Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 18(3): 610–625.
DOI: 10.3934/cam.2026025
Received: 07 September 2025
Revised: 24 March 2026
Accepted: 06 May 2026
Published: 03 August 2026

Research article

The boundedness of pullback attractors for delayed non-Newtonian fluids

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Abstract: We investigated the boundedness of pullback attractors for non-Newtonian fluids with delay in two-dimensional bounded domains. Under suitable assumptions on the delay term, we established the boundedness of the pullback attractor, defined in the phase space $\mathbb{L}^2$, which was bounded in $\mathbb{H}^4$ by means of refined energy estimates.

Keywords: boundedness; non-Newtonian fluid; pullback attractor
Mathematics Subject Classification: 35B41, 35B65, 35Q30

1. Introduction

Non-Newtonian fluids with delay arise naturally in many engineering applications, such as polymer processing and molten metal flows. Pullback attractors provide a fundamental framework for describing the long-term dynamics of such systems, and their properties play a crucial role in the design of practical control strategies. In this paper, we investigate the boundedness of pullback attractors for a class of delayed incompressible non-Newtonian fluid equations in a smoother space over two-dimensional bounded domains.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with smooth boundary $\partial\Omega$. We consider the following initial-boundary value problem

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot \mathrm{S}(e(u)) + f(t) + g(t, u_t), \text{ in } (\tau, +\infty) \times \Omega, \tag{1.1}$$

$$\nabla \cdot u = 0, \quad \text{in } (\tau, +\infty) \times \Omega, \tag{1.2}$$

$$u = 0, \quad \mathrm{S}_{ijk}\nu_j\nu_k = 0, \quad \text{on } (\tau, +\infty) \times \partial\Omega, \tag{1.3}$$

$$u(t, x)|_{t=\tau} = u^{in}, \quad x \in \Omega, \quad \tau \in \mathbb{R}, \tag{1.4}$$

$$u(t, x) = \phi^{in}(t - \tau, x), \quad (t, x) \in (\tau - h, \tau) \times \Omega, \tag{1.5}$$

where Eqs (1.1)-(1.2) describe the motion of delayed incompressible non-Newtonian fluids. Here, $u = u(t, x)$ denotes the velocity field, $p = p(t, x)$ the pressure, $f(t, x)$ the external force without delay, and $g(t, u_t)$ the delayed external force, where $u_t(s, x) = u(t + s, x)$ for $s \in (-h, 0)$, $t > \tau$, and $x \in \Omega$. The stress tensor $\mathrm{S}(e(u)) = (\mathrm{S}_{ij}(e(u)))_{2\times 2}$ is defined by

$$\mathrm{S}_{ij}(e(u)) = 2\mu_0(\varepsilon + |e|^2)^{-\alpha/2} e_{ij} - 2\mu_1 \Delta e_{ij}, \quad i, j = 1, 2,$$

where $e_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the strain-rate tensor, $|e|^2 = \sum_{i,j=1,2} |e_{ij}|^2$, and $\mu_0, \mu_1, \alpha \in (0, 1)$, $\varepsilon > 0$ are constants. The boundary condition (1.3) consists of the no-slip condition $u = 0$ and the traction moment condition $\mathrm{S}_{ijk}\nu_j\nu_k = 0$ where $\mathrm{S}_{ijk} = \frac{\partial e_{ij}}{\partial x_k}$ and $\nu = (\nu_1, \nu_2)$ denote the outward unit normal vector on $\partial\Omega$. Conditions (1.4)-(1.5) specify the initial data.

When the delay term is absent ($g(t, u_t) = 0$), Eqs (1.1)-(1.2) reduce to the well-known bipolar non-Newtonian fluid model [1], which describes the motion of molten plastics, polymer solutions, and other complex fluids. Extensive studies have been devoted to the well-posedness, regularity, and long-term behavior (via attractors) of this delay-free model [1–3].

For fluid models with delay ($g(t, u_t) \neq 0$), which are essential for modeling controlled fluid systems (e.g., wind tunnel experiments), Caraballo and Real [4, 5] pioneered the mathematical analysis of delayed Navier-Stokes equations. Their ideas have since been extended to other delayed fluid systems [6–10]. For the delayed non-Newtonian fluid model (1.1)-(1.2), Zhao et al. [10] first proved the unique existence of solutions and the existence of pullback attractors, followed by studies on stationary solutions and pullback attractors under different frameworks [11–13]. However, all these works focus only on the existence of pullback attractors in low-regularity spaces (e.g., $\mathbb{L}^2$ or $\mathbb{H}^2$), and the higher regularity (e.g., $\mathbb{H}^4$) which is crucial for understanding solution smoothness in practical applications remains unaddressed.

In infinite-dimensional dynamical systems, attractors (if they exist) typically exhibit higher regularity than the underlying phase space [14]. For delay-free fluid models, significant progress has been made in the boundedness of attractor in a smoother space. For example, the $\mathbb{H}^2$ and $\mathbb{H}^4$-boundedness of pullback attractors have been established for Navier-Stokes equations [15], micropolar fluid equations [16], and non-Newtonian fluid equations [17]. To the best of our knowledge, few results exist for the boundedness of pullback attractors in a smoother space for delayed fluid models, which motivates our current study.

Based on the existing existence results of pullback attractors [13, 18], the $\mathbb{H}^2$-boundedness of the attractor can be established via the improved energy estimates. To further enhance the regularity to $\mathbb{H}^4$ (as in the delay-free case [17]), the key point is to derive the boundedness of $\frac{du}{dt}$ in $L^\infty(\tau, T; L^2)$. However, the presence of the delay term $g(t, u_t)$ introduces technical difficulties in the energy estimates, as it obstructs the direct transfer of regularity. To overcome this difficulty, we impose on the delay term $g(t, u_t)$ some additional but natural hypotheses and develop refined energy estimates for $\frac{du}{dt}$.

The remainder of the paper is organized as follows. Section 2 introduces preliminary notations and recalls the existence of the pullback attractor for Eqs (1.1)-(1.5). Section 3 establishes the main result on the $\mathbb{H}^4$-boundedness of the pullback attractor.

Throughout this paper, $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and nonnegative integers, respectively. c or C denotes a generic constant that may vary across contexts. When dependence needs emphasis, we use notations like c_1 or $c(\cdot)$.

2. Preliminaries

In this section, we introduce necessary notations, transform Eqs (1.1)-(1.5) into a weak formation, and recall results on the global well-posedness and existence of pullback attractors associated with the system.

Function Spaces and Notations

Let $\mathbb{L}^p(\Omega)$ and $\mathbb{H}^m(\Omega)$ denote the 2D vector-valued Lebesgue spaces and Sobolev spaces, respectively, with norms $\|\cdot\|_{\mathbb{L}^p}$ and $\|\cdot\|_{\mathbb{H}^m}$. For the problem (1.1)-(1.5), we focus on the following subspaces of divergence-free vector fields

$$\begin{aligned}
\mathcal{V} &= \{\phi \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) : \phi = (\phi_1, \phi_2), \nabla \cdot \phi = 0\}, \\
H &= \overline{\mathcal{V}}^{\mathbb{L}^2(\Omega)} \quad \text{(endowed with norm } \|\cdot\|_H, \text{ dual space } H' = H), \\
V &= \overline{\mathcal{V}}^{\mathbb{H}^2(\Omega)} \quad \text{(endowed with norm } \|\cdot\|_V, \text{ dual space } V').
\end{aligned}$$

Here, $\overline{X}^Y$ denotes the closure of space X under the norm of Y. It follows that $V \hookrightarrow H = H' \hookrightarrow V'$, where the injections are dense and continuous. We use $(\cdot,\cdot)$ for the inner product in H and $\langle\cdot,\cdot\rangle$ for the dual product between V and V'. For simplicity, $\|\cdot\|_{\mathbb{L}^2}$ and $\|\cdot\|_H$ are abbreviated to $\|\cdot\|$ when no confusion arises. Additionally, we denote

$$\begin{aligned}
L^p(I;\mathbb{X}) &= \text{the space of strongly measurable functions on interval } I, \\
&\quad\ \text{with values in the Banach space } \mathbb{X}, \text{ endowed with the standard norm;} \\
C(I;\mathbb{X}) &= \text{the space of continuous functions on the interval } I, \\
&\quad\ \text{with values in the Banach space } \mathbb{X}, \text{ endowed with the uniform norm.}
\end{aligned}$$

Key Operators and Their Properties

To reformulate the system, we introduce three core operators capturing viscous, convective, and non-Newtonian effects.

The viscous operator A: Defined for $u, v \in V$ by

$$\langle Au, v\rangle = \sum_{i,j,k=1}^{2} \int_\Omega \frac{\partial e_{ij}(u)}{\partial x_k} \frac{\partial e_{ij}(v)}{\partial x_k} \mathrm{d}x,$$

where $e_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the strain tensor. A is a linear continuous operator both from V to V' and from $D(A)$ to H, with domain $D(A) = \mathbb{H}^4 \cap V$.

The convective operator B: First define the trilinear form for $u, v, w \in V$ by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \,\mathrm{d}x,$$

then set $\langle B(u, v), w\rangle = b(u, v, w)$, $\forall w \in V$. For brevity, $B(u) = B(u, u)$. $B(\cdot,\cdot)$ is continuous from $V \times V$ to V'.

Non-Newtonian operator N: Let $\mu(u) = 2\mu_0(\varepsilon + |e(u)|^2)^{-\alpha/2}$. For $u, v \in V$,

$$\langle N(u), v\rangle = \sum_{i,j=1}^{2} \int_\Omega \mu(u)e_{ij}(u)e_{ij}(v)\,\mathrm{d}x.$$

Then, $N(\cdot)$ is continuous from V to V'. For $u \in D(A)$, it extends to H via

$$\langle N(u), v\rangle = -\int_\Omega \{\nabla \cdot (\mu(u)e(u))\}v dx, \quad \forall v \in H.$$

The following lemma summarizes their fundamental estimates.

Lemma 2.1. *([1,10,19]) There exist constants $c_i > 0$ (depending only on Ω) such that for all $u, v, \omega \in V$,*

$$\begin{aligned}
&c_1\|u\|_V^2 \leqslant \langle Au, u\rangle \leqslant \|u\|_V^2, \quad |\langle Au, v\rangle| \leqslant c_2\|u\|_V\|v\|_V.\\
&|b(u, v, w)| \leqslant c_3\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2}.\\
&b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0.\\
&\langle N(u), u\rangle \geqslant 0, \quad |\langle N(u), v\rangle| \leqslant c_4\|u\|_V\|v\|_V.
\end{aligned}$$

Weak Formation of the System

Using the above operators, Eqs (1.1)-(1.5) are reformulated in the problem: for each $\tau \in \mathbb{R}$, to find $u \in L^\infty(\tau, T; H) \cap L^2(\tau - h, T; H) \cap L^2(\tau, T; V)$ for all $T > \tau$,

$$\frac{\partial u}{\partial t} + 2\mu_1 Au + B(u) + N(u) = f(t) + g(t, u_t), \tag{2.1}$$

in the sense of distributions $\mathcal{D}'(\tau, T; V')$ and

$$u(\tau, x) = u^{in}(x) = u^{in}, \tag{2.2}$$

$$u(t, x) = \phi^{in}(t - \tau, x) = \phi^{in}(s, x) = \phi^{in}, \quad t \in (\tau - h, \tau), s \in (-h, 0). \tag{2.3}$$

Assumptions and Existence Results

To analyze well-posedness and attractors, we introduce phase spaces for history data

$$L_H^2 = L^2(-h, 0; H), \quad L_V^2 = L^2(-h, 0; V), \quad L_{D(A)}^2 = L^2(-h, 0; D(A)),$$

and impose the following assumptions on external forces

Delay force $g : \mathbb{R} \times L_H^2 \mapsto \mathbb{L}^2$ is supposed to satisfy

(H1) *for any $\xi \in L_H^2$, the mapping $t \in \mathbb{R} \mapsto g(t, \xi) \in H$ is measurable and $g(t, 0) = 0$ for all $t \in \mathbb{R}$;*

(H2) *there is a constant $L_g > 0$ such that for any $t \in \mathbb{R}$, $\xi, \eta \in L_H^2$,*

$$\|g(t, \xi) - g(t, \eta)\| \leqslant L_g\|\xi - \eta\|_{L_H^2};$$

(H3) *there exists a constant $C_g \in (0, 2c_1\mu_1)$ such that*

$$\int_\tau^t \|g(s, u_s) - g(s, v_s)\|^2\mathrm{d}s \leqslant C_g^2 \int_{\tau-h}^t \|u(s) - v(s)\|^2\mathrm{d}s,$$

for all $\tau \leqslant t$, $u, v \in L^2(\tau - h, t; H)$. *Moreover, there exists a value* $\gamma \in (0, \min\{4c_1\mu_1 - 2C_g, 4c_1^2\mu_1\})$ *such that*

$$\int_\tau^t \mathrm{e}^{\gamma s}\|g(s, u_s)\|^2 \mathrm{d}s \leqslant C_g^2 \int_{\tau-h}^t \mathrm{e}^{\gamma s}\|u(s)\|^2 \mathrm{d}s,$$

for any $u \in L^2(\tau - h, t; H)$, $t \geqslant \tau$.

Non-delay force f satisfies

(H4) $f \in L_b^2(\mathbb{R}; H)$, *meaning* $f \in L_{loc}^2(\mathbb{R}; H)$ *such that*

$$\|f\|_{L_b^2}^2 = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s)\|^2 \mathrm{d}s < +\infty.$$

Under these assumptions, the following results hold.

Lemma 2.2. *([13, 18]) Suppose (H1)-(H4) hold and* $(u^{in}, \phi^{in}) \in H \times L_H^2$, *then there is a unique solution* u *to Eqs* (2.1)-(2.3) *satisfying for all* $T > \tau$,

$$u \in C([\tau, T]; H) \cap L^2(\tau - h, T; H) \cap L^2(\tau, T; V) \text{ and } u' \in L^2(\tau, T; V').$$

Moreover, the solution operators $\{U(t, \tau)\}_{t\geqslant\tau}$ *defined by*

$$U(t, \tau) : (u^{\mathrm{in}}, \phi^{\mathrm{in}}) \mapsto \big(u(t; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}}), u_t(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}})\big), \quad \forall\, t \geqslant \tau,$$

generate a continuous process in $H \times L_H^2$ *and possess a pullback absorbing set*

$$\mathcal{B}^H = \{(u, \phi) \in H \times L_V^2 \big| \, \|(u, \phi)\|_{H\times L_V^2}^2 \leqslant \mathcal{R}_0\},$$
$$R_0 = 1 + c(1 + e^{\gamma h})(1 + \frac{1}{\gamma})\|f\|_{L_b^2}^2.$$

Furthermore, the process $\{U(t, \tau)\}_{t\geqslant\tau}$ *possesses a unique pullback attractor in* $H \times L_H^2$, *i.e. a family* $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ *of compact subsets of* $H \times L_H^2$ *satisfies*

$$(i)\ U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t), \quad \forall\tau \leqslant t,$$

and for all $t \in \mathbb{R}$ *and any bounded subset* $\mathcal{B}$ *in* $H \times L_H^2$,

$$(ii)\ \lim_{\tau\to-\infty} \sup_{u_\tau\in\mathcal{B}} \inf_{v\in\mathcal{A}(t)} \|U(t, \tau)u_\tau - v\| = 0.$$

3. $\mathbb{H}^4$-boundedness of the pullback attractor

In this section, we aim to verify the $\mathbb{H}^4$-boundedness of the pullback attractor obtained in Theorem 2.2 by estimating the Galerkin approximation of solutions to Eqs (2.1)-(2.3). For this purpose, we need the following lemma.

Lemma 3.1. *([14]) Let* X, Y *be Banach spaces such that* X *is reflexive, and the inclusion* $X \subset Y$ *is continuous. Assume that* $\{u_n\}$ *is a bounded sequence in* $L^\infty(t_0, T; X)$ *such that* $u_n \rightharpoonup u$ *weakly in* $L^q(t_0, T; X)$ *for some* $q \in [1, +\infty)$ *and* $u \in C([t_0, T]; Y)$. *Then,* $u(t) \in X$ *and* $\|u(t)\|_X \leq \liminf_{n\to+\infty} \|u_n\|_{L^\infty(t_0,T;X)}$, *for all* $t \in [t_0, T]$.

It is well known that for $u \in D(A)$, $Au = P\Delta^2 u$ where P is the orthogonal projector in $\mathbb{L}^2(\Omega)$ onto H, the classical spectral theory of elliptic operators (see [19]) implies that A is an isomorphism from $D(A)$ onto H and its inverse A^{-1} is self-adjoint positive and compact in H, with a complete orthonormal basis in H of eigenvectors $\{w_n\}_{n=1}^{\infty} \subset D(A)$ whose span $\{w_1, w_2, \cdots, w_n, \cdots\}$ is dense in V, such that

$$Aw_n = \lambda_n w_n, \quad \forall\, n \in \mathbb{N},$$

where

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots, \quad \lambda_n \to +\infty \ \text{ as } \ n \to \infty.$$

Then, for each integer $n \geqslant 1$, we denote by

$$u_n(t) = u_n(t; \tau, u^{in}, \phi^{in}) = \sum_{j=1}^{n} \xi_{nj}(t) w_j$$

and

$$u_{nt}(\cdot) = u_n(t + \cdot; \tau, u^{in}, \phi^{in}),$$

the Galerkin approximation of the solution $u(t; \tau, u_\tau)$ to Eqs (2.1)-(2.3), which satisfies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(u_n(t), w_j) + 2\mu_1\langle Au_n(t), w_j\rangle + \langle B(u_n(t)), w_j\rangle \\
&\qquad\qquad + \langle N(u_n(t)), w_j\rangle = (f(t), w_j) + (g(t, u_{nt}), w_j), && (3.1)\\
&(u_n(\tau), w_j) = (u_\tau, w_j), \ j = 1, 2, \cdots, n. && (3.2)
\end{aligned}$$

3.1. $\mathbb{H}^2$-boundedness

According to Theorem 2.2, we just know that for any $T_1 < T_2$, $\cup_{t\in[T_1,T_2]}\mathcal{A}(t)$ is bounded in $H \times L_H^2$ with $\mathbb{L}^2$ regularity. In the following we first improve the $\mathbb{L}^2$-boundedness to the $\mathbb{H}^2$-boundedness based on an improved energy estimate.

Regarding the Galerkin approximation (as defined by Eqs (3.1)-(3.2)), we establish the following lemma.

Lemma 3.2. *Assume* (H1)-(H4) *hold, then for any bounded set $\mathcal{B} \subset H \times L_H^2$, $\tau \in \mathbb{R}$, $\epsilon > 0$, and $t > \tau + h + \epsilon$,*

$$\{u_n(r; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}})| \ \ r \in [\tau + \epsilon, t], (u^{\mathrm{in}}, \phi^{\mathrm{in}}) \in \mathcal{B}, n \geqslant 1\}, \tag{3.3}$$

$$\{u_{nr}(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}})| \ \ r \in [\tau + h + \epsilon, t], (u^{\mathrm{in}}, \phi^{\mathrm{in}}) \in \mathcal{B}, n \geqslant 1\}, \tag{3.4}$$

$$\{u_n(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}})| \ \ (u^{\mathrm{in}}, \phi^{\mathrm{in}}) \in \mathcal{B}, n \geqslant 1\}, \tag{3.5}$$

are bounded in V, L_V^2 and $L^2(\tau + \epsilon, t; D(A))$, respectively.

Proof. (A fundamental energy estimate) By multiplying Eq (3.1) by $\xi_{nj}(\theta)$ and summing them from $j = 1$ to n, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\|u_n(\theta)\|^2 + 4c_1\mu_1\|u_n(\theta)\|_V^2 \leqslant 2(f(\theta), u_n(\theta)) + 2(g(\theta, u_{n\theta}), u_n(\theta)), \tag{3.6}$$

where we have used Lemma 2.1. Integrating (3.6) for θ over $[\tau, r]$ gives

$$\|u_n(r)\|^2 + 4c_1\mu_1 \int_\tau^t \|u_n(\theta)\|_V^2 \mathrm{d}\theta \leqslant \|u^{\mathrm{in}}\|^2 + 2\int_\tau^t (f(\theta), u_n(\theta))\mathrm{d}\theta$$

$$+ 2\int_\tau^t (g(\theta, u_{n\theta}), u_n(\theta))\mathrm{d}\theta. \tag{3.7}$$

By the Young inequality, it holds

$$2\int_\tau^t (f(\theta), u_n(\theta))\mathrm{d}\theta \leqslant \gamma^{-1}\int_\tau^t \|f(\theta)\|^2\mathrm{d}\theta + \gamma\int_\tau^t \|u_n(\theta)\|_V^2\mathrm{d}\theta. \tag{3.8}$$

The Young inequality, (H3), and the fact $\|\cdot\| \leqslant \|\cdot\|_V$ imply

$$\begin{aligned} 2\int_\tau^t (g(\theta, u_{n\theta}), u_n(\theta))\mathrm{d}\theta &\leqslant 2\int_\tau^t \|g(\theta, u_{n\theta})\|\|u_n(\theta)\|\mathrm{d}\theta \\ &\leqslant \frac{1}{C_g}\int_\tau^t \|g(\theta, u_{n\theta})\|^2\mathrm{d}\theta + C_g\frac{1}{C_g}\int_\tau^t \|u_n(\theta)\|^2\mathrm{d}\theta \\ &\leqslant C_g\|\phi^{\mathrm{in}}\|_{L_H^2}^2 + 2C_g\int_\tau^t \|u_n(\theta)\|_V^2\mathrm{d}\theta. \end{aligned} \tag{3.9}$$

Combining Eqs (3.8) and (3.9) with Eq (3.7) deduces for all $\tau \leqslant r \leqslant t$,

$$\begin{aligned} &\|u_n(r)\|^2 + (4c_1\mu_1 - \gamma - 2C_g)\int_\tau^t \|u_n(\theta)\|_V^2\mathrm{d}\theta \\ &\qquad \leqslant C\left(\|(u^{\mathrm{in}}, \phi^{\mathrm{in}})\|_{H\times L_H^2} + \int_\tau^t \|f(\theta)\|^2\mathrm{d}\theta\right). \end{aligned} \tag{3.10}$$

(An improved energy estimate) By multiplying (3.1) by $A\xi_{nj}(\theta)$ and summing them from $j = 1$ to n, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\theta}\langle u_n(\theta), Au_n(\theta)\rangle + 4\mu_1\|Au_n(\theta)\| + 2\langle B(u_n(\theta)), Au_n(\theta)\rangle + 2\langle N(u_n(\theta)), Au_n(\theta)\rangle \\ &\qquad = 2(f(\theta), Au_n(\theta)) + 2(g(\theta, u_{n\theta}), Au_n(\theta)). \end{aligned} \tag{3.11}$$

By Lemma 2.1, the Young inequality, and the embedding $V \hookrightarrow \mathbb{H}^1 \hookrightarrow \mathbb{L}^2$, we have

$$\begin{aligned} &|\langle B(u_n(\theta)), Au_n(\theta)\rangle| \leqslant \|u_n(\theta)\|_{\mathbb{L}^4}\|\nabla u_n(\theta)\|_{\mathbb{L}^4}\|Au_n(\theta)\| \\ &\leqslant C\|u_n(\theta)\|^{1/2}\|\nabla u_n(\theta)\|^{1/2}\|\nabla u_n(\theta)\|^{1/2}\|\nabla^2 u_n(\theta)\|^{1/2}\|Au_n(\theta)\| \\ &\leqslant C\|u_n(\theta)\|_V^2\|Au_n(\theta)\| \leqslant \frac{1}{4}\mu_1\|Au_n(\theta)\|^2 + C\|u_n(\theta)\|_V^4. \end{aligned} \tag{3.12}$$

By Lemma 2.1 and the Young inequality, one has

$$\begin{aligned} &\left|\langle N(u_n(\theta)), Au_n(\theta)\rangle\right| \\ &\qquad \leqslant C\|u_n(\theta)\|_V\|Au_n(\theta)\| \leqslant \frac{\mu_1}{4}\|Au_n(\theta)\|^2 + C\|u_n(\theta)\|_V^2. \end{aligned} \tag{3.13}$$

By the Hölder inequality, it derives

$$|(f(\theta), Au_n(\theta))| \leqslant \frac{1}{4}\mu_1\|Au_n(\theta)\|^2 + C\|f(\theta)\|^2, \tag{3.14}$$

$$|g(\theta, u_{n\theta}), Au_n(\theta)| \leqslant \frac{1}{4}\mu_1 \|Au_n(\theta)\|^2 + C\|g(\theta, u_{n\theta})\|^2. \tag{3.15}$$

It follows from Eqs (3.11)-(3.15) and Lemma 2.1 that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Au_n(\theta), u_n(\theta)\rangle + 2\mu_1\|Au_n(\theta)\|^2 \\
&\qquad \leqslant C\|u_n(\theta)\|_V^4 + C\left(\|f(\theta)\|^2 + \|g(\theta, u_{n\theta})\|^2 + \|u_n(\theta)\|_V^2\right) \\
&\qquad \leqslant C\|u_n(\theta)\|_V^2\langle Au_n(\theta), u_n(\theta)\rangle + C\left(\|f(\theta)\|^2 + \|g(\theta, u_{n\theta})\|^2 + \|u_n(\theta)\|_V^2\right).
\end{aligned} \tag{3.16}$$

Set

$$\begin{aligned}
H_n(\theta) &= \langle Au_n(\theta), u_n(\theta)\rangle, \quad K_n(\theta) = C\|u_n(\theta)\|_V^2, \\
I_n(\theta) &= C\left(\|f(\theta)\|^2 + \|g(\theta, u_{n\theta})\|^2 + \|u_n(\theta)\|_V^2\right).
\end{aligned}$$

Then, Eq (3.16) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\theta}H_n(\theta) \leqslant K_n(\theta)H_n(\theta) + I_n(\theta). \tag{3.17}$$

Applying the Gronwall lemma to Eq (3.17) over $[s, r] \subset [\tau, t]$ yields

$$\begin{aligned}
H_n(r) &\leqslant \left(H_n(s) + \int_\tau^t I_n(\theta)\mathrm{d}\theta\right)\exp\left\{\int_\tau^t K_n(\theta)\mathrm{d}\theta\right\} \\
&= \left(\langle Au_n(s), u_n(s)\rangle + \int_\tau^t I_n(\theta)\mathrm{d}\theta\right)\exp\left\{\int_\tau^t K_n(\theta)\mathrm{d}\theta\right\}.
\end{aligned} \tag{3.18}$$

Integrating Eq (3.18) for s over $[\tau, r]$, one can see that for all $\tau + \epsilon \leqslant r \leqslant t$,

$$(r - \tau)H_n(r) \leqslant \left(\int_\tau^t \langle Au_n, u_n\rangle \mathrm{d}s + (t - \tau)\int_\tau^t I_n \mathrm{d}\theta\right)\exp\left\{\int_\tau^t K_n \mathrm{d}\theta\right\}. \tag{3.19}$$

To further estimate (3.19), by Lemma 2.1, (H3), and (3.10), we have

$$\begin{aligned}
&\int_\tau^t \langle Au_n(s), u_n(s)\rangle \mathrm{d}s + (t - \tau)\int_\tau^t I_n(\theta)\mathrm{d}\theta \\
&\leqslant \int_\tau^t \|u_n(\theta)\|_V^2 \mathrm{d}\theta + C(t - \tau)\int_\tau^t \left(\|u_n(\theta)\|_V^2 + \|f(\theta)\|^2 + \|g(\theta, u_{n\theta})\|^2\right)\mathrm{d}\theta \\
&\leqslant C(1 + t - \tau)\int_\tau^t \|u_n(\theta)\|_V^2 \mathrm{d}\theta + C(t - \tau)\left(\int_\tau^t \|f(\theta)\|^2 \mathrm{d}\theta + \int_{\tau-h}^t \|u_n(\theta)\|^2 \mathrm{d}\theta\right) \\
&\leqslant C(1 + t - \tau)\int_\tau^t \|u_n(\theta)\|_V^2 \mathrm{d}\theta + C(t - \tau)\left(\int_\tau^t \|f(\theta)\|^2 \mathrm{d}\theta + \|\phi^{\mathrm{in}}\|_{L_H^2}^2\right) \\
&\leqslant C(1 + t - \tau)\left(\|(u^{\mathrm{in}}, \phi^{\mathrm{in}})\|_{H\times L_H^2}^2 + \int_\tau^t \|f(\theta)\|^2 \mathrm{d}\theta\right),
\end{aligned} \tag{3.20}$$

$$\int_\tau^t K_n(\theta)\mathrm{d}\theta = C\int_\tau^t \|u_n\|_V^2 \mathrm{d}\theta \leqslant C\left(\|(u^{\mathrm{in}}, \phi^{\mathrm{in}})\|_{H\times L_H^2}^2 + \int_\tau^t \|f\|^2 \mathrm{d}\theta\right). \tag{3.21}$$

Therefore, by Lemma 2.1 and Eqs (3.19)-(3.21), we have for $\tau + \epsilon \leqslant r \leqslant t$ that

$$
\begin{aligned}
(r-\tau)\|u_n(r)\|_V^2 \leqslant C(1+t-\tau)&\left(\|(u^{\rm in},\phi^{\rm in})\|^2_{H\times L^2_H} + \int_\tau^t \|f(\theta)\|^2 \mathrm{d}\theta\right) \\
&\times \exp\left\{C(\|(u^{\rm in},\phi^{\rm in})\|^2_{H\times L^2_H} + \int_\tau^t \|f(\theta)\|^2 \mathrm{d}\theta)\right\},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|u_n(r)\|_V^2 \leqslant \frac{C(1+t-\tau)}{\epsilon}&\left(\|(u^{\rm in},\phi^{\rm in})\|^2_{H\times L^2_H} + \int_\tau^t \|f(\theta)\|^2 \mathrm{d}\theta\right) \\
&\times \exp\left\{C(\|(u^{\rm in},\phi^{\rm in})\|^2_{H\times L^2_H} + \int_\tau^t \|f(\theta)\|^2 \mathrm{d}\theta)\right\}.
\end{aligned}
\tag{3.22}
$$

Furthermore, for all $\tau + h + \epsilon \leqslant r \leqslant t$, one deduces

$$
\int_{r-h}^r \|u_n(\theta)\|_V^2 \mathrm{d}\theta \leqslant h \max_{\theta\in[\tau+\epsilon,t]} \|u_n(\theta)\|_V^2. \tag{3.23}
$$

Integrating Eq (3.16) for θ over $[\tau+\epsilon, t]$ and an analogue of (3.20) gives

$$
\begin{aligned}
2\mu_1 \int_{\tau+\epsilon}^t \|Au_n(\theta)\|^2 \mathrm{d}\theta &\leqslant \|u_n(\tau+\epsilon)\|_V^2 + C\left(\int_\tau^t \|f(\theta)\|^2\mathrm{d}\theta + \int_\tau^t \|g(\theta, u_{n\theta})\|^2 \mathrm{d}\theta\right) \\
&\quad + C\left(\int_\tau^t \|u_n(\theta)\|_V^2 \mathrm{d}\theta + \int_{\tau+\epsilon}^t \|u_n(\theta)\|_V^4 \mathrm{d}\theta\right) \\
&\leqslant \|u_n(\tau+\epsilon)\|_V^2 + C\left(\int_\tau^t \|f(\theta)\|^2\mathrm{d}\theta + \int_{\tau-h}^t \|u_n(\theta)\|^2 \mathrm{d}\theta\right) \\
&\quad + C\left(\int_\tau^t \|u_n(\theta)\|_V^2 \mathrm{d}\theta + \max_{[\tau+\epsilon,t]} \|u_n(\theta)\|_V^2 \int_\tau^t \|u_n(\theta)\|_V^2 \mathrm{d}\theta\right) \\
&\leqslant \|u_n(\tau+\epsilon)\|_V^2 + C\left(\|\phi^{\rm in}\|^2_{L^2_H} + \int_\tau^t \|f(\theta)\|^2\mathrm{d}\theta\right) \\
&\quad + C\left(1 + \max_{[\tau+\epsilon,t]} \|u_n(\theta)\|_V^2\right) \int_\tau^t \|u_n(\theta)\|_V^2 \mathrm{d}\theta.
\end{aligned}
\tag{3.24}
$$

Based on (3.22)-(3.24), one immediately concludes the validity of the results in Lemma 3.2. □

Thanks to Lemma 3.2, $\mathbb{H}^2$-boundedness of the pullback attractor is verified.

Theorem 3.3. *Assume* (H1)-(H4) *hold, then we have for any $T_1 < T_2$, $\cup_{t\in[T_1,T_2]}\mathcal{A}(t)$ is bounded in $V \times L^2_V$.*

Proof. For the solution $u(\cdot;\tau,u^{\rm in},\phi^{\rm in})$ and the approximate solutions $u_n(\cdot;\tau,u^{\rm in},\phi^{\rm in})$, we have the following facts:

$$
\left\{
\begin{array}{l}
u_n(\cdot;\tau,u^{\rm in},\phi^{\rm in}) \in L^\infty(\tau,t;H),\ u(\cdot;\tau,u^{\rm in},\phi^{\rm in}) \in C([\tau,t];H), \\
u_n(\cdot;\tau,u^{\rm in},\phi^{\rm in}) \rightharpoonup u(\cdot;\tau,u^{\rm in},\phi^{\rm in}) \ \text{ weakly in } \ L^2(\tau,t;V).
\end{array}
\right.
$$

Due to Lemmas 3.1 and 3.2, we obtain that for any bounded set $\mathcal{B} \subset H \times L^2_H$, $\tau \in \mathbb{R}$, $\epsilon > 0$, and $t > \tau + h + \epsilon$, the set

$$\cup_{r\in[\tau+h+\epsilon,t]} U(r,\tau)\mathcal{B} \text{ is a bounded set of } V \times L^2_V.$$

We set $\epsilon = 1$ and choose τ small enough such that $\tau + h < T_1 - 1$. Because $\mathcal{A}(\tau)$ is a bounded set in $H \times L^2_H$, then

$$\cup_{t\in[T_1,T_2]}\mathcal{A}(t) \subset \cup_{t\in[h+\tau+1,T_2]} U(t,\tau)\mathcal{A}(\tau)$$

is a bounded set of $V \times L^2_V$. □

3.2. $\mathbb{H}^4$-*boundedness*

Next, we improve the result of $\mathbb{H}^2$-boundedness in Theorem 3.3 to $\mathbb{H}^4$-boundedness based on the bounded estimate of $u_n' \in L^\infty(\tau+\epsilon, t; H), \epsilon > 0$.

To start, we establish a weak estimate with respect to u_n'.

Lemma 3.4. *Assume* (H1)-(H4) *hold, and we have*

$$\{u_n'(\cdot;\tau,u^{\text{in}},\phi^{\text{in}})|\ \ (u^{\text{in}},\phi^{\text{in}}) \in \mathcal{B}, n \geqslant 1\}, \tag{3.25}$$

is bounded in $L^2(\tau+\epsilon, t; H)$.

Proof. Multiplying Eq (3.1) by $\xi_{nj}'(\theta)$ and summing them for $j = 1$ to n, we obtain

$$\begin{aligned}\|u_n'(\theta)\|^2 + \mu_1 \frac{\mathrm{d}}{\mathrm{d}\theta}\langle Au_n(\theta), u_n(\theta)\rangle + \langle B(u_n(\theta)), u_n'(\theta)\rangle + \langle N(u_n(\theta)), u_n'(\theta)\rangle \\ = (f(\theta), u_n'(\theta)) + (g(\theta, u_{n\theta}), u_n'(\theta)).\end{aligned} \tag{3.26}$$

Similar to Eqs (3.12) and (3.13), one has

$$|\langle B(u_n(\theta)), u_n'(\theta)\rangle| \leqslant \frac{1}{8}\|u_n'(\theta)\|^2 + C\|u_n(\theta)\|_V^4, \tag{3.27}$$

$$|\langle N(u_n(\theta)), u_n'(\theta)\rangle| \leqslant \frac{1}{8}\|u_n'(\theta)\|^2 + C\|u_n(\theta)\|_V^2. \tag{3.28}$$

In addition, it's clear that

$$|(f(\theta), u_n'(\theta))| \leqslant \frac{1}{8}\|u_n'(\theta)\|^2 + C\|f(\theta)\|^2, \tag{3.29}$$

$$|(g(\theta, u_{n\theta}), u_n'(\theta))| \leqslant \frac{1}{8}\|u'(\theta)\|^2 + C\|g(\theta, u_{n\theta})\|^2. \tag{3.30}$$

Then, it follows from Eqs (3.26)-(3.30) that

$$\begin{aligned}&\|u_n'(\theta)\|^2 + 2\mu_1\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Au_n(\theta), u_n(\theta)\rangle \\ &\qquad \leqslant C\left(\|f(\theta)\|^2 + \|g(\theta, u_{n\theta})\|^2 + \|u_n(\theta)\|_V^2 + \|u_n(\theta)\|_V^4\right).\end{aligned} \tag{3.31}$$

Integrating Eq (3.31) for θ over $[\tau+\epsilon, t]$ and then calculating the resulting inequality in a quite similar manner as we did in obtaining Eq (3.24), it yields

$$\begin{aligned}\int_{\tau+\epsilon}^{t}\|u_n'(\theta)\|^2\mathrm{d}\theta &\leqslant C\left(\|u_n(\tau+\epsilon)\|_V^2+\int_\tau^t\|f(\theta)\|^2\mathrm{d}\theta+\int_\tau^t\|g(\theta,u_{n\theta})\|^2\mathrm{d}\theta\right)\\ &\quad+C\left(\int_\tau^t\|u_n(\theta)\|_V^2\mathrm{d}\theta+\int_{\tau+\epsilon}^t\|u_n(\theta)\|_V^4\mathrm{d}\theta\right)\\ &\leqslant C\left(\|u_n(\tau+\epsilon)\|_V^2+\|\phi^{\mathrm{in}}\|_{L_H^2}^2+\int_\tau^t\|f(\theta)\|^2\mathrm{d}\theta\right)\\ &\quad+C\left((1+\max_{[\tau+\epsilon,t]}\|u_n(\theta)\|_V^2)\int_\tau^t\|u_n(\theta)\|_V^2\mathrm{d}\theta\right). \end{aligned}\tag{3.32}$$

Therefore, Eq (3.32) combined with Lemma 3.2 shows that Lemma 3.4 holds. □

In order to improve the estimate in Lemma 3.4, we need some additional assumptions on the functions f and g. By the definition of the function $g(\cdot,\cdot)$, we have

$$(g(t,u_t))' = g_1'(t,u_t)+g_2'(t,u_t)u_t',$$

where $g_1' = \partial g(t,\xi)/\partial t$ and $g_2' = \partial g(t,\xi)/\partial\xi$, which are understood in the weak sense. We give the following assumptions:

(H5) assumes that $f\in W_b^{1,2}(\mathbb{R};H)$ and $\delta=\frac{c_1\mu_1}{3}-2\alpha\mu_0\varepsilon^{-\alpha/2}>0$.
(H6) assumes $g_i'(\cdot,\cdot):\mathbb{R}\times L_H^2\mapsto H(i=1,2)$ satisfies
- for any $\xi\in L_H^2$, the mapping $t\in\mathbb{R}\mapsto g_i'(t,\xi)\in H$ is measurable and $g_i'(t,0)=0$ for all $t\in\mathbb{R}$;
- for all $\tau\leqslant t$ and $u,v\in L^2(\tau-h,t;H)$, it holds that

$$\int_\tau^t\|g_1'(s,u_s)-g_1'(s,v_s)\|^2\mathrm{d}s\leqslant C_g^2\int_{\tau-h}^t\|u(s)-v(s)\|^2\mathrm{d}s;$$

- there is a constant $L_g'>0$ satisfying $L_g'R_0^{1/2}\leqslant c_1\mu_1$ such that

$$\|g_2'(t,\xi)-g_2'(t,\eta)\|\leqslant L_g'\|\xi-\eta\|_{L_H^2},\ \ \forall\, t\in\mathbb{R},\ \xi,\eta\in L_H^2.$$

Lemma 3.5. *Assume* (H1)-(H6) *hold, then for any bounded set $\mathcal{B}\subset H\times L_H^2$, there exists a $\tau_1(\mathcal{B},t)\in\mathbb{R}$ such that for any $\tau\leqslant\tau_1$, $\epsilon>0$, and $t>\tau+h+\epsilon$, we have*

$$\{u_n'(r;\tau,u^{\mathrm{in}},\phi^{\mathrm{in}})|\ \ r\in[\tau+\epsilon,t],(u^{\mathrm{in}},\phi^{\mathrm{in}})\in\mathcal{B},n\geqslant1\}$$

is bounded in H.

Proof. Differentiating Eq (3.1) with respect to θ, multiplying the resulting equality by $\xi_{nj}'(\theta)$, and summing them for $j=1$ to n, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\|u_n'(\theta)\|^2+4c_1\mu_1\|u_n'(\theta)\|_V^2+2b(u_n'(\theta),u_n(\theta),u_n'(\theta))+2\langle(N(u_n(\theta)))',u_n'(\theta)\rangle$$

$$\leqslant 2(f'(\theta), u_n'(\theta)) + 2(g_1'(\theta, u_{n\theta}), u_n'(\theta)) + 2(g_2'(\theta, u_{n\theta})u_{n\theta}', u_n'(\theta)). \tag{3.33}$$

By the Young inequality, we have

$$|2(f'(\theta), u_n'(\theta))| \leqslant \frac{c_1\mu_1}{4}\|u_n'(\theta)\|_V^2 + C\|f'(\theta)\|^2, \tag{3.34}$$

$$2|(g_1'(\theta, u_{n\theta}), u_n'(\theta))| \leqslant \frac{c_1\mu_1}{4}\|u_n'(\theta)\|_V^2 + C\|g_1'(\theta, u_{n\theta})\|^2, \tag{3.35}$$

$$2|b(u_n'(\theta), u_n(\theta), u_n'(\theta))| \leqslant \delta\|u_n'(\theta)\|_V^2 + C\|u_n'(\theta)\|^2\|u_n(\theta)\|_V^2, \tag{3.36}$$

where δ is the constant defined in (H5). Due to Theorem 2.2, there exists a $\tau_1(\mathcal{B}, t) \in \mathbb{R}$ such that $\|u_{n\theta}\|_{L_H^2}^2 \leqslant R_0$ for any $\tau \leqslant \tau_1$, and then we obtain

$$\begin{aligned} |2(g_2'(\theta, u_{n\theta})u_n'(\theta), u_n'(\theta))| &\leqslant 2\|g_2'(\theta, u_{n\theta})\|\,\|u_n'(\theta)\|_{\mathbb{L}^4(\Omega)}^2 \leqslant 2\|g_2'(\theta, u_{n\theta})\|\,\|u_n'(\theta)\|_V^2 \\ &\leqslant 2L_g'\|u_{n\theta}\|_{L_H^2}\|u_n'(\theta)\|_V^2 \leqslant 2c_1\mu_1\|u_n'(\theta)\|_V^2, \end{aligned} \tag{3.37}$$

where we have used the embedding $V \hookrightarrow \mathbb{L}^4(\Omega)$ and (H6). For the term $\langle (N(u))', u'\rangle$, we have

$$\begin{aligned} \langle (N(u_n(\theta)))', u_n'(\theta)\rangle = &\sum_{i,j=1}^{2}\int_\Omega (\mu(u_n(\theta)))' e_{ij}(u_n(\theta))e_{ij}(u_n'(\theta))\mathrm{d}x \\ &+ \sum_{i,j=1}^{2}\int_\Omega \mu(u_n(\theta))e_{ij}(u_n'(\theta))e_{ij}(u_n'(\theta))\mathrm{d}x \end{aligned} \tag{3.38}$$

and

$$\sum_{i,j=1}^{2}\int_\Omega \mu(u_n(\theta))e_{ij}(u_n'(\theta))e_{ij}(u_n'(\theta))\mathrm{d}x \geqslant 0. \tag{3.39}$$

By some computations, we have

$$\begin{aligned} &\Big|\sum_{i,j=1}^{2}\int_\Omega (\mu(u_n(\theta)))' e_{ij}(u_n(\theta))e_{ij}(u_n'(\theta))\mathrm{d}x\Big| \\ &= \Big|\sum_{i,j=1}^{2}\int_\Omega \alpha\mu_0(\varepsilon + |e(u_n(\theta))|^2)^{-\alpha/2-1}\Big(\sum_{k,l=1}^{2} e_{kl}(u_n(\theta))e_{kl}(u_n'(\theta))\Big)e_{ij}(u_n(\theta))e_{ij}(u_n'(\theta))\mathrm{d}x\Big| \\ &\leqslant \sum_{i,j=1}^{2}\int_\Omega \alpha\mu_0(\varepsilon + |e(u_n(\theta))|^2)^{-\alpha/2-1}|e(u_n(\theta))|^2\, e_{ij}(u_n'(\theta))e_{ij}(u_n'(\theta))\mathrm{d}x \\ &\leqslant \sum_{i,j=1}^{2}\int_\Omega \alpha\mu_0(\varepsilon + |e(u_n(\theta))|^2)^{-\alpha/2} e_{ij}(u_n'(\theta))e_{ij}(u_n'(\theta))\mathrm{d}x \\ &\leqslant \alpha\mu_0\varepsilon^{-\alpha/2}\|u_n'(\theta)\|_{\mathbb{H}^1(\Omega)}^2 \leqslant \alpha\mu_0\varepsilon^{-\alpha/2}\|u_n'(\theta)\|_V^2, \end{aligned} \tag{3.40}$$

where we have used the fact

$$\sum_{i,j=1}^{2}\int_\Omega e_{ij}(u)e_{ij}(u)\mathrm{d}x \leqslant C\|u\|_{\mathbb{H}^1(\Omega)}^2, \quad \forall\, u \in \mathbb{H}^1(\Omega).$$

Then, it follows from Eqs (3.33)-(3.40) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\|u_n'(\theta)\|^2 + c_1\mu_1\|u_n'(\theta)\|_V^2 \\ \leqslant C\left(\|f'(\theta)\|^2 + \|g_1'(\theta, u_{n\theta})\|^2 + \|u_n'(\theta)\|^2\|u_n(\theta)\|_V^2\right).\end{aligned} \tag{3.41}$$

Integrating Eq (3.41) for θ over $[s, r]$, we have for $\tau \leqslant s \leqslant r \leqslant t$ that

$$\begin{aligned}&\|u_n'(r)\|^2 + c_1\mu_1\int_s^r \|u_n'(\theta)\|_V^2\mathrm{d}\theta \\ &\leqslant \|u_n'(s)\|^2 + C\int_s^r \left(\|u_n'(\theta)\|^2\|u_n(\theta)\|_V^2 + \|f'(\theta)\|^2 + \|g_1'(\theta, u_{n\theta})\|^2\mathrm{d}\theta\right).\end{aligned} \tag{3.42}$$

By Eqs (3.10) and (H6), we have

$$\begin{aligned}\int_\tau^t \|g_1'(\theta, u_{n\theta})\|^2\mathrm{d}\theta &\leqslant C\int_{\tau-h}^t \|u_n(\theta)\|^2\mathrm{d}\theta \leqslant C\left(\|\phi^{\mathrm{in}}\|_{L_H^2}^2 + \int_\tau^t \|u_n(\theta)\|^2\mathrm{d}\theta\right) \\ &\leqslant C\left(\|(u^{\mathrm{in}}, \phi^{\mathrm{in}})\|_{H\times L_H^2}^2 + \int_\tau^t \|f(\theta)\|^2\mathrm{d}\theta\right).\end{aligned} \tag{3.43}$$

By inserting Eq (3.43) into Eq (3.42), we get for $\tau \leqslant s \leqslant r \leqslant t$ that

$$\begin{aligned}\|u_n'(r)\|^2 \leqslant\ & C\left(\|u_n'(s)\|^2 + \|(u^{\mathrm{in}}, \phi^{\mathrm{in}})\|_{H\times L_H^2}^2 + \int_\tau^t \|f(\theta)\|^2\mathrm{d}\theta\right) \\ &+ C\left(\int_\tau^t \|f'(\theta)\|^2\mathrm{d}\theta + \int_s^r \|u_n'(\theta)\|^2\|u_n(\theta)\|_V^2\mathrm{d}\theta\right).\end{aligned} \tag{3.44}$$

By the Gronwall lemma, we have for $\tau + \epsilon/2 \leqslant s \leqslant r \leqslant t$ that

$$\begin{aligned}\|u_n'(r)\|^2 \leqslant\ & C\left(\|u_n'(s)\|^2 + \|(u^{\mathrm{in}}, \phi^{\mathrm{in}})\|_{H\times L_H^2}^2 + \int_\tau^t \|f(\theta)\|^2\mathrm{d}\theta + \int_\tau^t \|f'(\theta)\|^2\mathrm{d}\theta\right) \\ &\times \exp\left\{C\int_{\tau+\epsilon/2}^t \|u_n(\theta)\|_V^2\mathrm{d}\theta\right\}.\end{aligned} \tag{3.45}$$

Integrating Eq (3.45) for s over $[\tau + \epsilon/2, r]$, we obtain for $\tau + \epsilon/2 \leqslant r \leqslant t$ that

$$\begin{aligned}&(r - \tau - \epsilon/2)\|u_n'(r)\|^2 \\ &\leqslant C\left(\int_{\tau+\epsilon/2}^t \|u_n'(s)\|^2\mathrm{d}s + (t-\tau)(\|(u^{\mathrm{in}}, \phi^{\mathrm{in}})\|_{H\times L_H^2}^2 + \int_\tau^t \|f(\theta)\|^2 + \|f'(\theta)\|^2\mathrm{d}\theta)\right) \\ &\quad\times \exp\left\{C\int_{\tau+\epsilon/2}^t \|u_n(\theta)\|_V^2\mathrm{d}\theta\right\}.\end{aligned} \tag{3.46}$$

In particular, we have for $\tau + \epsilon \leqslant r \leqslant t$ that

$$\begin{aligned}\|u_n'(r)\|^2 \leqslant\ & \frac{2C}{\epsilon}\left(\int_{\tau+\epsilon/2}^t \|u_n'(s)\|^2\mathrm{d}s + (t-\tau)(\|(u^{\mathrm{in}}, \phi^{\mathrm{in}})\|_{H\times L_H^2}^2 + \int_\tau^t (\|f(\theta)\|^2 + \|f'(\theta)\|^2)\mathrm{d}\theta)\right) \\ &\times \exp\left\{C\int_{\tau+\epsilon/2}^t \|u_n(\theta)\|_V^2\mathrm{d}\theta\right\}.\end{aligned} \tag{3.47}$$

By Lemma 3.4 and Eq (3.47), we complete the proof of this lemma. □

Lemma 3.6. *Assume* (H1)-(H6) *hold, then for any bounded set $\mathcal{B} \subset H \times L^2_H$, there exists a $\tau_1(\mathcal{B}, t) \in \mathbb{R}$ such that for any $\tau \leqslant \tau_1$, $\epsilon > 0$, and $t > \tau + h + \epsilon$, we have*

$$\{u_n(r; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}})| \;\; r \in [\tau + \epsilon, t], (u^{\mathrm{in}}, \phi^{\mathrm{in}}) \in \mathcal{B}, n \geqslant 1\}$$

is bounded in $D(A)$ and

$$\{u_{nr}(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}})| \;\; r \in [\tau + \epsilon + h, t], (u^{\mathrm{in}}, \phi^{\mathrm{in}}) \in \mathcal{B}, n \geqslant 1\}$$

is bounded in $L^2_{D(A)}$.

Proof. Multiplying Eq (3.1) by $A\xi_{nj}(\theta)$ and summing up the resulting inequality for $j = 1$ to n, we can obtain

$$\begin{aligned}(u_n'(\theta), Au_n(\theta)) + 2\mu_1\|Au_n(\theta)\|^2 + \langle B(u_n(\theta)), Au_n(\theta)\rangle + \langle N(u_n(\theta)), Au_n(\theta)\rangle \\ = (f(\theta), Au_n(\theta)) + (g(\theta, u_{n\theta}), Au_n(\theta)).\end{aligned} \tag{3.48}$$

By the Young inequality, we have

$$|(u_n'(\theta), Au_n(\theta))| \leqslant \frac{\mu_1}{8}\|Au_n(\theta)\|^2 + C\|u_n'(\theta)\|^2, \tag{3.49}$$

$$|(f(\theta), Au_n(\theta))| \leqslant \frac{\mu_1}{8}\|Au_n(\theta)\|^2 + C\|f(\theta)\|^2, \tag{3.50}$$

$$\|(g(\theta, u_{n\theta}), Au_n(\theta))\| \leqslant \frac{\mu_1}{4}\|Au_n(\theta)\|^2 + C\|g(\theta, u_{n\theta})\|^2. \tag{3.51}$$

Then, (H1)-(H2) imply for $\tau + h + \epsilon \leqslant \theta \leqslant t$ that

$$\|g(\theta, u_{n\theta})\|^2 \leqslant C\|u_{n\theta}(s)\|^2_{L^2_H} \leqslant C \max_{s\in[-h,0]} \|u_n(\theta + s)\|^2 = C \max_{s\in[\tau+\epsilon,t]} \|u_n(s)\|^2. \tag{3.52}$$

It follows from Eqs (3.12), (3.13), and (3.48)-(3.52) that for $\tau + h + \epsilon \leqslant \theta \leqslant t$,

$$\|Au_n(\theta)\|^2 \leqslant C\left(\|u_n'\|^2 + \|f\|^2 + \max_{s\in[\tau+\epsilon,t]} \|u_n(s)\|^2 + \|u_n\|_V^2 + \|u_n\|_V^4\right). \tag{3.53}$$

Noting that the embedding $W^{1,2}_b(\mathbb{R}; H) \hookrightarrow C(\mathbb{R}; H)$ is continuous, we have $f \in C(\mathbb{R}; H)$. Hence, by Lemma 3.1 and Eq (3.53), and we complete the proof. □

With the help of Lemma 3.6, we get the main result in this paper.

Theorem 3.7. *Assume* (H1)-(H6) *hold, and we have for any $T_1 < T_2$, $\cup_{t\in[T_1,T_2]}\mathcal{A}(t)$ is bounded in $D(A) \times L^2_{D(A)}$.*

Proof. For the solution $u(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}})$ and the approximate solutions $u_n(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}})$, we have the following facts:

$$\begin{cases} u_n(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}}) \in L^\infty(\tau + \epsilon, t; D(A)), \quad u(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}}) \in C([\tau + \epsilon, t]; H), \\ u_n(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}}) \rightharpoonup u(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}}) \;\text{ weakly in }\; L^2(\tau + \epsilon + h, t; D(A)).\end{cases}$$

Due to Lemmas 3.1 and 3.6, we obtain that for any bounded set $\mathcal{B} \subset H \times L^2_H$, there exists a $\tau_1(\mathcal{B}, t) \in \mathbb{R}$ such that for any $\tau \leqslant \tau_1, \epsilon > 0$ and $t > \tau + h + \epsilon$, the set

$$\cup_{r\in[\tau+h+\epsilon,t]} U(r,\tau)\mathcal{B} \text{ is bounded in } D(A) \times L^2_{D(A)}.$$

We set $\epsilon = 1$ and choose τ small enough such that $\tau + h < T_1 - 1$. Thanks to the boundness of $\mathcal{A}(\tau) \in H \times L^2_H$, it follows that the set

$$\cup_{t\in[T_1,T_2]} \mathcal{A}(t) \subset \cup_{t\in[h+\tau+1,T_2]} U(t,\tau)\mathcal{A}(\tau)$$

is a bounded set of $D(A) \times L^2_{D(A)}$. □

Author contributions

Tingting Liao: writing-original draft, writing-review, editing and formal analysis; Guowei Liu: conceptualization, supervision, writing review and editing; Tianfang Wu: methodology, writing-review and editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We thank the handling editor and the anonymous referees for their careful reading and valuable suggestions. The work is supported by the Natural Science Foundation of Chongqing (Grant No. CSTB2024NSCQ-MSX1089), the China Postdoctoral Science Foundation (Grant No. 2022M722105), the Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No. KJQN202200563).

Conflict of interest

The authors declare there is no conflict of interest.

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