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Research article

Liouville-type theorems for a class of semilinear elliptic equations with nonlinear gradient terms on smooth metric measure spaces

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Abstract: In this paper, we consider the following f-Laplacian equation with a gradient term and a nonlinear term

$$\Delta_f u + W(u)|\nabla u|^2 + F(u) = 0$$

on smooth metric measure spaces with nonnegative m-Bakry-Émery Ricci curvature, where $W(t)$ is a continuous function for $t > 0$ and $F(t)$ is a differentiable function in $(0, \infty)$. Under certain assumptions on W and F, by using the Saloff-Coste's Sobolev inequality and the Nash-Moser iteration method, we derive some Liouville-type theorems for positive solutions to the above equation. Our results extend those of Lu (2025) and McCoy (2007).

Keywords: nonlinear elliptic equation; Nash-Moser iteration; Liouville theorem
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1. Introduction

In this paper, we consider the following elliptic equation

$$\Delta_f u + W(u)|\nabla u|^2 + F(u) = 0 \tag{1.1}$$

on a complete smooth metric measure space $(M^n, g, e^{-f}dv)$, where $\Delta_f u := e^f \operatorname{div}\left(e^{-f}\nabla u\right)$ is the f-Laplacian, W is a continuous function, and F is a differentiable function. Under certain conditions on W and F, we prove some Liouville-type theorems for positive solutions of (1.1) on a complete smooth metric measure space $(M^n, g, e^{-f}dv)$ with nonnegative m-Bakry-Émery Ricci curvature.

It is widely recognized that Liouville-type theorems for nonlinear elliptic equations play a crucial role in the analysis of nonlinear partial differential equations (PDEs) [1–5]. When $W = 0$ and $F = u^\beta$ in $\mathbb{R}^n$, equation (1.1) reduces to the Lane-Emden equation

$$\Delta u + u^\beta = 0 \quad \text{on} \quad \mathbb{R}^n. \tag{1.2}$$

Gidas and Spruck [6] proved that if $n \geq 3$ and $1 < \beta < \frac{n+2}{n-2}$, then any nonnegative classical solution u of (1.2) is trivial in $\mathbb{R}^n$. Actually, in [6], the above problem is discussed in a framework of a general Riemann manifold and also for equations having a nonlinear term

$$\Delta u + F(u) = 0 \quad \text{on} \quad M^n. \tag{1.3}$$

More specifically, they showed the following general Liouville-type theorems:

Theorem 1.1. *[6, Theorem 6.1] Let M^n $(n > 2)$ be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature and u be a nonnegative C^2 solution of* (1.3). *Assume that the nonlinearity F satifies the following:*
(i) For some $\beta \in (0, \frac{n+2}{n-2})$,

$$\left(u^{-\beta} F(u)\right)' \leq 0.$$

(ii) For $n \geq 4$, assume

$$cu^{\theta} \leq F(u),$$

where $\theta \in (1, \frac{n+2}{n-2})$ and $c > 0$.
(iii) With some $c > 0$ and any $\rho \in \mathbb{R}$,

$$\Phi(t) = \int_0^t u^{\rho} F(u) du \leq cF(t) t^{1+\rho}.$$

Then $u \equiv 0$.

Beyond the well-known result of Gidas and Spruck for equation (1.2), it is worth noting that the same conclusion holds for the case $\beta \in (0, 1]$, thanks to the effectiveness of supersolutions of equation (1.2) established by Mitidieri and Pohozaev in [7].

Very recently, Lu [8] eliminated conditions (ii) and (iii) in Theorem 1.1 for solutions to equation (1.3) and derived the following Liouville-type theorem:

Theorem 1.2. *[8, Theorem 1.5] Let (M^n, g) be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. If u is a positive solution of* (1.3) *on M^n and there exists $\beta \in (1, \frac{n+3}{n-1})$ such that*

$$\left(u^{-\beta} F(u)\right)' \leq 0 \quad on \quad (0, \infty), \tag{1.4}$$

then u is constant.

When $W \neq 0$, equation (1.1) involves gradient terms. For nearly three decades, Liouville-type theorems for this class of equations have drawn significant attention from researchers worldwide [9–13]. McCoy [14] proved that if $(M^n, g, e^{-f} dv) = (M^n, g)$ and $W = -\eta t^{-1}$ with $\eta \neq 1$, then any solution u of (1.1) is constant. The key for the proof of this Liouville-type theorem is the Bernstein-type estimates, i.e., McCoy considered the following equation:

$$\Delta u - \eta u^{-1} |\nabla u|^2 + F(u) = 0, \quad \eta \neq 1 \quad \text{on} \quad M^n \tag{1.5}$$

and derived the following Liouville-type theorems:

Theorem 1.3. *[14, Theorem 2.1] Let (M, g) be a complete noncompact Riemannian manifold of dimension $n \geq 2$ with nonnegative Ricci curvature. Suppose the function $u : M \to \mathbb{R}^+$ satisfies* (1.5)*, where $\eta \neq 1$ is any constant and where the differentiable function F has a root at some $C > 0$ and satisfies the differential inequality*

$$F' \leq \left(\frac{n+1-2\eta}{n-1}\right)\frac{F}{u}. \tag{1.6}$$

Then u is identically constant.

On the other hand, Goffi [15] explored local integral gradient bounds for distributional solutions to a broad class of partial differential inequalities with divergence-form diffusion and power-law first-order terms. Filippucci et al. [16] employed the Bernstein method combined with Keller-Osserman's estimates to study the local and global properties of positive solutions to the equation

$$-\Delta_p u = |u|^{q-1}u + \widetilde{W}|\nabla u|^r$$

on a domain $\Omega \subset \mathbb{R}^N$, where $p > 1$, $q, r > 0$, and $\widetilde{W} \in \mathbb{R}$. As a result, they obtained several a priori estimates and Liouville-type theorems.

The above Liouville-type theorems were proved in [8, 14] using Bernstein and Yau's method (see [17–22]). Aside from the previously mentioned Bernstein and Yau's method, the Nash-Moser technique offers a refined approach for analyzing linear elliptic and parabolic PDEs in Euclidean space $\mathbb{R}^n$ or on a complete manifold M (see [23–31]).

In this paper, we use Saloff-Coste's Sobolev inequality and Nash-Moser iteration method to study some Liouville-type theorems of (1.1) on a complete smooth metric measure space $(M^n, g, e^{-f}dv)$. Our aim is to extend some work of [8] and [14] to general gradient nonlinearities.

In order to state our results, we need to introduce some notations. Let (M, g) be an n-dimensional complete smooth Riemannian manifold and f be a $C^2(M)$ function. We denote ∇, Δ, and ∇^2 the gradient, Laplacian, and Hessian operator on M with respect to g, respectively. Ric and R denote Ricci curvature and scalar curvature, respectively. An n-dimensional *smooth metric measure space* $(M, g, e^{-f}dv)$ is an n-dimensional complete Riemannian manifold (M, g) (without boundary) endowed with a weighted measure $e^{-f}dv$, and dv is the Riemannian volume element of the metric g. On a smooth metric measure space $(M, g, e^{-f}dv)$, we let

$$\mathrm{Ric}_f = \mathrm{Ric} + \nabla^2 f \tag{1.7}$$

stand for the Bakry-Émery Ricci curvature, which is also called ∞-Bakry-Émery Ricci curvature, i.e., the $m = \infty$ case of the following m-Bakry-Émery Ricci curvature defined by

$$\mathrm{Ric}_f^m = \mathrm{Ric}_f - \frac{1}{m-n}\nabla f \otimes \nabla f \tag{1.8}$$

with $m \geq n$ as a constant, and $m = n$ if and only if f is a constant. We define the f-Laplacian

$$\Delta_f := \Delta - \nabla f \cdot \nabla,$$

which is self-adjoint operator with respect to $e^{-f}dv$. Gradient estimates for f-Laplacian equations on Riemannian manifolds are a core technique in geometric analysis. For instance, by using the gradient estimates, Futaki et al. [32] derived a lower bound for the first eigenvalue $\lambda_1(\Delta_f)$ of the f-Laplacian on a compact manifold. Alternatively, on noncompact manifolds, the gradient estimates can be applied to prove Liouville-type theorems for positive solutions of the equation $\Delta_f u = 0$ (see [33]). The equation $\mathrm{Ric}_f = \lambda g$ for some constant λ is just the gradient Ricci soliton equation, which plays an important role in the study of Ricci flow. The equation $\mathrm{Ric}_f^m = \lambda g$ corresponds to the quasi-Einstein equation (cf. [34]), which has been studied by many authors. For the smooth metric measure spaces, we refer readers to the excellent work [35] of Wei and Wylie.

Next, we use the following change of variable. Let $W \in C^0(0,\infty)$ and set

$$\varpi(t) = \int_1^t W(s)\,ds, \quad t > 0. \tag{1.9}$$

Consider an auxiliary function G defined as follows.

(1) If

$$\int_0^1 e^{\varpi(s)} ds < \infty, \tag{1.10}$$

we define G by

$$G(t) := \int_0^t e^{\varpi(s)} ds, \quad t > 0. \tag{1.11}$$

(2) If

$$\int_1^\infty e^{\varpi(s)} ds < \infty, \tag{1.12}$$

we define G by

$$G(t) := \int_t^\infty e^{\varpi(s)} ds, \quad t > 0. \tag{1.13}$$

(3) If

$$\int_0^1 e^{\varpi(s)} ds = \infty \quad \text{and} \quad \int_1^\infty e^{\varpi(s)} ds = \infty, \tag{1.14}$$

we fix a positive real number l and define G as follows:

$$G(t) := \int_l^t e^{\varpi(s)} ds, \quad t \geq l, \tag{1.15}$$

$$G(t) := \int_t^l e^{\varpi(s)} ds, \quad 0 < t \leq l. \tag{1.16}$$

Our first main result is stated as follows.

Theorem 1.4. *Let $\beta \in \left(1, \frac{m+3}{m-1}\right)$ be a real number and $m \geq 3$. Let $\left(M^n, g, e^{-f}dv\right)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f^m \geq 0$. Assume that u is a positive solution of* (1.1).

(1) Suppose

(i) *W satisfies* (1.10), *G is defined by* (1.11), *and $u > 0$ or*

(ii) *W satisfies* (1.14), *G is defined by* (1.15), *and $u > l$.*

Moreover, let F satisfy

$$F'(t) \leq \left[\beta \frac{e^{\varpi(t)}}{G(t)} - W(t)\right] F(t), \quad \forall t \in I, \tag{1.17}$$

where $I = (0, \infty)$ in case of (i) or $I = (l, \infty)$ in case of (ii). Then $u \equiv C$ for some root C of $F(t) = 0$.
(2) Alternatively, suppose
(i) W satisfies (1.12), G is defined by (1.13), and $u > 0$ or
(ii) W satisfies (1.14), G is defined by (1.16), and $0 < u < l$.
Moreover, let F satisfy

$$F'(t) \leq \left[-\beta \frac{e^{\varpi(t)}}{G(t)} - W(t) \right] F(t), \quad \forall t \in J, \tag{1.18}$$

where $J = (0, \infty)$ in case of (i) or $J = (0, l)$ in case of (ii). Then $u \equiv C$ for some root C of $F(t) = 0$.

Remark 1.1. (i) In the case $W = 0$ and f is a constant, the assumption (1.17) is just (1.4) with $1 < \beta < \frac{n+3}{n-1}$. So, Lu's Liouville-type theorem (see Theorem 1.2) can be recovered by the above Theorem 1.4.

(ii) In the Liouville-type theorem by Lu (see Theorem 1.2), β is required to be positive. In our theorem, we do not require these constraints. In the proof of Theorem 2.1, β can be negative.

Remark 1.2. We introduce a transformation method that reduces the equation (1.1) to a form without the gradient term (see the equation (3.3)), allowing the application of Nash-Moser iteration techniques. Moreover, this transformation process is reversible. For the transformation $G = H(u)$, we require that G be a strictly monotonic function with $G > 0$. Therefore, we define the auxiliary function G by (1.11), (1.13), (1.15), or (1.16).

Remark 1.3. To my knowledge, there are very few results for the critical case under the given conditions. Nevertheless, by additionally assuming positivity or negativity of the nonlinear term F, we can establish results in the critical case.

Remark 1.4. In the case $\left(M^n, g, e^{-f} dv\right) = (M^n, g)$, $\beta = \frac{n+1}{n-1} \in \left(1, \frac{n+3}{n-1}\right)$, and $W = -\eta t^{-1}$ with $\eta \in (-\infty, 1) \cup (1, +\infty)$, assumption (1.17) is just (1.6). So, McCoy's Liouville-type theorem (see Theorem 1.3) can be recovered by the above Theorem 1.4.

Next, we use Theorem 1.4 to prove a Liouville-type theorem for positive solutions of (1.5) with $\eta = 1$.

Corollary 1.5. *Let $\beta \in \left(1, \frac{m+3}{m-1}\right)$ be a real number and $m \geq 3$. Let $\left(M^n, g, e^{-f} dv\right)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f^m \geq 0$. Assume that u is a positive solution of*

$$\Delta_f u - u^{-1} |\nabla u|^2 + F(u) = 0.$$

Assume that

$$F'(t) \leq \left(1 + \beta \frac{1}{\log t}\right) \frac{F(t)}{t}$$

holds for $t > l$ or for $0 < t < l$, respectively. Then $u \equiv C$ for some root C of $F(t) = 0$.

Remark 1.5. Our result may be considered as a supplement to the result of McCoy in [14, Theorem 2.1] (also see Theorem 1.3).

In the end, we derive some applications of Theorem 1.4.

Corollary 1.6. *Let $\beta \in \left(1, \frac{m+3}{m-1}\right)$ be a real number and $m \geq 3$. Let $\left(M^n, g, e^{-f} dv\right)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f^m \geq 0$. Assume that u is a positive solution of*

$$\Delta_f u + W(u) |\nabla u|^2 + \sum_{i=1}^{k} a_i u^{\alpha_i} - \sum_{j=1}^{l} b_j u^{\beta_j} = 0, \tag{1.19}$$

where $a_i > 0$, $b_j > 0$, and α_i and β_j are constants. Let $\max\limits_{1\le i\le k}(\alpha_i) \le \left(\pm\beta\frac{e^{\varpi(u)}}{G(u)} - W(u)\right)u \le \min\limits_{1\le j\le l}\left(\beta_j\right)$*. Then* $u \equiv C$ *for some root* C *of* $\sum_{i=1}^{k} a_i u^{\alpha_i} - \sum_{j=1}^{l} b_j u^{\beta_j} = 0$.

Remark 1.6. The special types of equation (1.19) include the static Fisher-Kolmogorov-Petrovsky-Piskunov (KPP) equation ($W(u) = 0$, $k = l = 1$, $a_1 = b_1 > 0$, $\alpha_1 = 1, \beta_1 = 2$), the Allen-Cahn equation ($W(u) = 0$, $k = l = 1$, $a_1 = b_1 = \alpha_1 = 1, \beta_1 = 3$), the static Newell-Whitehead equation ($W(u) = 0$, $k = l = 1$, $a_1 > 0$, $b_1 > 0$, $\alpha_1 = 1, \beta_1 = 3$), and the Lichnerowicz equation ($W(u) = 0$, $k = 2$, $l = 1$, $a_1 > 0$, $a_2 > 0$, $b_1 > 0$ $\alpha_1 = 1$, $\alpha_2 < 1, \beta_1 > 1$). Corollary 1.6 can apply to the above cases.

The rest of this paper is organized as follows. In Section 2, we prove Liouville-type theorems for (2.2) by the classical Nash-Moser iteration technique. In Section 3, we apply Theorem 2.1 to prove Theorem 1.4, and the proofs of related corollaries are provided.

2. Liouville-type theorems for $\Delta_f u + F(u) = 0$

The main result of this section is the following.

Theorem 2.1. *Let* $\left(M^n, g, e^{-f}dv\right)$ *be an n-dimensional complete smooth metric measure space with* $\mathrm{Ric}_f^m \ge 0$. *If F satisfies*

$$\beta F(t) - tF'(t) \ge 0, \quad \forall t > 0, \tag{2.1}$$

where $\beta \in \left(1, \frac{m+3}{m-1}\right)$*, then for any positive solution satisfying*

$$\Delta_f u + F(u) = 0, \tag{2.2}$$

$u(x)$ *must be a constant.*

In order to prove Theorem 2.1, we need the following result.

Theorem 2.2. *Let* $\left(M^n, g, e^{-f}dv\right)$ *be an n-dimensional complete smooth metric measure space with* $\mathrm{Ric}_f^m \ge -(m-1)\kappa g$*, where* κ *is a nonnegative constant. Assume for some constant* $\delta_0 < 1$ *that*

$$\frac{1}{m-1}\Big[(m+1)F(t) + 2\delta_0|F(t)|\Big] - tF'(t) \ge 0, \quad \forall t > 0. \tag{2.3}$$

If u is a positive solution to (2.2) *on a geodesic ball* $B_R(o) \subset M$*, then there exists a positive constant* C_{m,δ_0^+} *depending only on m, and* $\delta_0^+ = \max\{0, \delta_0\} \in [0, 1)$ *such that*

$$\sup_{B_{R/2}(o)} \frac{|\nabla u|}{u} \le C_{m,\delta_0^+}\left(\frac{1}{R} + \sqrt{\kappa}\right), \quad \forall R > 0.$$

Remark 2.1. Inequality (2.3) in Theorem 2.2 characterizes the subcritical condition that the nonlinear function F satisfies. In fact, when $F(t) \ge 0$, we can apply inequality (2.3) to obtain

$$\frac{m+1+2\delta_0}{m-1}F(t) - tF'(t) \ge 0.$$

When $F(t) < 0$, we can apply inequality (2.3) to obtain

$$\frac{m+1-2\delta_0}{m-1}F(t) - tF'(t) \ge 0.$$

Since $\delta_0 \in (0,1)$, we have
$$\frac{m+1\pm 2\delta_0}{m-1} \in (1, \frac{m+3}{m-1}).$$
This corresponds to β in inequality (2.1).

Moreover, the approach employed in the proof of Theorem 2.2 is applicable to semilinear elliptic equations with weighted p-Laplacians.

Proof of Theorem 2.2. To prove Theorem 2.2, we need the following pointwise differential inequality.

Lemma 2.3. *Let $\left(M^n, g, e^{-f}dv\right)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$, where κ is a nonnegative constant. Assume the condition* (2.3) *holds. If u is a positive weak solution to* (2.2) *on a geodesic ball $B_R(o) \subset M$, then the following pointwise inequality holds for every $\alpha \geq \alpha_0$:*

$$\Delta_f h^\alpha \geq 2\alpha\gamma_0 h^{\alpha+1} - 2\alpha(m-1)\kappa h^\alpha - \alpha a_1 |\nabla h| h^{\alpha-\frac{1}{2}}, \tag{2.4}$$

provided $\{\nabla u \neq 0\}$, where $\gamma_0 = \frac{1-\delta_0^+}{m-1}$, $a_1 = \frac{2(m-2)}{m-1}$, and α_0 is defined in (2.20).

Remark 2.2. Inequality (2.4) in Lemma 2.3 gives a very meticulous estimation of $\Delta_f|\nabla v|^{2\alpha}$. The introduction of the parameter α extends the applicability of the Cheng-Yau-type gradient estimate in equation (2.2). The larger the value of α, the higher the accuracy of the estimate for $\Delta_f|\nabla v|^{2\alpha}$.

Proof. Let
$$v = \ln u. \tag{2.5}$$

Then with respect to (2.2), we have

$$\Delta_f v = -|\nabla v|^2 - u^{-1}F(u).$$

Let $h = |\nabla v|^2$, thus we get
$$\Delta_f v = -h - u^{-1}F(u). \tag{2.6}$$

For any $\alpha > 1$, we have
$$\Delta_f h^\alpha = \alpha(\alpha-1)h^{\alpha-2}|\nabla h|^2 + \alpha h^{\alpha-1}\Delta_f h.$$

Using the Bochner formula,

$$\frac{1}{2}\Delta_f h = |\nabla^2 v|^2 + \langle \nabla v, \nabla \Delta_f v\rangle + \mathrm{Ric}_f(\nabla v, \nabla v), \tag{2.7}$$

so we get

$$\Delta_f h^\alpha = \alpha(\alpha-1)h^{\alpha-2}|\nabla h|^2 + 2\alpha h^{\alpha-1}\left(|\nabla^2 v|^2 + \langle \nabla v, \nabla \Delta_f v\rangle + \mathrm{Ric}_f(\nabla v, \nabla v)\right). \tag{2.8}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal frame of TM over a domain where $h \neq 0$, with $e_1 = \frac{\nabla v}{|\nabla v|}$. If we write $\nabla v = \sum_{i=1}^n v_i e_i$, it is straightforward to observe that $v_1 = |\nabla v| = h^{1/2}$ and $v_i = 0$ for all $2 \leq i \leq n$. The following identities are then satisfied:

$$v_{11} = \frac{1}{2}h^{-1}\langle \nabla v, \nabla h\rangle \leq \frac{|\nabla h|}{2h^{\frac{1}{2}}}, \tag{2.9}$$

$$\sum_{i=2}^{n} v_{ii} = -h - u^{-1}F(u) - v_{11} + \langle \nabla f, \nabla v \rangle. \tag{2.10}$$

By applying the Cauchy inequality, we obtain

$$\begin{aligned} |\nabla^2 v|^2 \geq & \sum_{i=1}^{n} v_{1i}^2 + \sum_{i=2}^{n} v_{ii}^2 \\ \geq & v_{11}^2 + \frac{1}{n-1}\left(\sum_{i=2}^{n} v_{ii}\right)^2. \end{aligned} \tag{2.11}$$

Then, making use of the inequality

$$(a+b)^2 \geq \frac{a^2}{1+\xi} - \frac{b^2}{\xi}, \quad \xi = \frac{m-n}{n-1},$$

from (2.11), we can deduce that

$$\begin{aligned} |\nabla^2 v|^2 \geq & v_{11}^2 + \frac{1}{n-1}\left(-h - u^{-1}F(u) - v_{11} + \langle \nabla f, \nabla v \rangle\right)^2 \\ \geq & v_{11}^2 + \frac{1}{n-1}\left(\frac{n-1}{m-1}(-h - u^{-1}F(u) - v_{11})^2 - \frac{n-1}{m-n}\langle \nabla f, \nabla v \rangle^2\right) \\ = & \frac{1}{m-1}h^2 + \frac{1}{m-1}u^{-2}F^2(u) + \frac{m}{m-1}v_{11}^2 + \frac{2}{m-1}hu^{-1}F(u) + \frac{2}{m-1}hv_{11} \\ & + \frac{2}{m-1}u^{-1}F(u)v_{11} - \frac{1}{m-n}\langle \nabla f, \nabla v \rangle^2. \end{aligned} \tag{2.12}$$

Then, from (2.6), we derive the following:

$$\begin{aligned} \langle \nabla v, \nabla \Delta_f v \rangle &= \langle \nabla v, \nabla(-h - u^{-1}F(u)) \rangle \\ &= -\langle \nabla v, \nabla h \rangle + \left(u^{-1}F(u) - F'(u)\right)h \\ &= -2hv_{11} + \left(u^{-1}F(u) - F'(u)\right)h. \end{aligned} \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.7) gives us

$$\begin{aligned} \frac{1}{2}\Delta_f h \geq & \frac{1}{m-1}h^2 + \frac{1}{m-1}u^{-2}F^2(u) + \frac{m}{m-1}v_{11}^2 + \frac{2}{m-1}hu^{-1}F(u) + \frac{2}{m-1}hv_{11} \\ & + \frac{2}{m-1}u^{-1}F(u)v_{11} - 2hv_{11} + \left(u^{-1}F(u) - F'(u)\right)h + \mathrm{Ric}_f^m(\nabla v, \nabla v) \\ \geq & \frac{1}{m-1}h^2 + \frac{1}{m-1}u^{-2}F^2(u) + \frac{m}{m-1}v_{11}^2 + \frac{2(2-m)}{m-1}hv_{11} \\ & + \frac{2}{m-1}u^{-1}F(u)v_{11} + \left(\frac{m+1}{m-1}u^{-1}F(u) - F'(u)\right)h - (m-1)\kappa h. \end{aligned} \tag{2.14}$$

For arbitrary $\delta \in [0,1)$, we have

$$\frac{\delta}{m-1}h^2 + \frac{\delta}{m-1}u^{-2}F^2(u) - \frac{2\delta}{m-1}u^{-1}|F(u)|h \geq 0, \tag{2.15}$$

and
$$\frac{1-\delta}{m-1}u^{-2}F^2(u)+\frac{2}{m-1}u^{-1}F(u)v_{11}\geq -\frac{1}{(m-1)(1-\delta)}v_{11}^2, \tag{2.16}$$
then we obtain
$$\begin{aligned}\frac{1}{2}\Delta_f h\geq&\frac{1-\delta}{m-1}h^2+\frac{1-\delta}{m-1}u^{-2}F^2(u)+\frac{2\delta}{m-1}u^{-1}|F(u)|h+\frac{m}{m-1}v_{11}^2+\frac{2(2-m)}{m-1}hv_{11}\\&+\frac{2}{m-1}u^{-1}F(u)v_{11}+\left(\frac{m+1}{m-1}u^{-1}F(u)-F'(u)\right)h-(m-1)\kappa h\\\geq&\frac{1-\delta}{m-1}h^2+\frac{m-m\delta-1}{(m-1)(1-\delta)}v_{11}^2+\frac{2(2-m)}{m-1}hv_{11}\\&+\left(\frac{m+1}{m-1}F(u)+\frac{2\delta}{m-1}|F(u)|-uF'(u)\right)u^{-1}h-(m-1)\kappa h.\end{aligned} \tag{2.17}$$
Observe that for $\delta=\delta_0^+=\max\{0,\delta_0\}\in[0,1)$, the following holds:
$$\frac{1}{2}\Delta_f h\geq\frac{1-\delta_0^+}{m-1}h^2+\frac{m-m\delta_0^+-1}{(m-1)(1-\delta_0^+)}v_{11}^2+\frac{2(2-m)}{m-1}hv_{11}-(m-1)\kappa h. \tag{2.18}$$
Combing (2.8) and (2.18), we obtain
$$\begin{aligned}&\frac{h^{1-\alpha}}{\alpha}\Delta_f h^\alpha\\\geq&(\alpha-1)h^{-1}|\nabla h|^2+2\left[\frac{1-\delta_0^+}{m-1}h^2+\frac{m-m\delta_0^+-1}{(m-1)(1-\delta_0^+)}v_{11}^2+\frac{2(2-m)}{m-1}hv_{11}-(m-1)\kappa h\right]\\\geq&4(\alpha-1)v_{11}^2+\frac{2-2\delta_0^+}{m-1}h^2+2\frac{m-m\delta_0^+-1}{(m-1)(1-\delta_0^+)}v_{11}^2+\frac{4(2-m)}{m-1}hv_{11}-2(m-1)\kappa h\\\geq&\frac{2-2\delta_0^+}{m-1}h^2+2\left(2\alpha-2+\frac{m-m\delta_0^+-1}{(m-1)(1-\delta_0^+)}\right)v_{11}^2-\frac{2(m-2)}{m-1}h^{\frac{1}{2}}|\nabla h|-2(m-1)\kappa h.\end{aligned} \tag{2.19}$$
There exists a sufficiently large constant α_0 such that
$$2\alpha_0-2+\frac{m-m\delta_0^+-1}{(m-1)(1-\delta_0^+)}\geq 0. \tag{2.20}$$
When $\alpha\geq\alpha_0$, we have
$$\frac{h^{1-\alpha}}{\alpha}\Delta_f h^\alpha\geq\frac{2-2\delta_0^+}{m-1}h^2-\frac{2(m-2)}{m-1}h^{\frac{1}{2}}|\nabla h|-2(m-1)\kappa h.$$
Thus, the proof of Lemma 2.3 is completed. □

To prove Theorem 2.2, we also need Saloff-Coste's Sobolev type inequality.

Lemma 2.4. *([36, 37]) Let $\left(M^n,g,e^{-f}dv\right)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f^m\geq-(m-1)\kappa g$, where κ is a nonnegative constant and $B_R(o)$ is a geodesic ball. For $m>2$, there exists a constant C_m depending only on m such that for all $\phi\in C_0^\infty(B_R(o))$, we have*
$$\left(\int_{B_R}|\phi|^{\frac{2m}{m-2}}e^{-f}dv\right)^{\frac{m-2}{m}}\leq e^{C_m(1+\sqrt{\kappa}R)}V_f^{-\frac{2}{m}}R^2\int_{B_R}\left(|\nabla\phi|^2+R^{-2}\phi^2\right)e^{-f}dv,$$
where $V_f=\int_{B_R}e^{-f}dv$.

We also need an integral inequality concerning $h = |\nabla v|^2$, which constitutes a key step in the Nash-Moser iteration.

Lemma 2.5. *[38, Lemma 3.1] Let $\left(M^n, g, e^{-f}dv\right)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$, where κ is a nonnegative constant and $B_R(o)$ is a geodesic ball. Assume condition (2.3) holds. Let u be a positive weak solution to (2.2) for any nonnegative cutoff function τ and positive number t. Then there exist constants a_2, a_3, and a_4 depending only on m and δ_0^+ such that for all $t > t_0 = C_{m,\delta_0^+}\left(1 + \sqrt{\kappa}R\right)$, there holds true that*

$$\begin{aligned}&\gamma_0\int_{B_R} h^{\alpha_0+t+1}\tau^2 e^{-f}dv + \frac{a_2}{t}e^{-t_0}V_f^{\frac{2}{m}}R^{-2}\left(\int_{B_R}|h^{\frac{\alpha_0+t}{2}}\tau|^{\frac{2m}{m-2}}e^{-f}dv\right)^{\frac{m-2}{m}}\\ \leq & a_4t_0^2R^{-2}\int_{B_R}h^{\alpha_0+t}\tau^2e^{-f}dv + \frac{a_3}{t}\int_{B_R}h^{\alpha_0+t}|\nabla\tau|^2e^{-f}dv.\end{aligned} \tag{2.21}$$

We also need the L_f^γ norm of h in $B_{\frac{3R}{4}}(o)$, which determines the initial data for the Nash-Moser iteration.

Lemma 2.6. *[38, Lemma 3.2] Let $\left(M^n, g, e^{-f}dv\right)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$, where κ is a nonnegative constant and $B_R(o)$ is a geodesic ball. Assume condition (2.3) holds. If u is a positive weak solution to (2.2) for $\gamma = (\alpha_0 + t_0)\frac{m}{m-2}$, there exists a positive constant $a_7 = a_7(m, \delta_0^+) > 0$ such that*

$$\left(\int_{B_{3R/4}} h^\gamma e^{-f}dv\right)^{\frac{1}{\gamma}} \leq a_7 V_f^{\frac{1}{\gamma}}\frac{t_0^2}{R^2}. \tag{2.22}$$

At this point, we are able to provide the proof of Theorem 2.2 through the Nash-Moser iteration method.

Proof. For convenience, we introduce the following notation:

$$\left\|h^{\frac{\alpha_0+t}{2}}\tau\right\|^2_{L_f^{\frac{2m}{m-2}}(B_R)} := \left(\int_{B_R}|h^{\frac{\alpha_0+t}{2}}\tau|^{\frac{2m}{m-2}}e^{-f}dv\right)^{\frac{m-2}{m}}.$$

Next, we remove the first term from (2.21), and consequently, we get

$$\begin{aligned}&\frac{a_2}{t}e^{-t_0}V_f^{\frac{2}{m}}R^{-2}\left(\int_{B_R}|h^{\frac{\alpha_0+t}{2}}\tau|^{\frac{2m}{m-2}}e^{-f}dv\right)^{\frac{m-2}{m}}\\ \leq & \frac{a_4t_0^2}{R^2}\int_{B_R}h^{\alpha_0+t}\tau^2e^{-f}dv + \frac{a_3}{t}\int_{B_R}h^{\alpha_0+t}|\nabla\tau|^2e^{-f}dv\\ \leq & \int_{B_R}\left(\frac{a_4t_0^2}{R^2}\tau^2 + \frac{a_3}{t}|\nabla\tau|^2\right)h^{\alpha_0+t}e^{-f}dv,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\left(\int_{B_R}|h^{\frac{\alpha_0+t}{2}}\tau|^{\frac{2m}{m-2}}e^{-f}dv\right)^{\frac{m-2}{m}}\\ \leq & e^{t_0}V_f^{-\frac{2}{m}}\int_{B_R}\left(\frac{a_4}{a_2}t_0^2t\tau^2 + \frac{a_3}{a_2}R^2|\nabla\tau|^2\right)h^{\alpha_0+t}e^{-f}dv.\end{aligned} \tag{2.23}$$

Then, we choose an increasing sequence $\{\gamma_k\}_{k=1}^{\infty}$ where

$$\gamma_1 = \gamma \quad \text{and} \quad \gamma_{k+1} = \frac{m\gamma_k}{m-2}, \quad k = 1, 2, \ldots,$$

as well as a decreasing sequence $\{r_k\}_{k=1}^{\infty}$ satisfying

$$r_k = \frac{R}{2} + \frac{R}{4^k}, \quad k = 1, 2, \ldots.$$

Next, we can pick $\{\tau_k\}_{k=1}^{\infty} \subset C_0^{\infty}(B_R(o))$ such that

$$\tau_k \in C_0^{\infty}(B_{r_k}(o)), \quad \tau_k = 1 \text{ in } B_{r_{k+1}}(o), \quad \text{and} \quad |\nabla \tau_k| \leq \frac{C4^k}{R}.$$

By setting $t = t_k$, $\tau = \tau_k$, and $t_k + \alpha_0 = \gamma_k$ in (2.23), we deduce that

$$\begin{aligned}
&\left(\int_{B_{r_k}} |h^{\frac{\alpha_0+t_k}{2}}\tau_k|^{\frac{2m}{m-2}} e^{-f} dv\right)^{\frac{m-2}{m}} \\
\leq& e^{t_0} V_f^{-\frac{2}{m}} \int_{B_{r_k}} \left(\frac{a_4}{a_2} t_0^2 t_k \tau_k^2 + \frac{a_3}{a_2} R^2 |\nabla \tau_k|^2\right) h^{\alpha_0+t_k} e^{-f} dv \\
\leq& e^{t_0} V_f^{-\frac{2}{m}} \int_{B_{r_k}} \left(\frac{a_4}{a_2} t_0^2 t_k \tau_k^2 + \frac{a_3}{a_2} C^2 16^k\right) h^{\alpha_0+t_k} e^{-f} dv \\
\leq& e^{t_0} V_f^{-\frac{2}{m}} \left(\frac{a_4}{a_2} t_0^2 t_k + \frac{a_3}{a_2} C^2 16^k\right) \int_{B_{r_k}} h^{\alpha_0+t_k} e^{-f} dv \\
\leq& e^{t_0} V_f^{-\frac{2}{m}} \left(\frac{a_4}{a_2} t_0^2 \left(t_0 + \alpha_0\right) \left(\frac{m}{m-2}\right)^k + \frac{a_3}{a_2} C^2 16^k\right) \int_{B_{r_k}} h^{\alpha_0+t_k} e^{-f} dv \\
\leq& e^{t_0} V_f^{-\frac{2}{m}} \left(\frac{a_4}{a_2} t_0^2 \left(t_0 + \alpha_0\right) 16^k + \frac{a_3}{a_2} C^2 16^k\right) \int_{B_{r_k}} h^{\alpha_0+t_k} e^{-f} dv.
\end{aligned} \tag{2.24}$$

There exists some constant a_8 such that

$$\left(\int_{B_{r_k}} |h^{\frac{\alpha_0+t_k}{2}}\tau_k|^{\frac{2m}{m-2}} e^{-f} dv\right)^{\frac{m-2}{m}} \leq a_8 t_0^3 16^k e^{t_0} V_f^{-\frac{2}{m}} \int_{B_{r_k}} h^{\gamma_k} e^{-f} dv. \tag{2.25}$$

Then, by raising both sides of (2.25) to the power of $1/\gamma_k$, we obtain

$$\left(\int_{B_{r_k}} |h^{\frac{\alpha_0+t_k}{2}}\tau_k|^{\frac{2m}{m-2}} e^{-f} dv\right)^{\frac{m-2}{m\gamma_k}} \leq (a_8 t_0^3 e^{t_0} V_f^{-\frac{2}{m}})^{\frac{1}{\gamma_k}} 16^{\frac{k}{\gamma_k}} \left(\int_{B_{r_k}} h^{\gamma_k} e^{-f} dv\right)^{\frac{1}{\gamma_k}}. \tag{2.26}$$

Thus, by the definition of γ_{k+1} and being $\alpha_0 + t_k = \gamma_k$, we reach

$$\left(\int_{B_{r_{k+1}}} h^{\gamma_{k+1}} e^{-f} dv\right)^{\frac{1}{\gamma_{k+1}}} \leq (a_8 t_0^3 e^{t_0} V_f^{-\frac{2}{m}})^{\frac{1}{\gamma_k}} 16^{\frac{k}{\gamma_k}} \left(\int_{B_{r_k}} h^{\gamma_k} e^{-f} dv\right)^{\frac{1}{\gamma_k}},$$

which implies that

$$\|h\|_{L_f^{\gamma_{k+1}}(B_{r_{k+1}})} \leq \left(a_8 t_0^3 e^{t_0} V_f^{-\frac{2}{m}}\right)^{\frac{1}{\gamma_k}} 16^{\frac{k}{\gamma_k}} \|h\|_{L_f^{\gamma_k}(B_{r_k})}. \tag{2.27}$$

By iteration, we get

$$\|h\|_{L_f^{\gamma_{k+1}}(B_{r_{k+1}})} \leq \left(a_8 t_0^3 e^{t_0} V_f^{-\frac{2}{m}}\right)^{\sum_{k=1}^{\infty}\frac{1}{\gamma_k}} 16^{\sum_{k=1}^{\infty}\frac{k}{\gamma_k}} \|h\|_{L_f^{\gamma_1}(B_{3R/4})}. \tag{2.28}$$

We observe that

$$\sum_{k=1}^{\infty}\frac{1}{\gamma_k} = \frac{m}{2\gamma_1} \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{k}{\gamma_k} = \frac{m^2}{4\gamma_1}$$

Then, letting $k \to \infty$ in (2.28) yields the following inequality:

$$\|h\|_{L_f^{\beta_{k+1}}(B_{k+1})} \leq \left(a_8 t_0^3 e^{t_0}\right)^{\frac{m}{2\beta_1}} 16^{\frac{m^2}{4\beta_1}} V_f^{-\frac{1}{\beta_1}} \|h\|_{L_f^{\beta_1}(B_{3R/4})}. \tag{2.29}$$

There exists $a_9 > 0$ such that

$$a_9 \geq \left(a_8 t_0^3 e^{t_0}\right)^{\frac{m}{2\beta_1}} 16^{\frac{m^2}{4\beta_1}}.$$

Thus, we have

$$\|h\|_{L_f^{\infty}(B_{R/2})} \leq a_9 V_f^{-\frac{1}{\beta_1}} \|h\|_{L_f^{\beta_1}(B_{3R/4})}. \tag{2.30}$$

Based on (2.22), we obtain

$$\|h\|_{L_f^{\infty}(B_{R/2})} \leq a_{10}\frac{(1+\sqrt{K}R)^2}{R^2}, \tag{2.31}$$

where $a_{10} = a_9 a_7 c^2_{m,\delta_0^+}$. Recalling $h = |\nabla v|^2$ and $v = \ln u$, we arrive at the required estimate. Thus, we finish proving Theorem 2.2. □

Now we prove Theorem 2.1.

Proof of Theorem 2.1. If the conditions in Theorem 2.2 are satisfied, then for some constant $\delta_0 < 1$, there holds

$$\frac{1}{m-1}\Big[(m+1)F(t) + 2\delta_0 |F(t)|\Big] - tF'(t) \geq 0, \quad \forall t > 0.$$

When $F(t) \geq 0$, let $\beta = \frac{1}{m-1}(m+1+2\delta_0)$, and according to $\delta_0 < 1$, we have $\beta < \frac{m+3}{m-1}$ and

$$\begin{aligned} 0 &\leq \frac{1}{m-1}\Big[(m+1)F(t) + 2\delta_0 |F(t)|\Big] - tF'(t) \\ &= \frac{1}{m-1}\Big[(m+1) + 2\delta_0\Big]F(t) - tF'(t) \\ &= \beta F(t) - tF'(t). \end{aligned}$$

When $F(t) \leq 0$, let $\beta = \frac{1}{m-1}(m+1-2\delta_0)$, and according to $\delta_0 < 1$, we have $\beta > 1$ and

$$\begin{aligned} 0 &\leq \frac{1}{m-1}\Big[(m+1)F(t) + 2\delta_0 |F(t)|\Big] - tF'(t) \\ &= \frac{1}{m-1}\Big[(m+1) - 2\delta_0\Big]F(t) - tF'(t) \\ &= \beta F(t) - tF'(t). \end{aligned}$$

To conclude,

$$\frac{1}{m-1}\Big[(m+1)F(t)+2\delta_0|F(t)|\Big]-tF'(t)\geq 0$$

and

$$\beta F(t)-tF'(t)\geq 0$$

are equivalent for any F under the condition $\beta\in\left(1,\frac{m+3}{m-1}\right)$.

From the conclusion of Theorem 2.1, when $\kappa=0$, for any $x\in B_{R/2}(o)\subset M$, we obtain

$$\sup_{B_{R/2}(o)}\frac{|\nabla u|}{u}\leq\frac{C_{m,\delta_0}}{R}. \tag{2.32}$$

Letting $R\to\infty$ in (2.32), we have

$$\nabla u=0,\quad \forall x\in M. \tag{2.33}$$

Therefore, u is a positive constant on M.

3. Proof of Theorem 1.4 and related corollaries

Following the approach in [39, Theorem 2], we apply Theorem 2.1 to demonstrate Theorem 1.4 and the associated corollaries.

Let $u\in C^3(M)$ and G be the function defined by (1.11), (1.13), (1.15), or (1.16). Clearly, in all cases, G is a monotone function, and thus G has an inverse function $\hat{G}$. Set $H=G(u)$. Then a computation shows that

$$\Delta_f H=G'(u)\Delta_f u+G''(u)|\nabla u|^2. \tag{3.1}$$

Proof of (i) of case (1) in Theorem 1.4. Assuming (1.10) holds and G is defined by (1.11), then G is a monotone function, and its inverse function is denoted by $\hat{G}$. Furthermore, we observe that $u>0$ implies $H>0$. We define $\tilde{F}$ by

$$\tilde{F}(t):=F\left(\hat{G}(t)\right)e^{\varpi\left(\hat{G}(t)\right)},\quad t>0. \tag{3.2}$$

Combing (3.1) and (1.1), we have

$$\begin{aligned}\Delta_f H&=G'(u)\Delta_f u+G''(u)|\nabla u|^2\\&=e^{\varpi(u)}\Big[\Delta_f u+W(u)|\nabla u|^2\Big]\\&=-e^{\varpi(u)}F(u)\\&=-\tilde{F}(H).\end{aligned} \tag{3.3}$$

Therefore, by making the change of variable $H=G(u)$, (1.1) can be transformed into $\Delta_f H+\tilde{F}(H)=0$. Since $H>0$, it follows Theorem 2.1 that if $\tilde{F}$ satisfies

$$\tilde{F}'(H)\leq\beta\frac{\tilde{F}(H)}{H}, \tag{3.4}$$

with $\beta\in\left(1,\frac{m+3}{m-1}\right)$, then all positive solutions of $\Delta_f H+\tilde{F}(H)=0$ are constant.

Meanwhile, from (3.4), we have

$$F'\left(\hat{G}(H)\right)\hat{G}'(H)e^{\varpi\left(\hat{G}(H)\right)} + F\left(\hat{G}(H)\right)e^{\varpi\left(\hat{G}(H)\right)}\varpi'\left(\hat{G}(H)\right)\hat{G}'(H) \leq \beta\frac{F\left(\hat{G}(H)\right)e^{\varpi\left(\hat{G}(H)\right)}}{H}. \tag{3.5}$$

By dividing both sides of (3.5) by $\hat{G}'(H)e^{\varpi\left(\hat{G}(H)\right)}$, we have

$$F'\left(\hat{G}(H)\right) \leq \left[\frac{\beta}{\hat{G}'(H)H} - \varpi'\left(\hat{G}(H)\right)\right]F\left(\hat{G}(H)\right).$$

Because $H = G(u)$ and $u = \hat{G}(H)$, by applying (1.10) and (1.11), we have

$$F'(u) \leq \left[\beta\frac{e^{\varpi(u)}}{G(u)} - W(u)\right]F(u).$$

Therefore, if F satisfies the above inequality with $\beta \in \left(1, \frac{m+3}{m-1}\right)$, then the solution of (1.1) is a constant. Thus, the proof of (i) of case (1) in Theorem 1.4 is completed.

Proof of (ii) of case (1) in Theorem 1.4. Supposing (1.14) is satisfied and G is given by (1.15), then G is a monotone function, and its inverse function is denoted by $\hat{G}$. Moreover, $u > l$ implies $H > 0$. We define $\tilde{F}$ by (3.2). Similar to the proof of (i) of case (1) in Theorem 1.4, by making the change of variable $H = G(u)$, (1.1) can be transformed into $\Delta_f H + \tilde{F}(H) = 0$. By Theorem 2.1, it follows that if $\tilde{F}$ satisfies

$$\tilde{F}'(H) \leq \beta\frac{\tilde{F}(H)}{H},$$

with $\beta \in \left(1, \frac{m+3}{m-1}\right)$, then every positive weak solution of $\Delta_f H + \tilde{F}(H) = 0$ is constant. Through calculation, we can obtain

$$F'\left(\hat{G}(H)\right) \leq \left[\frac{\beta}{\hat{G}'(H)H} - \varpi'\left(\hat{G}(H)\right)\right]F\left(\hat{G}(H)\right).$$

Since $H = G(u)$ and $u = \hat{G}(H)$, by using (1.10) and (1.15), we obtain

$$F'(u) \leq \left[\beta\frac{e^{\varpi(u)}}{G(u)} - W(u)\right]F(u).$$

Therefore, if F satisfies the above inequality with $\beta \in \left(1, \frac{m+3}{m-1}\right)$, then the solution of (1.1) is a constant. Then we complete the proof of (ii) in case (1).

Proof of (i) of case (2) in Theorem 1.4. Assuming (1.12) is satisfied and G is given by (1.13), then G is a monotone function, and its inverse function is denoted by $\hat{G}$. Moreover, $u > 0$ implies $H > 0$. We define $\tilde{F}$ by (3.2). By repeating the proof of part (i) of case (1) in Theorem 1.4, when $H = G(u)$, (1.1) can be transformed into $\Delta_f H + \tilde{F}(H) = 0$. In light of Theorem 2.1, it follows that if $\tilde{F}$ satisfies

$$\tilde{F}'(H) \leq \beta\frac{\tilde{F}(H)}{H},$$

with $\beta \in \left(1, \frac{m+3}{m-1}\right)$, then every positive solution of $\Delta_f H + \tilde{F}(H) = 0$ is constant. Then we can obtain

$$F'\left(\hat{G}(H)\right) \leq \left[\frac{\beta}{\hat{G}'(H)H} - \varpi'\left(\hat{G}(H)\right)\right]F\left(\hat{G}(H)\right).$$

Since $H = G(u)$ and $u = \hat{G}(H)$, by using (1.10) and (1.13), we obtain

$$F'(u) \leq \Big[-\beta \frac{e^{\varpi(u)}}{G(u)} - W(u) \Big] F(u)\,.$$

Consequently, if F satisfies the inequality with $\beta \in \left(1, \frac{m+3}{m-1}\right)$, then the solution of (1.1) is a constant. This completes the proof of part (i) for case (2) in Theorem 1.4.

Proof of (ii) of case (2) in Theorem 1.4. Assuming (1.14) is satisfied and G is given by (1.16), then G is a monotone function, and its inverse function is denoted by $\hat{G}$. Moreover, $0 < u < l$ implies $H > 0$. We define $\tilde{F}$ by (3.2). According to the proof of (i) of case (1) in Theorem 1.4, if $H = G(u)$, (1.1) can be transformed into $\Delta_f H + \tilde{F}(H) = 0$. It follows Theorem 2.1 that when $\tilde{F}$ satisfies

$$\tilde{F}'(H) \leq \beta \frac{\tilde{F}(H)}{H},$$

with $\beta \in \left(1, \frac{m+3}{m-1}\right)$, then every positive solution of $\Delta_f H + \tilde{F}(H) = 0$ is constant. Through calculation, it is easy to obtain

$$F'\left(\hat{G}(H)\right) \leq \Big[\frac{\beta}{\hat{G}'(H)H} - \varpi'\left(\hat{G}(H)\right) \Big] F\left(\hat{G}(H)\right).$$

Because $H = G(u)$ and $u = \hat{G}(H)$, by using (1.10) and (1.16), we have

$$F'(u) \leq \Big[-\beta \frac{e^{\varpi(u)}}{G(u)} - W(u) \Big] F(u)\,.$$

Therefore, if F satisfies the above inequality with $\beta \in \left(1, \frac{m+3}{m-1}\right)$, then the solution of (1.1) is a constant. Hence, (ii) of case (2) in Theorem 1.4 follows. Thus, we complete the proof of Theorem 1.4.

In what follows, we prove Corollary 1.5.

Proof of corollary 1.5. It is easy to see that $W = -t^{-1}$ satisfies (1.14) and $\varpi(t) = -\log t$.

If G is defined by (1.15), we have

$$G(t) = \log \frac{t}{l}, \quad t > l.$$

If G is defined by (1.16), we have

$$G(t) = \log \frac{l}{t}, \quad 0 < t < l.$$

From Theorem 1.4, the condition on F changes to

$$F'(t) \leq [1 + \beta \frac{1}{\log t}] \frac{F(t)}{t},$$

which must hold for $t > l$ or for $0 < t < l$, respectively. Hence, we have completed the proof of Corollary 1.5.

Proof of corollary 1.6. When $F(u) = \sum_{i=1}^{k} a_i u^{\alpha_i} - \sum_{j=1}^{l} b_j u^{\beta_j}$, where $a_i > 0$, $b_j > 0$, and α_i and β_j are constants, by applying Theorem 1.4, we obtain the following:

(i) If $F(u)$ satisfies (1.17), we have

$$\sum_{i=1}^{k}\alpha_i a_i u^{\alpha_i-1}-\sum_{j=1}^{l}\beta_j b_j u^{\beta_j-1}$$
$$\leq\Big[\beta\frac{e^{\varpi(u)}}{G(u)}-W(u)\Big]\Big[\sum_{i=1}^{k}a_i u^{\alpha_i}-\sum_{j=1}^{l}b_j u^{\beta_j}\Big],$$

then we have

$$\sum_{i=1}^{k}\left[\alpha_i-\left(\beta\frac{e^{\varpi(u)}}{G(u)}-W(u)\right)u\right]a_i u^{\alpha_i-1}$$
$$\leq\sum_{j=1}^{l}\left[\beta_j-\left(\beta\frac{e^{\varpi(u)}}{G(u)}-W(u)\right)u\right]b_j u^{\beta_j-1}.$$

(ii) If $F(u)$ satisfies (1.18), we have

$$\sum_{i=1}^{k}\alpha_i a_i u^{\alpha_i-1}-\sum_{j=1}^{l}\beta_j b_j u^{\beta_j-1}$$
$$\leq\Big[-\beta\frac{e^{\varpi(u)}}{G(u)}-W(u)\Big]\Big[\sum_{i=1}^{k}a_i u^{\alpha_i}-\sum_{j=1}^{l}b_j u^{\beta_j}\Big],$$

then we have

$$\sum_{i=1}^{k}\left[\alpha_i-\left(-\beta\frac{e^{\varpi(u)}}{G(u)}-W(u)\right)u\right]a_i u^{\alpha_i-1}$$
$$\leq\sum_{j=1}^{l}\left[\beta_j-\left(-\beta\frac{e^{\varpi(u)}}{G(u)}-W(u)\right)u\right]b_j u^{\beta_j-1}.$$

Thus, we have

$$\sum_{i=1}^{k}\left[\alpha_i-\left(\pm\beta\frac{e^{\varpi(u)}}{G(u)}-W(u)\right)u\right]a_i u^{\alpha_i-1}$$
$$\leq\sum_{j=1}^{l}\left[\beta_j-\left(\pm\beta\frac{e^{\varpi(u)}}{G(u)}-W(u)\right)u\right]b_j u^{\beta_j-1}.$$

Now we set $\alpha_0=\max\limits_{1\leq i\leq k}(\alpha_i)$, $\beta_0=\min\limits_{1\leq j\leq l}\left(\beta_j\right)$. Let $\alpha_0\leq\left(\pm\beta\frac{e^{\varpi(u)}}{G(u)}-W(u)\right)u\leq\beta_0$. Then $u\equiv C$ for some root C of $\sum_{i=1}^{k}a_i u^{\alpha_i}-\sum_{j=1}^{l}b_j u^{\beta_j}=0$. Hence, we have proved Corollary 1.6.

4. Conclusions

In this paper, we establish Liouville-type theorems for positive solutions of equation (1.1) on smooth metric measure spaces with nonnegative m-Bakry-Émery Ricci curvature. By combining the Saloff-Coste Sobolev inequality with the Nash-Moser iteration, we derive gradient estimates that lead to the triviality of solutions under suitable assumptions on W and F. Our main result extends and unifies the works of Lu and McCoy, covering a wider range of parameters and general gradient nonlinearities. As applications, we treat the critical case $\eta = 1$ and polynomial-type nonlinearities. The method is robust and may be extended to parabolic problems or settings with lower regularity.

Author contributions

Fanqi Zeng: Writing-original draft, methodology; Cheng Jin: Formal Analysis, writing-review, editing, methodology.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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