0) = \exp(-2x^2 - 2y^2) \exp\left(\frac{i}{\exp(x+y) + \exp(-x-y)}\right), \quad (x, y) \in \Omega = (-4, 4)^2.$$

Following the parameter setup adopted in [4], we fix the equation's coefficients as $\alpha = 0.5$, $\nu = \kappa = \beta = 1$, and $\gamma = 3$ throughout the computation. Obviously, the exact solution is unknown for this problem. In the numerical simulation, we employ the developed numerical scheme with spatial mesh size $h = 4\sqrt{2}/25$ and time step $\tau = 1/100$ to compute the numerical solution.

Figure 3 displays the three-dimensional distribution of density $|u_h|^2$ at four representative time instants $T = 0.2, 1.2, 1.6, 2.4$. It can be observed that the initial Gaussian pulse continuously diffuses and

expands over the spatial domain as time elapses, and such evolving behaviors are in good consistency with the numerical results given in [4], which verifies the reliability of our proposed numerical scheme.

5. Conclusions

Based on the Crank–Nicolson scheme, this paper proposes a fully implicit discontinuous Galerkin fully discrete scheme. The primary innovation of this work resides in the development of a novel analytical technique through which the unique solvability and optimal error estimates of the numerical solution are rigorously derived. This work's proof framework can be effectively utilized as a reference for analyzing fully implicit numerical schemes applied to a broader class of nonlinear partial differential equations. For future research, several promising directions are worthy of exploration. First, extending the proposed analytical reasoning technique to three-dimensional settings merits further investigation. Second, this study only employs the second-order Crank–Nicolson scheme for analytical purposes. In fact, it remains an open question whether the developed analytical method can be generalized to high-order temporal discretization schemes such as the spectral collocation method and the discontinuous Galerkin time-stepping method. Third, we intend to derive rigorous infinity-norm error estimates which constitute another meaningful research topic for our subsequent work.

Author contributions

Zhen Guan performed the theoretical derivations and drafted the initial manuscript. Xianxian Cao and Junjun Wang designed and implemented the computational programs. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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