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Research article

Doubly nonlinear equation with perturbation governed by hypergraph Laplacian

Masahiro Ikeda[1,2] **and Shun Uchida**[3,*]

[1] Graduate School of Information Science and Technology, The University of Osaka, 1-5 Yamadaoka Suita-shi, Osaka 565-0871, Japan

[2] Center for Advanced Intelligence Project, RIKEN, 1-4-1 Nihonbashi, Chuo-ku, Tokyo 103-0027, Japan

[3] Faculty of Science and Technology, Oita University, 700 Dannoharu, Oita 870-1192, Japan

* **Correspondence:** Email: shunuchida@oita-u.ac.jp.

Abstract: The hypergraph Laplacian, which was introduced to study the structure of complicated networks in the information science, is defined as a set-valued operator on a finite-dimensional Euclidean space. In previous works, we have shown that a nonlinear ordinary differential equation governed by the hypergraph Laplacian describes the diffusion process on a hypergraph, and its behavior closely resembles the heat equation. In this paper, we consider some generalized doubly nonlinear equation governed by the hypergraph Laplacian in order to deal with various diffusion phenomena on discrete domains.

Keywords: hypergraph Laplacian; nonlinear evolution inclusion; doubly nonlinear equation; set-valued differential equation; subdifferential; solvability
Mathematics Subject Classification: Primary 34G25; Secondary 05C65, 34A12, 34A34, 47H04

1. Introduction

In this paper, we are concerned with the following doubly nonlinear ordinary differential equation:

$$\frac{d}{dt}\beta(x(t)) + \partial\varphi_{G,p}(x(t)) \ni F(\beta(x(t))) + h(t),$$

where $x : [0, T] \to \mathbb{R}^N$ is an unknown function, $h : [0, T] \to \mathbb{R}^N$ is a given function, $\beta, F : \mathbb{R}^N \to \mathbb{R}^N$ are given continuous functions, and $\beta : \mathbb{R}^N \to \mathbb{R}^N$ is a maximal monotone graph. Moreover, $\partial\varphi_{G,p} : \mathbb{R}^N \to 2^{\mathbb{R}^N}$, which is defined as a set-valued operator on $\mathbb{R}^N$, is the hypergraph Laplacian (a precise definition will be given in Section 2.2). The hypergraph Laplacian was introduced by Yuichi Yoshida in [1] and is used to study the structure of networks represented by hypergraphs in the information science (see, e.g., [2–6]).

In [7,8], we showed that the solution to a nonlinear ordinary differential equation $\frac{d}{dt}x(t)+\partial\varphi_{G,p}(x(t)) \ni 0$ describes the diffusion process on a hypergraph and its behavior closely resembles that of the solution to the classical heat equation $\partial_t u - \Delta u = 0$. Hence, by replacing the Laplacian in parabolic-type PDEs with the hypergraph Laplacian, we might be able to introduce various diffusion models on discrete domains, whose behavior is similar to parabolic type PDEs. For instance, when we consider the Allen–Cahn type equation $\frac{d}{dt}x(t) + \partial\varphi_{G,p}(x(t)) \ni W'(x(t))$, where W is a double-well potential, the behavior of its solution might resemble that of the Allen–Cahn type PDE (see, e.g., [9, 10]). Then, we can observe two phase separation on hypergraphs, which might imply the clustering or the grouping of hypergraph networks and help to investigate the geometric structure of networks. Moreover, we might obtain some hints about how to study the asymptotic behavior of solutions to the doubly nonlinear parabolic equation through the investigation of $\frac{d}{dt}\beta(x) + \partial\varphi_{G,p}(x) \ni F(\beta(x)) + h$ as a toy model. In order to apply our results to various diffusion models on discrete domains, here we consider some generalized equation and show its solvability.

The solvability of doubly nonlinear equations is a classical problem, and numerous results have been obtained (see, e.g., Ch. 11 of [11] and Ch. III of [12]). However, to the best of our knowledge, they deal with equations without perturbation terms or assume some growth conditions of β and perturbations F in most previous studies. Hence, in this paper, we consider doubly nonlinear equations with a perturbation term $F(\beta(x))$ and aim to show the existence of solutions without any growth conditions on β and the perturbation term F as much as possible. In [13], we deal with the same type of doubly nonlinear parabolic equation:

$$\partial_t\beta(u(y,t)) - \Delta_p u(y,t) = F(\beta(u(y,t))) + h(y,t), \tag{1.1}$$

where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. In that paper, by using

$$\int_\Omega \beta(u(y,t))(-\Delta_p u(y,t))dy \geq 0,$$

which is formally derived from the integration by parts, we have shown the solvability of (1.1) without any growth conditions of the nonlinear term $\beta : \mathbb{R} \to \mathbb{R}$ and the perturbation term $F : \mathbb{R} \to \mathbb{R}$. Analogous to this, if $\beta : \mathbb{R}^N \to \mathbb{R}^N$ and the hypergraph Laplacian satisfy

$$\beta(x(t)) \cdot \partial\varphi_{G,p}(x(t)) \geq 0, \tag{1.2}$$

we might expect to follow essentially the same strategy of proofs as those in [13]. However, since we can not use the integration by parts in the inner product, it is difficult to determine whether (1.2) holds or not without any assumptions for the nonlinearity $\beta : \mathbb{R}^N \to \mathbb{R}^N$. To cope with this difficulty, we introduce some assumptions (β.1)–(β.3) in Section 3. The most important assumption is (β.1), which ensures the validity of (1.2) and makes it possible to use the same technique as in the previous study. These assumptions (β.1)–(β.3) seem to be natural. For example, $\beta(x) = \|x\|^{q-2}x$ satisfies all assumptions (β.1)–(β.3), so we can treat a typical example

$$\frac{d}{dt}\|x(t)\|^{q-2}x(t) + \partial\varphi_{G,p}(x(t)) \ni F_0(x(t)) + h(t) \tag{1.3}$$

with $q > 1$ in our results, where $F_0(x) := F(\|x\|^{q-2}x)$. For a future problem, the large time behavior of solutions to this equation is of interest, for example, the extinction and blow-up of solutions in finite time might occur by analogy with the behavior of solutions to the porous medium equation and the fast diffusion equation (see, e.g., [14]).

In the next section, we define some notations and the hypergraph Laplacian. In Section 3, we state our assumptions on β and present the main theorems. We give our proofs in Section 4.

2. Preliminaries

In order to state our main results, here we give some notations and their basic properties. Henceforth, $\|\cdot\|$ denotes the canonical norm in the Euclidean space $\mathbb{R}^N$, and we shall use the following notations for Lebesgue spaces and Sobolev spaces. Let $r \in [1, \infty)$. The Lebesgue space $L^r(0, T; \mathbb{R}^N)$ is the space of measurable functions $x : [0, T] \to \mathbb{R}^N$ such that

$$\|x\|_{L^r} := \left(\int_0^T \|x(t)\|^r dt \right)^{1/r} < \infty$$

and $L^\infty(0, T; \mathbb{R}^N)$ is the space of measurable functions $x : [0, T] \to \mathbb{R}^N$ such that

$$\|x\|_{L^\infty} := \operatorname*{ess\,sup}_{0 \le t \le T} \|x(t)\| < \infty.$$

Moreover, the space $W^{1,r}(0, T; \mathbb{R}^N)$ consists of functions in $L^r(0, T; \mathbb{R}^N)$ whose derivative $\frac{d}{dt}x$ in the distributional sense also belongs to $L^r(0, T; \mathbb{R}^N)$, which is endowed with the norm $\|x\|_{W^{1,r}} := \|x\|_{L^r} + \|\frac{d}{dt}x\|_{L^r}$. Furthermore, let $q' := q/(q-1)$ be the Hölder conjugate exponent of $q > 1$.

2.1. Subdifferential

Let $\phi : \mathbb{R}^N \to (-\infty, +\infty]$ (i.e., $\phi(x)$ possibly takes $+\infty$ with some $x \in \mathbb{R}^N$) be a lower semi-continuous convex function and $\phi \not\equiv +\infty$ (i.e., $\phi(x) < +\infty$ with some $x \in \mathbb{R}^N$). We call the set $D(\phi) := \{x \in \mathbb{R}^N; \phi(x) < +\infty\}$ the effective domain of ϕ. Then, $\eta \in \mathbb{R}^N$ is called the subgradient of ϕ at $x \in \mathbb{R}^N$ if

$$\eta \cdot (y - x) \le \phi(y) - \phi(x) \quad \forall y \in \mathbb{R}^N \tag{2.1}$$

and the subdifferential operator $\partial\phi$ is defined by

$$\partial\phi(x) := \{\eta \in \mathbb{R}^N; \ \ \eta \cdot (y - x) \le \phi(y) - \phi(x) \quad \forall y \in \mathbb{R}^N\}. \tag{2.2}$$

Here, $D(\partial\phi) := \{x \in \mathbb{R}^N; \partial\phi(x) \neq \emptyset\}$ is the domain of the subdifferential operator. Note that $\partial\phi : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ possibly becomes a set-valued mapping in general. By the definition of the subgradient, it is obvious that $0 \in \partial\phi(x)$ if and only if ϕ attains its minimum at x (other basic properties of the subdifferential can be found in, e.g., [15–17]).

The subgradient is one of the generalizations of the usual (Fréchet) derivative. Indeed, if ϕ is differentiable at x in the usual sense, then its derivative $D\phi(x)$ coincides with the subgradient (i.e., $\partial\phi(x) = \{D\phi(x)\}$). However, several properties of the usual differential do not hold in the case of the subdifferential. A typical example is that $\partial(\phi + \psi)(x)$ does not coincide with $\partial\phi(x) + \partial\psi(x)$ in general. On the other hand, the following classical result holds (see, e.g., Theorem 2.10 of [15]).

Lemma 1. *Let $\phi, \psi : \mathbb{R}^N \to (-\infty, +\infty]$ be lower semicontinuous convex functions and $\phi, \psi \not\equiv +\infty$. If $(\mathrm{int}D(\phi)) \cap D(\psi) \neq \emptyset$, then $\partial(\phi + \psi) = \partial\phi + \partial\psi$ holds.*

Throughout this paper, we assume that $\beta : \mathbb{R}^N \to \mathbb{R}^N$ is a single-valued continuous function and has a primitive function j. More precisely, we assume that there exists some lower semi-continuous convex function $j : \mathbb{R}^N \to (-\infty, +\infty]$ with $j \not\equiv +\infty$ such that $\beta(x) = \partial j(x)$ for every $x \in D(\partial j)$ and j is continuously Fréchet differentiable on $D(\partial j)$ (then, $\beta(x)$ coincides with the Fréchet derivative of $j(x)$ on $D(\partial j)$).

By the maximal monotonicity of the subdifferential operator, we can see the following demiclosedness property (see, e.g., Proposition 3.4 of [15]).

Lemma 2. *Let $\{x_n\}_{n\in\mathbb{N}}$ weakly converge to x in $L^q(0,T;\mathbb{R}^N)$ and $\{\beta(x_n)\}_{n\in\mathbb{N}}$ strongly converge to ζ in $L^{q'}(0,T;\mathbb{R}^N)$ with some $q > 1$. Then, $\zeta = \beta(x)$ for a.e. $t \in (0,T)$.*

Next, we define the Legendre-Fenchel transform of j by

$$j^*(\sigma) = \sup_{s\in\mathbb{R}^N} (\sigma \cdot s - j(s)) \,.$$

If j is a convex lower semi-continuous function with $D(j) \neq \emptyset$, then j^* is also convex lower semi-continuous and satisfies $D(j^*) \neq \emptyset$. Moreover, the subdifferential of j^* coincides with the inverse operator of ∂j:

$$\partial j^*(\sigma) = \beta^{-1}(\sigma) := \{s \in \mathbb{R}^N;\ \sigma = \beta(s)\}.$$

Then, by $x \in \beta^{-1}(\beta(x)) = \partial j^*(\beta(x))$ and the definition of the subgradient, we get

$$(\beta(y) - \beta(x)) \cdot x \leq j^*(\beta(y)) - j^*(\beta(x)). \tag{2.3}$$

When $\beta(0) = 0$, then $0 \in \beta^{-1}(0)$ holds, namely, j^* attains its minimum at $x = 0$. Then, we can assume that $j^*(0) = 0$ and $j^* \geq 0$ without loss of generality. Furthermore, the following property of a chain rule holds.

Lemma 3. *Assume $x \in L^p(0,T;\mathbb{R}^N)$, $\zeta \in L^\infty(0,T;\mathbb{R}^N)$, $j^*(\zeta) \in L^1(0,T;\mathbb{R})$, and $\beta(x) \in W^{1,p'}(0,T;\mathbb{R}^N)$. If $\zeta(t) = \beta(x(t))$ for a.e. $t \in (0,T)$, then*

$$j^*(\zeta(t)) - j^*(\beta(x(0))) = \int_0^t \frac{d}{ds}\beta(x(s)) \cdot x(s)ds \tag{2.4}$$

holds for almost any $t \in (0,T]$.

Proof. The same type of identity in Sobolev spaces has been shown in, e.g., Lemma 1.5 of [18] and Lemma 2 of [13] (see also [19] for a chain rule inequality in metric spaces). We can prove Lemma 3 by almost the same procedure as in previous results. However, for the sake of self-completeness and convenience, here we give a proof of this lemma in $\mathbb{R}^N$-setting by following the arguments in those results.

Let $x(t) = x(0)$ for any $t < 0$. Then, by (2.3), for arbitrary $\theta > 0$,

$$\begin{aligned} &j^*(\beta(x(s))) - j^*(\beta(x(s-\theta))) \leq (\beta(x(s)) - \beta(x(s-\theta))) \cdot x(s), \\ &j^*(\beta(x(s))) - j^*(\beta(x(s-\theta))) \geq (\beta(x(s)) - \beta(x(s-\theta))) \cdot x(s-\theta). \end{aligned} \tag{2.5}$$

Integrating the first inequality of (2.5) over $(0,t)$, we have

$$\int_0^t j^*(\beta(x(s)))ds - \int_0^t j^*(\beta(x(s-\theta)))ds = \int_{t-\theta}^t j^*(\beta(x(s)))ds - \theta j^*(\beta(x(0))),$$

namely,

$$\frac{1}{\theta}\int_{t-\theta}^{t} j^*(\beta(x(s)))ds - j^*(\beta(x(0))) \le \frac{1}{\theta}\int_0^t (\beta(x(s)) - \beta(x(s-\theta)))\cdot x(s)ds.$$

Since $\beta(x) = \zeta$ holds for a.e. $t \in (0,T)$, we obtain

$$\frac{1}{\theta}\int_{t-\theta}^{t} j^*(\zeta(s))ds - j^*(\beta(x(0))) \le -\int_0^t \frac{\beta(x(s-\theta)) - \beta(x(s))}{\theta}\cdot x(s)ds.$$

Since $\beta(x)$ is differentiable for a.e. $t \in (0,T)$, we obtain by taking the limit as $\theta \to 0$,

$$\begin{aligned}\limsup_{\theta\to 0}\frac{1}{\theta}\int_{t-\theta}^{t} j^*(\zeta(s))ds - j^*(\beta(x(0))) &\le -\liminf_{\theta\to 0}\int_0^t \frac{\beta(x(s-\theta)) - \beta(x(s))}{\theta}\cdot x(s)ds \\ &\le \int_0^t \frac{d}{ds}\beta(x(s))\cdot x(s)ds,\end{aligned}$$

where we used Fatou's lemma. Next, integrating the second inequality of (2.5) over $(0,t)$, we get

$$\begin{aligned}\frac{1}{\theta}\int_{t-\theta}^{t} j^*(\beta(x(s)))ds - j^*(\beta(x(0))) &\ge \frac{1}{\theta}\int_0^t (\beta(x(s)) - \beta(x(s-\theta)))\cdot x(s-\theta)ds \\ &= \frac{1}{\theta}\int_{-\theta}^{t-\theta} (\beta(x(s+\theta)) - \beta(x(s)))\cdot x(s)ds\end{aligned}$$

with $\theta > 0$. Then, we have

$$\frac{1}{\theta}\int_{t-\theta}^{t} j^*(\zeta(s))ds - j^*(\beta(x(0))) \ge \int_{-\theta}^{t-\theta} \frac{\beta(x(s+\theta)) - \beta(x(s))}{\theta}\cdot x(s)ds.$$

Hence, by Fatou's lemma,

$$\liminf_{\theta\to 0}\frac{1}{\theta}\int_{t-\theta}^{t} j^*(\zeta(s)))ds - j^*(\beta(x(0))) \ge \int_0^t \frac{d}{ds}\beta(x(s))\cdot x(s)ds,$$

and then the limit of $\frac{1}{\theta}\int_{t-\theta}^{t} j^*(\zeta(s)))ds$ as $\theta \to +0$ exists and satisfies

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_{t-\theta}^{t} j^*(\zeta(s))ds = j^*(\beta(x(0))) + \int_0^t \frac{d}{ds}\beta(x(s))\cdot x(s)ds. \tag{2.6}$$

Similarly, we have by (2.3),

$$\begin{aligned} j^*(\beta(x(s+\theta))) - j^*(\beta(x(s))) &\le (\beta(x(s+\theta)) - \beta(x(s)))\cdot x(s+\theta), \\ j^*(\beta(x(s+\theta))) - j^*(\beta(x(s))) &\ge (\beta(x(s+\theta)) - \beta(x(s)))\cdot x(s). \end{aligned} \tag{2.7}$$

Integrating the first inequality of (2.7) over $(-\theta, t)$, we get

$$\int_{-\theta}^{t} j^*(\beta(x(s+\theta)))ds - \int_{-\theta}^{t} j^*(\beta(x(s)))ds = \int_{t}^{t+\theta} j^*(\beta(x(s)))ds - \theta j^*(\beta(x(0)))$$

and then

$$\frac{1}{\theta}\int_t^{t+\theta} j^*(\zeta(s))ds - j^*(\beta(x(0))) \leq \frac{1}{\theta}\int_{-\theta}^{t}(\beta(x(s+\theta)) - \beta(x(s)))\cdot x(s+\theta)ds$$
$$= -\int_0^{t+\theta}\frac{\beta(x(s-\theta)) - \beta(x(s))}{\theta}\cdot x(s)ds.$$

Hence,

$$\limsup_{\theta\to 0}\frac{1}{\theta}\int_t^{t+\theta} j^*(\zeta(s))ds - j^*(\beta(x(0))) \leq \int_0^t \frac{d}{ds}\beta(x(s))\cdot x(s)ds.$$

By the same argument as before, we can derive from the second inequality of (2.7)

$$\liminf_{\theta\to 0}\frac{1}{\theta}\int_t^{t+\theta} j^*(\zeta(s))ds - j^*(\beta(x(0))) \geq \int_0^t \frac{d}{ds}\beta(x(s))\cdot x(s)ds.$$

Therefore,

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_t^{t+\theta} j^*(\zeta(s))ds = j^*(\beta(x(0))) + \int_0^t \frac{d}{ds}\beta(x(s))\cdot x(s)ds. \tag{2.8}$$

By (2.6) and (2.8), $t \mapsto \int_0^t j^*(\zeta(s))ds$ is differentiable for every $t > 0$ and the assertion holds for every Lebesgue point of $t \mapsto j^*(\zeta(t))$. □

2.2. Hypergraph Laplacian

Next, we define an operator called the hypergraph Laplacian. A *(weighted) hypergraph* G is defined as a triplet consisting of

- a finite set $V = \{1, 2, \dots, N\}$,
- a family of subsets E with more than one element of V, that is, $\#e \geq 2$ for each $e \in E$ ($\#e$ stands for the number of elements of e),
- a function $w : E \to (0, \infty)$.

This $G = (V, E, w)$ describes a model of a network in which the vertices numbered from 1 to N are connected by each $e \in E$. Here, $e \in E$, called a *hyperedge*, can be regarded as a group of multiple vertices. For instance, models of co-authorship relationships among researchers and communities in social media can be described by hypergraphs (see Figure 1). In this case, $w : E \to (0, \infty)$, called the weight of the hyperedge, describes the strength of each connection $e \in E$.

In order to study the structure of a hypergraph, the hypergraph Laplacian was introduced in [1] in the following way (see also [7, 8, 20]). For each hyperedge $e \in E$, we define $f_e : \mathbb{R}^N \to [0, \infty)$ by

$$f_e(x) := \max_{i,j\in e} |x_i - x_j| \tag{2.9}$$

and

$$\varphi_{G,p}(x) := \frac{1}{p}\sum_{e\in E} w(e)(f_e(x))^p \qquad p \in [1, \infty). \tag{2.10}$$

When we regard the i-th component x_i of $x \in \mathbb{R}^N$ as the heat or the number of particles on the i-th node of V, then f_e represents the largeness of heat/density gradient along the hyperedge $e \in E$ and $\varphi_{G,p}$ can be interpreted as the p-Dirichlet energy associated with the hypergraph G.

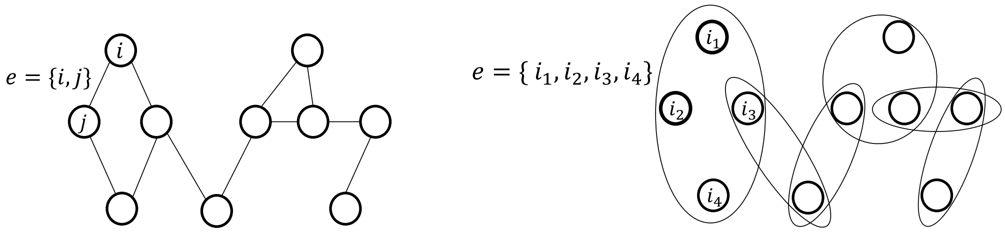

Figure 1. The set $V = \{1, \ldots, N\}$ corresponds to the set of vertices labeled from 1 to N. When every $e \in E$ consists of two elements, each $e \in E$ represents a line segment connecting two nodes and then G describes a usual graph (left figure). On the other hand, if e has more than two elements, $e \in E$ can be regarded as a group of multiple vertices and G describes a hypergraph (right figure).

Since f_e and $\varphi_{G,p}$ are continuous and convex on $\mathbb{R}^N$, we can define the subdifferential operators of them and represent the subgradients of them by

$$\partial f_e(x) = \operatorname*{argmax}_{b \in B_e} b \cdot x = \left\{ b_e \in B_e;\ \ b_e \cdot x = \max_{b \in B_e} b \cdot x \right\}, \tag{2.11}$$

$$\begin{aligned} \partial \varphi_{G,p}(x) &= \sum_{e \in E} w(e)(f_e(x))^{p-1} \partial f_e(x) \\ &= \left\{ \sum_{e \in E} w(e)(f_e(x))^{p-1} b_e;\ b_e \in \operatorname*{argmax}_{b \in B_e} b \cdot x \right\}, \end{aligned} \tag{2.12}$$

where $B_e \subset \mathbb{R}^N$ is defined by

$$\begin{aligned} B_e &:= \operatorname{conv}\{\mathbf{1}_i - \mathbf{1}_j;\ i, j \in e\} \\ &= \operatorname{conv}\left\{ (\ldots, 0, \overset{\overset{i}{\vee}}{1}, 0, \ldots, 0, \overset{\overset{j}{\vee}}{-1}, 0, \ldots) \in \mathbb{R}^N;\ i, j \in e \right\} \end{aligned} \tag{2.13}$$

and $\mathbf{1}_i$ is the i-th unit vector of the canonical basis of $\mathbb{R}^N$. Namely, B_e is the convex hull of the set which consists of vectors $\mathbf{1}_i - \mathbf{1}_j$ with respect to the indices i, j belonging to e. By the definition of B_e, we obtain $f_e(x) = \max_{b \in B_e} b \cdot x$, and then its subdifferential coincides with (2.11) by Danskin-Bertseka's theorem (see, e.g., Proposition 2.54 of [21]). Thanks to the subdifferential formula of the composition of functionals (see, e.g., Corollary 3.5 of [22] and also Proposition 2.2 of [8]), we obtain the formula (2.12). Then, the subdifferential operator $\partial \varphi_{G,p}$ is called the *hypergraph (p-)Laplacian*, where $1 \le p < \infty$.

Here we state some properties of the hypergraph Laplacian for later use. First, we can easily see that $\varphi_{G,p}(x) \ge 0$ for every $x \in \mathbb{R}^N$. Since the hypergraph Laplacian can be written as a subdifferential operator, we can see the following (see, e.g., Lemma 1.2 of [23] and Proposition 2.7 of [15]).

Lemma 4. *Let $\{x_n\}_{n \in \mathbb{N}}$ weakly converge to x in $L^q(0, T; \mathbb{R}^N)$ and $\{\eta_n\}_{n \in \mathbb{N}}$ weakly converge to η in $L^{q'}(0, T; \mathbb{R}^N)$ with some $q > 1$. If $\eta_n(t) \in \partial \varphi_{G,p}(x_n(t))$ for a.e. $t \in (0, T)$ and*

$$\limsup_{n \to \infty} \int_0^T (\eta_n(t) - \eta(t)) \cdot (x_n(t) - x(t)) dt \le 0,$$

then $\eta(t) \in \partial \varphi_{G,p}(x(t))$ holds for a.e. $t \in (0, T)$.

By the definition (2.11), we have $x \cdot b_e = f_e(x)$ for every $x \in \mathbb{R}^N$ and $b_e \in \partial f_e(x)$. Then, by (2.12), we get

$$\eta \cdot x = p\varphi_{G,p}(x) \quad \forall x \in \mathbb{R}^N, \ \ \forall \eta \in \partial\varphi_{G,p}(x). \tag{2.14}$$

Moreover, $f_e(x) \leq 2\|x\|$ and $\|b\| \leq 2$ hold for every $x \in \mathbb{R}^N$, $b \in B_e$, and $e \in E$ by the definitions (2.9) and (2.13). Then, there exists some positive constant $C > 0$ such that

$$\varphi_{G,p}(x) \leq C\|x\|^p, \qquad \|\eta\| \leq C\|x\|^{p-1} \tag{2.15}$$

for every $x \in \mathbb{R}^N$ and $\eta \in \partial\varphi_{G,p}(x)$.

2.3. Constraint term

In [8], an evolution equation $\frac{d}{dt}x(t) + \partial\varphi_{G,p}(x(t)) \ni 0$ is considered, and we show that the behavior of its solutions $x : [0, \infty) \to \mathbb{R}^N$ closely resembles that of the heat equation $\partial_t u - \Delta u = 0$ with the homogeneous Neumann boundary condition. Namely, each component $x_1(t), \ldots, x_N(t)$ of $x(t)$ moves freely and tends to the mean value of the initial data. On the other hand, in [7], we consider the case where the heat on several nodes is fixed as some given value. This situation can be described by an equation with a constraint term, and we show that the behavior of solutions is similar to that of $\partial_t u - \Delta u = 0$ with the homogeneous Dirichlet boundary condition, i.e., $x(t)$ tends to 0 as $t \to \infty$.

In this paper, we are concerned with both the unconstrained case and the constrained case (referred to as the "Neumann boundary condition case" and the "Dirichlet boundary condition case," respectively). For the latter case, we here state some notations to describe a constraint term. We set $N = n + m$. Let the first n components of $x(t)$ move freely, and the last m components of $x(t)$ be fixed as $x_{n+j}(t) \equiv 0$ (see Figure 2). In order to represent this situation, we define a set $K_0 \subset \mathbb{R}^N$ by

$$K_0 := \{x \in \mathbb{R}^N; x = (x_1, \ldots, x_n, 0, \ldots, 0) \ \ x_i \in \mathbb{R} \ \ i = 1, \ldots, n\}$$

and an indicator function $I_{K_0} : \mathbb{R}^N \to [0, \infty]$ by

$$I_{K_0}(x) := \begin{cases} 0 & \text{if } x \in K_0, \\ +\infty & \text{if } x \notin K_0. \end{cases}$$

Since I_{K_0} is a lower semi-continuous convex function, we can define the subdifferential of I_{K_0}, and then the situation where the last m components of $x(t)$ are fixed as $x_{n+j}(t) \equiv 0$ can be described by the constraint term ∂I_{K_0}.

According to Lemma 2.1 of [7], the constraint term ∂I_{K_0} can be characterized by

$$\partial I_{K_0}(x) = \{\xi \in \mathbb{R}^N; \xi = (0, \ldots, 0, \xi_{n+1}, \ldots, \xi_{n+m}), \ \ \xi_{n+j} \in \mathbb{R}, \ \ j = 1, \ldots, m\}.$$

Hence, if $x \in K_0$ and $\xi \in \partial I_{K_0}(y)$ with an arbitrary $y \in \mathbb{R}^N$, we have $x \cdot \xi = 0$ since $x = (x_1, \ldots, x_n, 0 \ldots, 0)$ and $\xi = (0, \ldots, 0, \xi_{n+1}, \ldots, \xi_{n+m})$.

By the definition, $D(\varphi_{G,p}) = \mathbb{R}^N$ holds, and then we can deduce $\partial(\varphi_{G,p} + I_{K_0}) = \partial\varphi_{G,p} + \partial I_{K_0}$ by Lemma 1. Hence, from Lemma 1.2 of [23] and Proposition 2.7 of [15], we can derive the following.

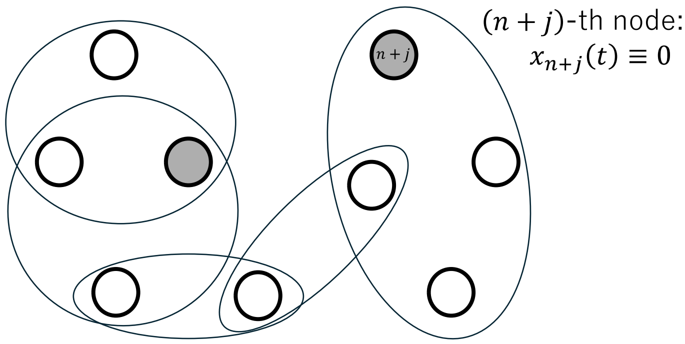

Figure 2. We consider the case where the heat on some vertices (colored in the above) is given as $x_{n+j}(t) \equiv 0$.

Lemma 5. *Let $\{x_n\}_{n\in\mathbb{N}}$ weakly converge to x in $L^q(0,T;\mathbb{R}^N)$, and $\{Y_n\}_{n\in\mathbb{N}}$ weakly converge to Y in $L^{q'}(0,T;\mathbb{R}^N)$ with some $q > 1$. If $Y_n(t) \in \partial\varphi_{G,p}(x_n(t)) + \partial I_{K_0}(x_n(t))$ for a.e. $t \in (0,T)$ and*

$$\limsup_{n\to\infty} \int_0^T (Y_n(t) - Y(t)) \cdot (x_n(t) - x(t))dt \leq 0,$$

then $Y(t) \in \partial\varphi_{G,p}(x(t)) + \partial I_{K_0}(x(t))$ for a.e. $t \in (0,T)$.

Moreover, the following Poincaré-type inequality for the hypergraph Laplacian holds (see Theorem 2.2 of [7]).

Lemma 6. *There exists some constant $c > 0$ such that*

$$\varphi_{G,p}(x) + I_{K_0}(x) \geq c\|x\|^p \qquad \forall x \in \mathbb{R}^N.$$

3. Main theorems

We first deal with the following Cauchy problem of the doubly nonlinear equation without constraint term:

$$\begin{cases} \dfrac{d}{dt}\beta(x(t)) + \partial\varphi_{G,p}(x(t)) \ni F(\beta(x(t))) + h(t), \\ x(0) = x_0. \end{cases} \tag{3.1}$$

In this problem, we assume the following two conditions:

(β.1) Let $\beta : \mathbb{R}^N \to \mathbb{R}^N$ be continuous (possibly satisfy $D(\beta) \neq \mathbb{R}^N$). Suppose that there exists some lower semi-continuous convex function $j : \mathbb{R}^N \to (-\infty, +\infty]$ with $j \not\equiv +\infty$ such that j is continuously Fréchet differentiable on $D(\partial j)$ and $\beta(x) = Dj(x)$ for every $x \in D(\partial j) = D(\beta)$. Moreover, assume $0 \in D(\beta)$, $0 \in D(j)$, and there is some non-negative function $B : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfying

$$\beta(x) = \begin{cases} B(x)x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(β.2) There exist some positive constants $\alpha, c_0, C_0 > 0$ such that

$$j(x) \geq c_0\|x\|^{1+\alpha} - C_0 \quad \forall x \in \mathbb{R}^N.$$

Theorem 1. *(without constraint term) Let the parameter of the hypergraph Laplacian p satisfy $p \geq 1$ and assume (β.1) and (β.2). Suppose that $F : \mathbb{R}^N \to \mathbb{R}^N$ is continuous. Then, for every $h \in L^\infty(0,T;\mathbb{R}^N)$ and $x_0 \in D(\beta)$, there exists $T' \in (0,T]$ such that* (3.1) *possesses at least one time-local solution between* $[0,T']$ *satisfying*

$$x \in L^\infty(0,T';\mathbb{R}^N), \quad \beta(x) \in W^{1,\infty}(0,T';\mathbb{R}^N).$$

We also consider the following problem with the constraint term ∂I_{K_0}, which corresponds to the zero-Dirichlet boundary condition case:

$$\begin{cases} \dfrac{d}{dt}\beta(x(t)) + \partial\varphi_{G,p}(x(t)) + \partial I_{K_0}(x(t)) \ni F(\beta(x(t))) + h(t), \\ x(0) = x_0. \end{cases} \tag{3.2}$$

In this case, we add the following condition:

(β.3) $(\mathrm{int}D(j)) \cap K_0 \neq \emptyset$ and $D(\beta) \cap K_0 \neq \emptyset$.

Theorem 2. *(with constraint term) Let the parameter of the hypergraph Laplacian p satisfy $p \geq 1$ and assume (β.1) and (β.3). When $p = 1$, assume in addition (β.2). Suppose that $F : \mathbb{R}^N \to \mathbb{R}^N$ is continuous. Then, for every $h \in L^\infty(0,T;\mathbb{R}^N)$ and $x_0 \in D(\beta) \cap K_0$, there exists $T' \in (0,T]$ such that* (3.2) *possesses at least one time-local solution between* $[0,T']$ *satisfying*

$$x \in L^p(0,T';\mathbb{R}^N), \quad \beta(x) \in W^{1,p'}(0,T';\mathbb{R}^N)$$

if $p > 1$, and

$$x \in L^\infty(0,T';\mathbb{R}^N), \quad \beta(x) \in W^{1,\infty}(0,T';\mathbb{R}^N)$$

if $p = 1$.

Remark 1. *A typical example which satisfies (β.1)–(β.3) is*

$$j(x) = \frac{1}{q}\|x\|^q, \quad \beta(x) = \|x\|^{q-2}x$$

with $q > 1$ (note that $D(j) = D(\beta) = \mathbb{R}^N$ in this case). Hence, our results can be applied to (1.3)*. Another example which satisfies $D(\beta) \neq \mathbb{R}^N$ and (β.1)–(β.3) is*

$$j(x) = \begin{cases} -\dfrac{1}{2}\log(1 - \|x\|^2) & \text{if } \|x\| < 1, \\ +\infty & \text{if } \|x\| \geq 1, \end{cases} \qquad \beta(x) = \begin{cases} \dfrac{x}{1 - \|x\|^2} & \text{if } \|x\| < 1, \\ \emptyset & \text{if } \|x\| \geq 1, \end{cases}$$

(note that $j(x) \geq 0 \geq \|x\|^q - 1$ holds for any $\|x\| < 1$ and $(\mathrm{int}D(j)) \cap K_0 = D(\beta) \cap K_0 \ni 0$).

4. Proofs of main theorems

4.1. Theorem 1 (without constraint term)

We begin with the following equation:

$$\beta(x) + \tau\partial\varphi_{G,p}(x) \ni g, \tag{4.1}$$

where $\tau > 0$ and $g \in \mathbb{R}^N$ are given. By the definition of the subdifferential and $D(\partial\varphi_{G,p}) = \mathbb{R}^N$, if the lower semi-continuous convex functional defined by

$$\Phi(x) := j(x) + \tau\varphi_{G,p}(x) - g \cdot x$$

possesses a minimizer x, this satisfies

$$0 \in \partial\Phi(x) := Dj(x) + \tau\partial\varphi_{G,p}(x) - g = \beta(x) + \tau\partial\varphi_{G,p}(x) - g,$$

namely, a minimizer of Φ becomes a solution to (4.1) satisfying $x \in D(\beta)$ (here we use Lemma 1). By using (β.2), we have

$$\Phi(x) \geq c_0\|x\|^{1+\alpha} - C_0 - \|g\|\|x\| \geq c\|x\|^{1+\alpha} - C_0 - C\|g\|^{\frac{1+\alpha}{\alpha}}$$

with some constants $c, C > 0$. This implies the existence of a minimizing sequence of Φ which is relatively compact. Then, the limit of such a minimizing sequence exists and becomes a global minimizer of Φ by the lower semicontinuity of Φ. Multiplying (4.1) by $\beta(x) = B(x)x$, we obtain

$$\|\beta(x)\|^2 + p\tau B(x)\varphi_{G,p}(x) \leq \|g\|\|\beta(x)\| \tag{4.2}$$

since $x \cdot \eta = p\varphi_{G,p}(x)$ holds for every $x \in \mathbb{R}^N$ and $\eta \in \partial\varphi_{G,p}(x)$. Since $B(x) \geq 0$, we get from (4.2)

$$\|\beta(x)\| \leq \|g\|. \tag{4.3}$$

Based on this fact, next we consider the following time discretization of (3.1):

$$\begin{cases} \dfrac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} + \partial\varphi_{G,p}(x_\tau^{i+1}) \ni F^M(\beta(x_\tau^i)) + h_\tau^i, \\ x_\tau^0 = x_0, \end{cases} \tag{4.4}$$

where $\tau = T/\mu$ ($\mu \in \mathbb{N}$), $i = 0, \ldots, \mu - 1$, and $h_\tau^i = \frac{1}{\tau}\int_{i\tau}^{(i+1)\tau} h(t)dt$. Moreover,

$$F^M(z) := \begin{cases} M\dfrac{F(z)}{\|F(z)\|} & \text{if } \|F(z)\| > M, \\ F(z) & \text{if } \|F(z)\| \leq M. \end{cases} \tag{4.5}$$

Equation (4.4) is equivalent to

$$\beta(x_\tau^{i+1}) + \tau\partial\varphi_{G,p}(x_\tau^{i+1}) \ni \beta(x_\tau^i) + \tau F^M(\beta(x_\tau^i)) + \tau h_\tau^i.$$

Hence, for any given $x_\tau^i \in D(\beta)$, there exists at least one $x_\tau^{i+1} \in D(\beta)$. Therefore, by fixing $x_0 \in D(\beta)$, we can obtain a sequence $x_\tau^1, \ldots, x_\tau^\mu \in D(\beta)$ by induction.

Let $\eta_\tau^{i+1} \in \mathbb{R}^N$ be a section of $\partial\varphi_{G,p}(x_\tau^{i+1})$ satisfying (4.4). Namely, η_τ^{i+1} satisfies

$$\eta_\tau^{i+1} \in \partial\varphi_{G,p}(x_\tau^{i+1}), \qquad \frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} + \eta_\tau^{i+1} = F^M(\beta(x_\tau^i)) + h_\tau^i. \tag{4.6}$$

According to (4.3) with $g = \beta(x_\tau^i) + \tau F^M(\beta(x_\tau^i)) + \tau h_\tau^i$, we get

$$\|\beta(x_\tau^{i+1})\| \le \|\beta(x_\tau^i)\| + \tau\|F^M(\beta(x_\tau^i))\| + \tau\|h_\tau^i\| \le \|\beta(x_\tau^i)\| + \tau M + \tau\|h\|_{L^\infty},$$

which yields from $T = \tau\mu$

$$\max_{i=1,\ldots,\mu-1} \|\beta(x_\tau^{i+1})\| \le \|\beta(x_0)\| + T(M + \|h\|_{L^\infty}). \tag{4.7}$$

Multiplying (4.6) by x_τ^{i+1}, we have

$$j(x_\tau^{i+1}) \le j(0) + (\|\beta(x_\tau^i)\| + \tau M + \tau\|h\|_{L^\infty})\|x_\tau^{i+1}\|,$$

where we use the definition of the subdifferential

$$\beta(x_\tau^{i+1}) \cdot x_\tau^{i+1} = \beta(x_\tau^{i+1}) \cdot (x_\tau^{i+1} - 0) \ge j(x_\tau^{i+1}) - j(0)$$

and (2.14). By (4.7) and the assumption (β.2), we get

$$c_0\|x_\tau^{i+1}\|^{1+\alpha} - C_0 \le j(0) + C\|x_\tau^{i+1}\|,$$

which immediately implies

$$\max_{i=1,\ldots,\mu} \|x_\tau^i\| \le C_1, \tag{4.8}$$

where and henceforth $C_1 > 0$ stands for a general constant which is independent of μ and τ. By (2.15), we obtain

$$\max_{i=1,\ldots,\mu} \|\eta_\tau^i\| \le C_1. \tag{4.9}$$

From (4.6), we also have

$$\max_{i=0,\ldots,\mu-1} \left\| \frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} \right\| \le C_1. \tag{4.10}$$

Multiplying (4.6) by x_τ^{i+1} again and using (2.3), (4.8), and (4.10), we get

$$\frac{1}{\tau}(j^*(\beta(x_\tau^{i+1})) - j^*(\beta(x_\tau^i))) \le C_1,$$

which yields

$$\max_{i=1,\ldots,\mu} j^*(\beta(x_\tau^i)) \le TC_1 + j^*(\beta(x_0)). \tag{4.11}$$

Here, for a given sequence $w_\tau := \{w_\tau^0, w_\tau^1, \dots, w_\tau^\mu\}$, we define

$$\Pi_\tau w_\tau(t) := \begin{cases} w_\tau^{i+1} & \text{if } t \in (i\tau, (i+1)\tau], \\ w_\tau^0 & \text{if } t = 0, \end{cases}$$
$$\Lambda_\tau w_\tau(t) := \begin{cases} \dfrac{w_\tau^{i+1} - w_\tau^i}{\tau}(t - i\tau) + w_\tau^i & \text{if } t \in (i\tau, (i+1)\tau], \\ w_\tau^0 & \text{if } t = 0. \end{cases} \tag{4.12}$$

Then, (4.7) implies

$$\sup_{0 \le t \le T} \|\Pi_\tau \beta(x_\tau(t))\| \le \|\beta(x_0)\| + TC_1. \tag{4.13}$$

Moreover, we can see from (4.8)–(4.10) and (4.13) that

$$\sup_{0 \le t \le T} \|\Pi_\tau x_\tau(t)\| + \sup_{0 \le t \le T} \|\Lambda_\tau \beta(x_\tau(t))\| + \sup_{0 \le t \le T} \|\Pi_\tau \eta_\tau(t)\| + \sup_{0 \le t \le T} \left\| \frac{d}{dt} \Lambda_\tau \beta(x_\tau(t)) \right\| \le C_1.$$

Then, there exist some subsequences of $\{\Pi_\tau x_\tau\}_{\tau>0}$, $\{\Pi_\tau \beta(x_\tau)\}_{\tau>0}$, $\{\Pi_\tau \eta_\tau\}_{\tau>0}$ (we omit relabeling), and some $x, \zeta, \eta : [0, T] \to \mathbb{R}^N$ such that

$$\begin{aligned} &\Pi_\tau x_\tau \rightharpoonup x && *\text{-weakly in } L^\infty(0, T; \mathbb{R}^N), \\ &\Lambda_\tau \beta(x_\tau) \to \zeta && \text{strongly in } C([0, T]; \mathbb{R}^N), \\ &\Pi_\tau \eta_\tau \rightharpoonup \eta && *\text{-weakly in } L^\infty(0, T; \mathbb{R}^N), \end{aligned}$$

and

$$\frac{d}{dt} \Lambda_\tau \beta(x_\tau) \rightharpoonup \frac{d}{dt} \zeta \qquad * \text{-weakly in } L^\infty(0, T; \mathbb{R}^N)$$

as $\tau \to 0$.

If $t \in (i\tau, (i+1)\tau]$,

$$\begin{aligned} \|\Pi_\tau \beta(x_\tau(t)) - \Lambda_\tau \beta(x_\tau(t))\| &= \left\| \beta(x_\tau^{i+1}) - \frac{\beta(x^{i+1}) - \beta(x^i)}{\tau}(t - i\tau) - \beta(x_\tau^i) \right\| \\ &= ((i+1)\tau - t) \left\| \frac{\beta(x^{i+1}) - \beta(x^i)}{\tau} \right\| \end{aligned}$$

and by (4.10),

$$\int_{i\tau}^{(i+1)\tau} \|\Pi_\tau \beta(x_\tau(t)) - \Lambda_\tau \beta(x_\tau(t))\|^q dt \le C_1 \int_{i\tau}^{(i+1)\tau} ((i+1)\tau - t)^q dt \le C_1 \tau^{q+1}$$

with any $q > 1$. Hence,

$$\int_0^T \|\Pi_\tau \beta(x_\tau(t)) - \Lambda_\tau \beta(x_\tau(t))\|^q dt \le C_1 T \tau^q \to 0$$

as $\tau \to 0$, namely,

$$\Pi_\tau \beta(x_\tau) \to \zeta \qquad \text{strongly in } L^q(0, T; \mathbb{R}^N) \quad \forall q > 1.$$

This and Lemma 2 imply that $\zeta = \beta(x)$ a.e $t \in (0, T)$. By the dominated convergence theorem, we have

$$\Pi_\tau F^M(\beta(x_\tau)) \to F^M(\beta(x)) \qquad \text{strongly in } L^q(0, T; \mathbb{R}^N) \quad \forall q > 1$$

and

$$\Pi_\tau h_\tau \to h \qquad \text{strongly in } L^q(0, T; \mathbb{R}^N) \quad \forall q > 1.$$

Therefore, (4.6) converges to

$$\frac{d}{dt}\beta(x) + \eta = F^M(\beta(x)) + h$$

$*$-weakly in $L^\infty(0, T; \mathbb{R}^N)$. We have to show $\eta(t) \in \partial\varphi_{G,p}(x(t))$ for a.e. $t \in (0, T)$.

Let $t = T' \leq T$ be a time at which ζ satisfies (2.4). Namely, assume

$$j^*(\zeta(T')) - j^*(\beta(x(0))) = \int_0^{T'} \frac{d}{dt}\beta(x(t)) \cdot x(t)dt.$$

Moreover, assume $m\tau < T' \leq (m+1)\tau$ with some $m = 0, \ldots, \mu - 1$. Since j^* is convex,

$$\begin{aligned}
j^*(\Lambda_\tau\beta(x_\tau(T'))) =& j^*\left(\frac{\beta(x_\tau^{m+1}) - \beta(x_\tau^m)}{\tau}(T' - m\tau) + \beta(x_\tau^m)\right) \\
=& j^*\left(\frac{T' - m\tau}{\tau}\beta(x_\tau^{m+1}) + \frac{(m+1)\tau - T'}{\tau}\beta(x_\tau^m)\right) \\
\leq& \frac{T' - m\tau}{\tau} j^*\left(\beta(x_\tau^{m+1})\right) + \frac{(m+1)\tau - T'}{\tau} j^*\left(\beta(x_\tau^m)\right) \\
\leq& j^*\left(\beta(x_\tau^{m+1})\right) + j^*\left(\beta(x_\tau^m)\right).
\end{aligned}$$

By (4.11), the RHS is uniformly bounded, and then $\{j^*(\Lambda_\tau\beta(x_\tau(T')))\}_{\tau>0}$ is also uniformly bounded. Extract a convergent subsequence and let its limit be Θ. Since j^* is lower semi-continuous convex and $\Lambda_\tau\beta(x_\tau(T')) \to \zeta(T')$ as $\tau \to 0$, we have

$$j^*(\zeta(T')) \leq \liminf_{\tau\to 0} j^*(\Lambda_\tau\beta(x_\tau(T'))) = \Theta. \tag{4.14}$$

Multiplying (4.6) by x_τ^{n+1}, we get

$$j^*(\beta(x_\tau^{n+1})) - j^*(\beta(x_\tau^n)) + \tau\eta_\tau^{n+1} \cdot x_\tau^{n+1} \leq \tau F^M(\beta(x_\tau^n)) \cdot x_\tau^{n+1} + \tau h_\tau^n \cdot x_\tau^{n+1}.$$

Summing this from $n = 0$ to $m - 1$, we have

$$\begin{aligned}
&j^*(\beta(x_\tau^m)) - j^*(\beta(x_0)) + \int_0^{m\tau} \Pi_\tau\eta_\tau(t) \cdot \Pi_\tau x_\tau(t)dt \\
&\leq \int_0^{m\tau} \Pi_\tau h_\tau(t - \tau) \cdot \Pi_\tau x_\tau(t)dt + \int_0^{m\tau} \Pi_\tau F^M(\beta(x_\tau(t - \tau))) \cdot \Pi_\tau x_\tau(t)dt.
\end{aligned}$$

Moreover,

$$\frac{T' - m\tau}{\tau} j^*(\beta(x_\tau^{m+1})) - \frac{T' - m\tau}{\tau} j^*(\beta(x_\tau^m)) + (T' - m\tau)\eta_\tau^{m+1} \cdot x_\tau^{m+1}$$

$$\leq (T' - m\tau)h_\tau^m \cdot x_\tau^{m+1} + (T' - m\tau)F^M(\beta(x_\tau^m)) \cdot x_\tau^{m+1}.$$

Then, we obtain

$$\begin{aligned}
&\frac{T' - m\tau}{\tau} j^*(\beta(x_\tau^{m+1})) + \frac{(m+1)\tau - T'}{\tau} j^*(\beta(x_\tau^m)) - j^*(\beta(x_0)) + \int_0^{T'} \Pi_\tau \eta_\tau(t) \cdot \Pi_\tau x_\tau(t)dt \\
&\leq \int_0^{T'} \Pi_\tau h_\tau(t-\tau) \cdot \Pi_\tau x_\tau(t)dt + \int_0^{T'} \Pi_\tau F^M(\beta(x_\tau(t-\tau))) \cdot \Pi_\tau x_\tau(t)dt.
\end{aligned}$$

By the convexity of j^*,

$$\begin{aligned}
&\frac{T' - m\tau}{\tau} j^*(\beta(x_\tau^{m+1})) + \frac{(m+1)\tau - T'}{\tau} j^*(\beta(x_\tau^m)) \\
&\geq j^*\left(\frac{T' - m\tau}{\tau}\beta(x_\tau^{m+1}) + \frac{(m+1)\tau - T'}{\tau}\beta(x_\tau^m)\right) \\
&= j^*\left(\frac{\beta(x_\tau^{m+1}) - \beta(x_\tau^m)}{\tau}(T' - m\tau) + \beta(x_\tau^m)\right) = j^*\left(\Lambda_\tau \beta(x_\tau(T'))\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&j^*\left(\Lambda_\tau \beta(x_\tau(T'))\right) - j^*(\beta(x_0)) + \int_0^{T'} \Pi_\tau \eta_\tau(t) \cdot \Pi_\tau x_\tau(t)dt \\
&\leq \int_0^{T'} \Pi_\tau h_\tau(t-\tau) \cdot \Pi_\tau x_\tau(t)dt + \int_0^{T'} \Pi_\tau F^M(\beta(x_\tau(t-\tau))) \cdot \Pi_\tau x_\tau(t)dt,
\end{aligned}$$

which yields, as $\tau \to 0$,

$$\begin{aligned}
&\Theta - j^*(\beta(x_0)) + \limsup_{\tau \to 0} \int_0^{T'} \Pi_\tau \eta_\tau(t) \cdot \Pi_\tau x_\tau(t)dt \\
&\leq \int_0^{T'} h(t) \cdot x(t)dt + \int_0^{T'} F^M(\beta(x(t))) \cdot x(t)dt.
\end{aligned}$$

By (4.14) and Lemma 3,

$$\begin{aligned}
&\limsup_{\tau \to 0} \int_0^{T'} \Pi_\tau \eta_\tau(t) \cdot \Pi_\tau x_\tau(t)dt \\
&\leq \int_0^{T'} h(t) \cdot x(t)dt + \int_0^{T'} F^M(\beta(x(t))) \cdot x(t)dt - j^*(\zeta(T')) + j^*(\beta(x_0)) \\
&= \int_0^{T'} \left(F^M(\beta(x(t))) + h(t) - \frac{d}{dt}\beta(x(t))\right) \cdot x(t)dt = \int_0^{T'} \eta(t) \cdot x(t)dt,
\end{aligned}$$

i.e.,

$$\limsup_{\tau \to 0} \int_0^{T'} (\Pi_\tau \eta_\tau(t) - \eta(t)) \cdot (\Pi_\tau x_\tau(t) - x(t))dt \leq 0.$$

Hence, by Lemma 4, we obtain $\eta(t) \in \partial\varphi_{G,p}(x(t))$ almost everywhere in $(0, T')$. Therefore, x is a solution to $\frac{d}{dt}\beta(x) + \partial\varphi_{G,p}(x) \ni F^M(\beta(x)) + h$ between $[0, T']$.

Finally, we show that this solution is also a time-local solution to (3.1). By (4.13), the solution satisfies

$$\operatorname*{ess\,sup}_{0\le t\le T} \|\beta(x(t))\| \le \|\beta(x_0)\| + T(M + \|h\|_{L^\infty}).$$

Here we fix $M = 4\|\beta(x_0)\|$ and

$$T'' = \min\left\{T', \frac{\|\beta(x_0)\|}{4\|\beta(x_0)\| + \|h\|_{L^\infty}}\right\}.$$

Then,

$$\operatorname*{ess\,sup}_{0\le t\le T''} \|\beta(x(t))\| \le \|\beta(x_0)\| + \|\beta(x_0)\| \le M$$

and $F^M(\beta(x)) = F(\beta(x))$ holds within $[0, T'']$. Therefore, the solution x to $\frac{d}{dt}\beta(x) + \partial\varphi_{G,p}(x) \ni F^M(\beta(u)) + h$ satisfies (3.1) when $[0, T'']$.

4.2. Theorem 2 (with constraint term)

We begin with the following equation:

$$\beta(x) + \tau\partial\varphi_{G,p}(x) + \tau\partial I_{K_0}(x) \ni g, \tag{4.15}$$

where $\tau > 0$ and $g \in \mathbb{R}^N$ are given. This equation has a solution $x \in D(\beta) \cap D(\partial I_{K_0}) = D(\beta) \cap K_0$ when the functional defined by

$$\Phi(x) = j(x) + \tau\varphi_{G,p}(x) + \tau I_{K_0}(x) - g \cdot x$$

possesses a minimizer. Indeed, from (β.3) and Lemma 1, a minimizer x satisfies

$$\begin{aligned} 0 \in \partial\Phi(x) &= \partial(j + \tau\varphi_{G,p} + \tau I_{K_0})(x) - g = \partial j(x) + \tau\partial\varphi_{G,p}(x) + \tau\partial I_{K_0}(x) - g \\ &= \beta(x) + \tau\partial\varphi_{G,p}(x) + \tau\partial I_{K_0}(x) - g, \end{aligned}$$

which is equivalent to (4.15). If $p > 1$, then Φ possesses a minimizer without any assumptions on j and β by Lemma 6. When $p = 1$, we can show that there is a minimizer of Φ by assuming the condition (β.2) (its proof is exactly the same as that in Section 4.1).

Multiply (4.15) by $\beta(x)$. Since $\beta(x) = B(x)x$, we have

$$\beta(x) \cdot \eta = B(x)x \cdot \eta = pB(x)\varphi_{G,p}(x) \ge 0$$

for any $\eta \in \partial\varphi_{G,p}(x)$. Moreover, since the solution to (4.15) satisfies $x \in K_0$ and

$$\partial I_{K_0}(x) = \{\xi \in \mathbb{R}^N;\ \ \xi = (0, \dots, 0, \xi_{n+1}, \dots, \xi_{n+m})\ \ \xi_{n+j} \in \mathbb{R}\ \ j = 1, \dots, m\}$$

and $x = (x_1, \dots, x_n, 0 \dots, 0)$, we can see that

$$\beta(x) \cdot \xi = B(x)x \cdot \xi = 0$$

for any $\xi \in \partial I_{K_0}(x)$. Hence, we can show that the solution to (4.15) satisfies

$$\|\beta(x)\| \le \|g\|. \tag{4.16}$$

Next, we consider the following problem, which is the time discretization of (3.2):

$$\begin{cases} \dfrac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} + \partial\varphi_{G,p}(x_\tau^{i+1}) + \partial I_{K_0}(x_\tau^{i+1}) \ni F^M(\beta(x_\tau^i)) + h_\tau^i, \\ x_\tau^0 = x_0, \end{cases} \tag{4.17}$$

where $\tau = T/\mu$ ($\mu \in \mathbb{N}$), $i = 0, \ldots, \mu - 1$, $h_\tau^i = \frac{1}{\tau}\int_{i\tau}^{(i+1)\tau} h(t)dt$, and F^M is defined by (4.5). This equation is equivalent to

$$\beta(x_\tau^{i+1}) + \tau\partial\varphi_{G,p}(x_\tau^{i+1}) + \tau\partial I_{K_0}(x_\tau^{i+1}) \ni \beta(x_\tau^i) + \tau F^M(\beta(x_\tau^i)) + \tau h_\tau^i.$$

Hence, for any given $x_\tau^i \in D(\beta)$, there exists a solution $x_\tau^{i+1} \in D(\beta) \cap K_0$ to (4.17). Then, for any given $x_0 \in D(\beta)$, we can obtain $x_\tau^1, \ldots, x_\tau^\mu \in D(\beta) \cap K_0$ inductively. For later use, we assume $x_0 \in D(\beta) \cap K_0$. Henceforth, let $\eta_\tau = \{\eta_\tau^1, \ldots, \eta_\tau^\mu\}$ and $\xi_\tau = \{\xi_\tau^1, \ldots, \xi_\tau^\mu\}$ be the sections of $\partial\varphi_{G,p}$ and ∂I_{K_0} satisfying (4.17). Namely, it holds that $\eta_\tau^{i+1} \in \partial\varphi_{G,p}(x_\tau^{i+1})$, $\xi_\tau^{i+1} \in \partial I_{K_0}(x_\tau^{i+1})$, and

$$\frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} + \eta_\tau^{i+1} + \xi_\tau^{i+1} = F^M(\beta(x_\tau^i)) + h_\tau^i. \tag{4.18}$$

By using (4.16) with $g = \beta(x_\tau^i) + \tau F^M(\beta(x_\tau^i)) + \tau h_\tau^i$, we obtain

$$\|\beta(x_\tau^{i+1})\| \le \|\beta(x_\tau^i)\| + \tau\|F^M(\beta(x_\tau^i))\| + \tau\|h_\tau^i\|,$$

namely,

$$\max_{i=1,\ldots,\mu} \|\beta(x_\tau^i)\| \le \|\beta(x_0)\| + T(M + \|h\|_{L^\infty}). \tag{4.19}$$

Multiplying (4.17) by $x_\tau^{i+1} \in K_0$, we have

$$\begin{aligned} \frac{1}{\tau}(j^*(\beta(x_\tau^{i+1})) - j^*(\beta(x_\tau^i))) + p\varphi_{G,p}(x_\tau^{i+1}) + I_{K_0}(x_\tau^{i+1}) &\le (F^M(\beta(x_\tau^i)) + h_\tau^i) \cdot x_\tau^{i+1} \\ &\le C_2\|x_\tau^{i+1}\|. \end{aligned} \tag{4.20}$$

Here and henceforth, $c_2, C_2 > 0$ are general constants independent of μ and τ. We have by Lemma 6,

$$j^*(\beta(x_\tau^{i+1})) - j^*(\beta(x_\tau^i)) + c_2\tau\|x_\tau^{i+1}\|^p \le \tau C_2\|x_\tau^{i+1}\|.$$

First, let $p > 1$. Then, by Young's inequality we obtain

$$\tau C_2\|x_\tau^{i+1}\| \le \varepsilon\tau\|x_\tau^{i+1}\|^p + \tau C_2$$

with an arbitrary small $\varepsilon > 0$, which yields

$$j^*(\beta(x_\tau^\mu)) + c_2\tau\sum_{i=1}^{\mu}\|x_\tau^i\|^p \le TC_2 + j^*(\beta(x_0)).$$

Since $j^* \ge 0$ and $j^*(\beta(x_0)) \le j^*(\beta(0)) + (\beta(x_0) - \beta(0)) \cdot x_0$,

$$\tau\sum_{i=0}^{\mu-1}\|x_\tau^{i+1}\|^p \le C_2 \tag{4.21}$$

holds when $p > 1$. Next, we consider the case where $p = 1$. In this case, we need to assume (β.2) in order to derive an estimate of x_τ^i. Indeed, by the definition of the subdifferential, we have

$$j(x_\tau^{i+1}) \le j(0) + \beta(x_\tau^{i+1}) \cdot (x_\tau^{i+1} - 0).$$

By using (β.2) and (4.19), we get

$$c_0 \|x_\tau^{i+1}\|^{1+\alpha} \le C_2(\|x_\tau^{i+1}\| + 1),$$

which yields

$$\max_{i=1,\dots,\mu} \|x_\tau^i\| \le C_2, \tag{4.22}$$

when $p = 1$ (note that this estimate for $p = 1$ is slightly different from that for $p > 1$, recall (4.21)). From (4.20) and (4.21), we can derive

$$\max_{i=1,\dots,\mu} j^*(\beta(x_\tau^i)) \le C_2. \tag{4.23}$$

Moreover, by (2.15), we have

$$\tau \sum_{i=1}^{\mu} \|\eta_\tau^i\|^{p'} \le C_2 \tag{4.24}$$

if $p > 1$, and

$$\max_{i=1,\dots,\mu} \|\eta_\tau^i\| \le C_2 \tag{4.25}$$

if $p = 1$.

Next, we multiply (4.18) by

$$\frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} = \frac{B(x_\tau^{i+1})x_\tau^{i+1} - B(x_\tau^i)x_\tau^i}{\tau}.$$

Since $x_\tau^{i+1}, x_\tau^i \in K_0$ for every $i = 0, \dots, \mu - 1$ (note that we assume $x_0 \in D(\beta) \cap K_0$), we have

$$\xi_\tau^{i+1} \cdot \frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} = \frac{B(x_\tau^{i+1})}{\tau} \xi_\tau^{i+1} \cdot x_\tau^{i+1} - \frac{B(x_\tau^i)}{\tau} \xi_\tau^{i+1} \cdot x_\tau^i = 0$$

with $\xi_\tau^{i+1} \in \partial I_{K_0}(x_\tau^{i+1})$. Hence,

$$\begin{aligned}
&\left\| \frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} \right\|^2 \le \| - \eta_\tau^{i+1} + F^M(\beta(x_\tau^i)) + h_\tau^i \| \left\| \frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} \right\| \\
\Rightarrow \quad &\left\| \frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} \right\| \le \|\eta_\tau^{i+1}\| + M + \|h\|_{L^\infty},
\end{aligned}$$

which yields, from (4.24) and (4.25),

$$\tau \sum_{i=0}^{\mu-1} \left\| \frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} \right\|^{p'} \le C_2 \tag{4.26}$$

if $p > 1$, and

$$\max_{i=0,\ldots,\mu-1} \left\| \frac{\beta(x_\tau^{i+1}) - \beta(x_\tau^i)}{\tau} \right\| \le C_2 \tag{4.27}$$

if $p = 1$. By (4.18),

$$\tau \sum_{i=1}^{\mu} \|\xi_\tau^i\|^{p'} \le C_2 \tag{4.28}$$

holds if $p > 1$, and

$$\max_{i=1,\ldots,\mu} \|\xi_\tau^i\| \le C_2 \tag{4.29}$$

holds if $p = 1$.

According to (4.21)–(4.29) and $\tau \sum_{i=1}^{\mu} = \int_0^T dt$, we obtain

$$\int_0^T \|\Pi_\tau x_\tau(t)\|^p dt + \int_0^T \|\Pi_\tau \eta_\tau(t)\|^{p'} dt + \int_0^T \|\Pi_\tau \xi_\tau(t)\|^{p'} dt + \int_0^T \left\| \frac{d}{dt} \Lambda_\tau \beta(x_\tau(t)) \right\|^{p'} dt \le C_2$$

if $p > 1$, and

$$\max_{0\le t\le T} \|\Pi_\tau x_\tau(t)\| + \max_{0\le t\le T} \|\Pi_\tau \eta_\tau(t)\| + \max_{0\le t\le T} \|\Pi_\tau \xi_\tau(t)\| + \max_{0\le t\le T} \left\| \frac{d}{dt} \Lambda_\tau \beta(x_\tau(t)) \right\| \le C_2$$

if $p = 1$, where Λ_τ and Π_τ are defined as (4.12). Moreover, from (4.19), we have

$$\sup_{0\le t\le T} \|\Pi_\tau \beta(x_\tau(t))\| \le \|\beta(x_0)\| + T(M + \|h\|_{L^\infty}) \tag{4.30}$$

and

$$\sup_{0\le t\le T} \|\Lambda_\tau \beta(x_\tau(t))\| \le C_2.$$

By the Ascoli–Arzela theorem, there exist some subsequence of $\{\Lambda_\tau \beta(x_\tau)\}_\tau$ (we omit relabeling) and some $\zeta : [0, T] \to \mathbb{R}^N$ such that

$$\Lambda_\tau \beta(x_\tau) \to \zeta \quad \text{strongly in } C([0, T]; \mathbb{R}^N).$$

By the same procedure as that in Section 4.1,

$$\Pi_\tau \beta(x_\tau) \to \zeta \qquad \text{strongly in } L^q(0, T; \mathbb{R}^N) \quad \forall q > 1$$

if $p = 1$. When $p > 1$,

$$\begin{aligned} \int_0^T \|\Pi_\tau \beta(x_\tau)(t) - \Lambda_\tau \beta(x_\tau)(t)\|^{p'} dt &= \sum_{i=0}^{\mu-1} \int_{i\tau}^{(i+1)\tau} \|\Pi_\tau \beta(x_\tau(t)) - \Lambda_\tau \beta(x_\tau(t))\|^{p'} dt \\ &\le \sum_{i=0}^{\mu-1} \int_{i\tau}^{(i+1)\tau} ((i+1)\tau - t)^{p'} \left\| \frac{\beta(x^{i+1}) - \beta(x^i)}{\tau} \right\|^{p'} dt \end{aligned}$$

$$\leq C_2 T \tau^{p'-1}.$$

Hence,

$$\Pi_\tau \beta(x_\tau) \to \zeta \qquad \text{strongly in } L^{p'}(0,T;\mathbb{R}^N).$$

In both cases, Lemma 2 implies that $\zeta = \beta(x)$ a.e $t \in (0,T)$. By the dominated convergence theorem, we have, as $\tau \to 0$,

$$\begin{aligned} &\Pi_\tau F^M(\beta(x_\tau)) \to F^M(\beta(x)) && \text{strongly in } L^q(0,T;\mathbb{R}^N) \quad \forall q > 1, \\ &\Pi_\tau h_\tau \to h && \text{strongly in } L^q(0,T;\mathbb{R}^N) \quad \forall q > 1. \end{aligned}$$

Moreover, from the uniform boundedness, we can derive with some $x, \eta, \xi : [0,T] \to \mathbb{R}^N$,

$$\begin{aligned} &\Pi_\tau x_\tau \rightharpoonup x && \text{weakly in } L^p(0,T;\mathbb{R}^N), \\ &\Pi_\tau \eta_\tau \rightharpoonup \eta && \text{weakly in } L^{p'}(0,T;\mathbb{R}^N), \\ &\Pi_\tau \xi_\tau \rightharpoonup \xi && \text{weakly in } L^{p'}(0,T;\mathbb{R}^N), \\ &\frac{d}{dt}\Lambda_\tau \beta(x_\tau) \rightharpoonup \frac{d}{dt}\beta(x) && \text{weakly in } L^{p'}(0,T;\mathbb{R}^N), \end{aligned}$$

if $p > 1$, and

$$\begin{aligned} &\Pi_\tau x_\tau \rightharpoonup x && * \text{-weakly in } L^\infty(0,T;\mathbb{R}^N), \\ &\Pi_\tau \eta_\tau \rightharpoonup \eta && * \text{-weakly in } L^\infty(0,T;\mathbb{R}^N), \\ &\Pi_\tau \xi_\tau \rightharpoonup \xi && * \text{-weakly in } L^\infty(0,T;\mathbb{R}^N), \\ &\frac{d}{dt}\Lambda_\tau \beta(x_\tau) \rightharpoonup \frac{d}{dt}\beta(x) && * \text{-weakly in } L^\infty(0,T;\mathbb{R}^N), \end{aligned}$$

if $p = 1$. Therefore, x, η, and ξ satisfy the following equation in $L^{p'}(0,T;\mathbb{R}^N)$:

$$\frac{d}{dt}\beta(x) + \eta + \xi = F^M(\beta(x)) + h.$$

Finally, we have to show that $\eta(t) + \xi(t) \in \partial\varphi_{G,p}(x(t)) + \partial I_{K_0}(x(t))$ for a.e. $t \in (0,T)$. Let $Y_\tau^i := \eta_\tau^i + \xi_\tau^i$. Then, $Y_\tau^i \in \partial(\varphi_{G,p} + I_{K_0})(x_\tau^i) = \partial\varphi_{G,p}(x_\tau^i) + \partial I_{K_0}(x_\tau^i)$ (recall Lemma 1 and $D(\varphi_{G,p}) = \mathbb{R}^N$). Furthermore,

$$\Pi_\tau Y_\tau \rightharpoonup Y := \eta + \xi \quad \text{weakly in } L^{p'}(0,T;\mathbb{R}^N).$$

By exactly the same argument as that in Section 4.1, we obtain

$$\begin{aligned} &j^*\left(\Lambda_\tau \beta(x_\tau(T'))\right) - j^*(\beta(x_0)) + \int_0^{T'} \Pi_\tau Y_\tau(t) \cdot \Pi_\tau x_\tau(t)dt \\ &\quad \leq \int_0^{T'} \Pi_\tau h_\tau(t-\tau) \cdot \Pi_\tau x_\tau(t)dt + \int_0^{T'} \Pi_\tau F^M(\beta(x_\tau(t-\tau))) \cdot \Pi_\tau x_\tau(t)dt, \end{aligned}$$

where $t = T' \leq T$ is a time at which ξ satisfies (2.4). Then,

$$\limsup_{\tau \to 0} \int_0^{T'} \Pi_\tau Y_\tau(t) \cdot \Pi_\tau x_\tau(t)dt$$

$$\le \int_0^{T'} h(t) \cdot x(t)dt + \int_0^{T'} F^M(\beta(x(t))) \cdot x(t)dt - j^*(\zeta(T')) + j^*(\beta(x_0))$$
$$= \int_0^{T'} \left(F^M(\beta(x(t))) + h(t) - \frac{d}{dt}\beta(x(t)) \right) \cdot x(t)dt = \int_0^{T'} Y(t) \cdot x(t)dt,$$

i.e.,

$$\limsup_{\tau \to 0} \int_0^{T'} (\Pi_\tau Y_\tau(t) - Y(t)) \cdot (\Pi_\tau x_\tau(t) - x(t))dt \le 0.$$

Hence, by Lemma 5, we obtain $Y(t) \in \partial\varphi_{G,p}(x(t)) + \partial I_{K_0}(x(t))$ for a.e. $t \in (0, T')$. Therefore, x is a solution to $\frac{d}{dt}\beta(x) + \partial\varphi_{G,p}(x) + \partial I_{K_0}(x) \ni F^M(\beta(x)) + h$ between $[0, T']$. By the same reason as that for Section 4.1, we can select some suitable M and $T'' \le T'$ so that $F^M(\beta(x(t))) = F(\beta(x(t)))$ holds between $[0, T'']$. Thus, x is a time-local solution to (3.2).

Author contributions

Masahiro Ikeda: Writing–review & editing, supervision, formal analysis, methodology; Shun Uchida: Writing–original draft, writing–review & editing, supervision, formal analysis, methodology.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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