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Research article

Stability of the 2D MHD system with fractional vertical dissipation and magnetic damping

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Abstract: Due to the importance in physical applications and mathematics, the stability problems of partially dissipative hydrodynamic system have attracted considerable attention in recent years and achieved significant progress. However, research on the stability analysis for the partial fractional dissipative system is relatively limited. The present study demonstrates the stability of a two-dimensional (2D) incompressible magnetohydrodynamic (MHD) system with fractional vertical dissipation and magnetic damping within the Sobolev space $H^2(\mathbb{R}^2)$. In view of the presence of fractional order operators, classical tools and techniques designed for the fully dissipated MHD system no longer work; therefore, we develop new anisotropic inequalities that involve fractional derivatives to address this issue. Our result extends and improves the work in [Nonlinearity, 2021, 34(4): 2527].

Keywords: MHD system; partial and fractional dissipation; magnetic damping; stability
Mathematics Subject Classification: 35B35, 35B40, 76E25

1. Introduction

As a fundamental theoretical model in continuum physics, the magnetohydrodynamic (MHD) system characterizes the dynamic behavior of conductive fluids including plasmas, electrolytes, and liquid metals (see, e.g., [1, 2]), and has become an indispensable mathematical tool in astrophysics, controlled thermonuclear fusion, geophysics, and many other scientific fields [3]. The MHD system includes the Navier-Stokes equations, which govern the changes of momentum and energy of fluids, and Maxwell's equations, which govern the physical properties of fluids. With the ongoing expansion of plasma research and its applications, investigations on the MHD system have also deepened. The basic issues include the existence, uniqueness, and stability of solutions. As research progresses, investigating the impact of different dissipation mechanisms (such as partial or fractional dissipation) on these fundamental properties has become a focus. A great number of works have addressed the stability issues of related systems [4–8], while the global well-posedness theory is established in [9, 10], and relevant existence results can be found in [11].

The standard MHD system includes fully dissipative terms, and its study has been relatively in-depth. However, due to the complexity of physical phenomena and practical applications, the assumption of full dissipation may not be applicable in some special physical cases; therefore, many researchers are dedicated to studying the partially dissipative MHD equations. Specifically, Feng, Hafeez, and Wu [12] established the stability of solutions to the following 2D MHD system with only vertical dissipation and magnetic damping in $H^2(\mathbb{R}^2)$,

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = \nu \partial_{22} \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \partial_1 \boldsymbol{b}, \quad \boldsymbol{x} \in \mathbb{R}^2, \quad t > 0, \\ \partial_t \boldsymbol{b} + \boldsymbol{u} \cdot \nabla \boldsymbol{b} + \eta \boldsymbol{b} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} + \partial_1 \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{b} = 0, \end{cases} \tag{1.1}$$

in which $\nu > 0$ and $\eta > 0$ are the coefficients of viscosity and magnetic diffusion, respectively, and the key variables are defined as follows: the velocity field $\boldsymbol{u} = (u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t))$, the magnetic field $\boldsymbol{b} = (b_1(\boldsymbol{x}, t), b_2(\boldsymbol{x}, t))$, and the scalar pressure $P = P(\boldsymbol{x}, t)$.

A key point to mention is that the partial dissipation term in System (1.1) is integer-order. Additionally, there are many other results focused on the stability analysis for the integer-order partial dissipative MHD system, with abundant findings for the 2D case [13–17] and further developments in the 3D setting [18–22].

However, integer-order differential equations may not be accurate enough to describe some complex physical processes, especially those that involve properties such as memory effects and non-local diffusion. In the study of the MHD system, fractional dissipation terms can be used to describe more complex interactions between magnetic fields and fluids, which is particularly important to study large-scale systems such as geophysical fluids and solar wind-magnetosphere interactions. There are relatively few studies on the stability of the fractional partial dissipative MHD system, and the related results can be found in [23–27].

Inspired by [12], it is natural to think that if we extend the integer-order dissipation of (1.1) to the fractional order, will the stability result still hold in the Sobolev space $H^2(\mathbb{R}^2)$? This article aims to address this question and provide a positive answer.

We investigate the stability of solutions to the following 2D MHD system with fractional vertical dissipation and magnetic damping:

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P + \nu \Lambda_2^{2\alpha} \boldsymbol{u} = \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \partial_1 \boldsymbol{b}, \quad \boldsymbol{x} \in \mathbb{R}^2, \quad t > 0, \\ \partial_t \boldsymbol{b} + \boldsymbol{u} \cdot \nabla \boldsymbol{b} + \eta \boldsymbol{b} = \boldsymbol{b} \cdot \nabla \boldsymbol{u} + \partial_1 \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{b} = 0, \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{b}(\boldsymbol{x}, 0) = \boldsymbol{b}_0(\boldsymbol{x}). \end{cases} \tag{1.2}$$

Here, the fractional partial operator $\Lambda_2^{2\alpha}$ with $\alpha > 0$ is defined via its Fourier transform as follows:

$$\widehat{\Lambda_2^{2\alpha} f}(\boldsymbol{\xi}) = |\xi_2|^{2\alpha} \hat{f}(\boldsymbol{\xi}).$$

The primary result established in our work is presented below.

Theorem 1.1. *Assume $\nu, \eta > 0$ and $\alpha \in (0, 1]$. For System (1.2) with the initial data $(\boldsymbol{u}_0, \boldsymbol{b}_0) \in H^2(\mathbb{R}^2)$ that satisfies $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$, there exists a constant $\epsilon_0 := \epsilon_0(\nu, \eta) > 0$ such that if $\epsilon \le \epsilon_0$ and*

$$\|\boldsymbol{u}_0\|_{H^2} + \|\boldsymbol{b}_0\|_{H^2} \le \epsilon, \tag{1.3}$$

then (1.2) possesses a unique global solution $(\boldsymbol{u}, \boldsymbol{b})$ that verifies

$$\|\boldsymbol{u}(t)\|_{H^2}^2 + \|\boldsymbol{b}(t)\|_{H^2}^2 + \int_0^t (\|\Lambda_2^\alpha \boldsymbol{u}(\tau)\|_{H^2}^2 + \|\boldsymbol{b}(\tau)\|_{H^2}^2 + \|\partial_1 \boldsymbol{u}(\tau)\|_{H^1}^2)\, d\tau \le C_0^2 \epsilon^2 \tag{1.4}$$

for some constant $C_0 > 0$ and all $t > 0$.

Remark 1.1. Feng, Hafeez, and Wu [12] established the stability of System (1.2) in the case when $\alpha = 1$. Under the same initial conditions and the same framework of the Sobolev space $H^2(\mathbb{R}^2)$, Theorem 1.1 extends the admissible range of the dissipation index α to $(0, 1]$.

The improvement of our result relies on the new analytic tools employed in our proof. In [12], the authors used the classical Sobolev inequality

$$\|f\|_{L^\infty(\mathbb{R})} \le C \|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}},$$

which is applicable to the study of integer-order dissipative problems. In contrast, we employ the following Sobolev inequality that involves a fractional derivative:

$$\|f\|_{L^\infty(\mathbb{R})} \le C(\|f\|_{L^2(\mathbb{R})} + \|\Lambda^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbb{R})}),$$

which allows us to handle fractional dissipative terms in a lower regularity framework.

Next, we sketch the proof framework for Theorem 1.1. The key techniques employed here include energy estimates and the bootstrapping argument. A natural starting point is to construct an appropriate energy functional, which consists of two parts:

$$\mathcal{E}(t) := \mathcal{E}_1(t) + \mathcal{E}_2(t), \tag{1.5}$$

where

$$\mathcal{E}_1(t) := \sup_{0 \le \tau \le t} \|(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{H^2}^2 + 2 \int_0^t \left(\nu \|\Lambda_2^\alpha \boldsymbol{u}(\tau)\|_{H^2}^2 + \eta \|\boldsymbol{b}(\tau)\|_{H^2}^2 \right) d\tau, \tag{1.6}$$

$$\mathcal{E}_2(t) := \int_0^t \|\partial_1 \boldsymbol{u}(\tau)\|_{H^1}^2\, d\tau. \tag{1.7}$$

We are going to prove that $\mathcal{E}(t)$ fulfills an appropriate energy inequality

$$\mathcal{E}(t) \le \mathcal{E}(0) + C \mathcal{E}(t)^{\frac{3}{2}}, \tag{1.8}$$

from which we can deduce a uniform global bound under the condition that the initial data are small enough. This allows us to conclude the stability result.

The remainder of this paper is structured as follows: Section 2 presents important anisotropic inequalities that involve fractional derivatives, and Section 3 is dedicated to proving Theorem 1.1.

2. Preliminaries

Next, we introduce essential interpolation inequalities of a 1D function on the whole line $\mathbb{R}$, which form a key basis to derive the anisotropic inequalities in $\mathbb{R}^2$.

Lemma 2.1. ([28]) *Take an arbitrary 1D function $f = f(x)$ as follows:*
(1) *suppose that $f(x) \in H^1(\mathbb{R})$, then*

$$\|f\|_{L^\infty(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}; \tag{2.1}$$

(2) *suppose that $f(x) \in H^{\frac{1}{2}+\epsilon}(\mathbb{R})$ with $\epsilon \in (0, +\infty)$, then*

$$\|f\|_{L^\infty(\mathbb{R})} \le C(\|f\|_{L^2(\mathbb{R})} + \|\Lambda^{\frac{1}{2}+\epsilon}f\|_{L^2(\mathbb{R})}); \tag{2.2}$$

(3) *suppose that $f(x) \in H^\alpha(\mathbb{R})$ for any $\alpha \in [0, 1]$, then*

$$\|f\|_{L^{\frac{2}{1-\alpha}}(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|\Lambda^\alpha f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{2.3}$$

As a consequence of (2.1)-(2.3), we derive the anisotropic inequalities in the lemma below.

Lemma 2.2. *Take an arbitrary function $f = f(x_1, x_2)$ on $\mathbb{R}^2$ as follows:*
(1) *suppose that $f \in H^\gamma(\mathbb{R}^2)$ with $\gamma \in [0, 1]$, then*

$$\|f\|_{L_{x_2}^{\frac{2}{1-\gamma}}L_{x_1}^2(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\Lambda_2^\gamma f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \tag{2.4}$$

and in particular, when $\gamma = 1$,

$$\|f\|_{L_{x_2}^\infty L_{x_1}^2(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}; \tag{2.5}$$

(2) *suppose that $f \in H^2(\mathbb{R}^2)$, then*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_1\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}, \tag{2.6}$$

while if $f \in H^{1+2\epsilon}(\mathbb{R}^2)$ with $\epsilon > 0$, then

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\left(\|f\|_{L^2(\mathbb{R}^2)} + \|\Lambda_1^{\frac{1}{2}+\epsilon}f\|_{L^2(\mathbb{R}^2)} + \|\Lambda_2^{\frac{1}{2}+\epsilon}f\|_{L^2(\mathbb{R}^2)} + \|\Lambda_1^{\frac{1}{2}+\epsilon}\Lambda_2^{\frac{1}{2}+\epsilon}f\|_{L^2(\mathbb{R}^2)}\right); \tag{2.7}$$

(3) *suppose that $f \in H^{1+\epsilon}(\mathbb{R}^2)$ with $\epsilon > 0$, then*

$$\|f\|_{L_{x_2}^{\frac{2}{\eta}}L_{x_1}^\infty(\mathbb{R}^2)} \le C\left(\|\Lambda_2^{\frac{1-\eta}{2}}f\|_{L^2(\mathbb{R}^2)} + \|\Lambda_2^{\frac{1-\eta}{2}}\Lambda_1^{\frac{1}{2}+\epsilon}f\|_{L^2(\mathbb{R}^2)}\right), \tag{2.8}$$

where $\eta \in (0, 1]$ and $\epsilon > 0$.

Proof. First, we prove (2.4). It follows from (2.3) that

$$
\begin{aligned}
\|f\|_{L_{x_2}^{\frac{2}{1-\gamma}} L_{x_1}^2(\mathbb{R}^2)} &\le \left\| \|f\|_{L_{x_2}^{\frac{2}{1-\gamma}}(\mathbb{R})} \right\|_{L_{x_1}^2(\mathbb{R})} \\
&\le C \left\| \|f\|_{L_{x_2}^2(\mathbb{R})}^{\frac{1}{2}} \|\Lambda_2^{\gamma} f\|_{L_{x_2}^2(\mathbb{R})}^{\frac{1}{2}} \right\|_{L_{x_1}^2(\mathbb{R})} \\
&\le C \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\Lambda_2^{\gamma} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.
\end{aligned}
$$

To prove (2.6), we use the Minkowski inequality and (2.1) to obtain the following:

$$
\begin{aligned}
\|f\|_{L_{x_2}^\infty L_{x_1}^\infty(\mathbb{R}^2)} &= \left\| \|f\|_{L_{x_2}^\infty(\mathbb{R})} \right\|_{L_{x_1}^\infty(\mathbb{R})} \\
&\le C \left\| \|f\|_{L_{x_2}^2(\mathbb{R})}^{\frac{1}{2}} \|\partial_2 f\|_{L_{x_2}^2(\mathbb{R})}^{\frac{1}{2}} \right\|_{L_{x_1}^\infty(\mathbb{R})} \\
&\le C \left\| \|f\|_{L_{x_1}^\infty(\mathbb{R})} \right\|_{L_{x_2}^2(\mathbb{R})}^{\frac{1}{2}} \left\| \|\partial_2 f\|_{L_{x_1}^\infty(\mathbb{R})} \right\|_{L_{x_2}^2(\mathbb{R})}^{\frac{1}{2}} \\
&\le C \left\| \|f\|_{L_{x_1}^2(\mathbb{R})}^{\frac{1}{2}} \|\partial_1 f\|_{L_{x_1}^2(\mathbb{R})}^{\frac{1}{2}} \right\|_{L_{x_2}^2(\mathbb{R})}^{\frac{1}{2}} \left\| \|\partial_2 f\|_{L_{x_1}^2(\mathbb{R})}^{\frac{1}{2}} \|\partial_1 \partial_2 f\|_{L_{x_1}^2(\mathbb{R})}^{\frac{1}{2}} \right\|_{L_{x_2}^2(\mathbb{R})}^{\frac{1}{2}} \\
&\le C \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_1 \partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.
\end{aligned}
$$

Similarly, for (2.7), we resort to (2.2) and obtain the following:

$$
\begin{aligned}
\|f\|_{L_{x_2}^\infty L_{x_1}^\infty(\mathbb{R}^2)} &= \left\| \|f\|_{L_{x_2}^\infty(\mathbb{R})} \right\|_{L_{x_1}^\infty(\mathbb{R})} \le C \left\| \|f\|_{L_{x_2}^2(\mathbb{R})} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_2}^2(\mathbb{R})} \right\|_{L_{x_1}^\infty(\mathbb{R})} \\
&\le C \left(\left\| \|f\|_{L_{x_1}^\infty(\mathbb{R})} \right\|_{L_{x_2}^2(\mathbb{R})} + \left\| \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^\infty(\mathbb{R})} \right\|_{L_{x_2}^2(\mathbb{R})} \right) \\
&\le C \left\| \|f\|_{L_{x_1}^2(\mathbb{R})} + \|\Lambda_1^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^2(\mathbb{R})} \right\|_{L_{x_2}^2(\mathbb{R})} \\
&\quad + C \left\| \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^2(\mathbb{R})} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^2(\mathbb{R})} \right\|_{L_{x_2}^2(\mathbb{R})} \\
&\le C \left(\|f\|_{L^2(\mathbb{R}^2)} + \|\Lambda_1^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbb{R}^2)} + \|\Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbb{R}^2)} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbb{R}^2)} \right).
\end{aligned}
$$

For (2.8), applying the Minkowski inequality, (2.2), together with the Sobolev embedding $\dot{H}^{\frac{1-\eta}{2}}(\mathbb{R}) \hookrightarrow L^{\frac{2}{\eta}}(\mathbb{R})$ with $\eta \in (0, 1]$, we obtain the following:

$$
\begin{aligned}
\|f\|_{L_{x_2}^{\frac{2}{\eta}} L_{x_1}^\infty(\mathbb{R}^2)} &= C \left\| \|f\|_{L_{x_1}^\infty(\mathbb{R})} \right\|_{L_{x_2}^{\frac{2}{\eta}}(\mathbb{R})} \\
&\le C \left\| \|f\|_{L_{x_1}^2(\mathbb{R})} + \|\Lambda_1^{\frac{1}{2}+\epsilon} f\|_{L_{x_1}^2(\mathbb{R})} \right\|_{L_{x_2}^{\frac{2}{\eta}}(\mathbb{R})} \\
&\le C \left(\left\| \|f\|_{L_{x_2}^{\frac{2}{\eta}}(\mathbb{R})} \right\|_{L_{x_1}^2(\mathbb{R})} + \left\| \|\Lambda_1^{\frac{1}{2}+\epsilon} f\|_{L_{x_2}^{\frac{2}{\eta}}(\mathbb{R})} \right\|_{L_{x_1}^2(\mathbb{R})} \right) \\
&\le C \left(\|\Lambda_2^{\frac{1-\eta}{2}} f\|_{L^2(\mathbb{R}^2)} + \|\Lambda_2^{\frac{1-\eta}{2}} \Lambda_1^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbb{R}^2)} \right).
\end{aligned}
$$

Hence, the proof for Lemma 2.2 is completed. □

3. Proof of Theorem 1.1

In this section, we dedicate our attention to demonstrating Theorem 1.1, with our primary efforts focused on establishing (1.8).

Proof of Theorem 1.1. Since $\|(\boldsymbol{u},\boldsymbol{b})\|_{H^2}$ is equivalent to $\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2}+\|(\boldsymbol{u},\boldsymbol{b})\|_{\dot{H}^2}$, we only need to bound the L^2 and $\dot{H}^2$-norms of $(\boldsymbol{u},\boldsymbol{b})$. Notably, in view of $\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{b}=0$, we have the following:

$$\|\boldsymbol{u}(t)\|_{L^2}^2+\|\boldsymbol{b}(t)\|_{L^2}^2+2\nu\int_0^t\|\Lambda_2^\alpha\boldsymbol{u}(\tau)\|_{L^2}^2\,d\tau+2\eta\int_0^t\|\boldsymbol{b}(\tau)\|_{L^2}^2\,d\tau=\|\boldsymbol{u}_0\|_{L^2}^2+\|\boldsymbol{b}_0\|_{L^2}^2. \tag{3.1}$$

For the $\dot{H}^2$-norm estimate, we take ∂_i^2 $(i=1,2)$ to (1.2), then dot with $(\partial_i^2\boldsymbol{u},\partial_i^2\boldsymbol{b})$ to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_i^2(\|\partial_i^2\boldsymbol{u}(t)\|_{L^2}^2+\|\partial_i^2\boldsymbol{b}(t)\|_{L^2}^2)+\nu\sum_i^2\|\partial_i^2\Lambda_2^\alpha\boldsymbol{u}\|_{L^2}^2+\eta\sum_i^2\|\partial_i^2\boldsymbol{b}\|_{L^2}^2\\&=:I_1+I_2+I_3+I_4+I_5,\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}I_1&=\sum_i^2\int\partial_i^2\partial_1\boldsymbol{b}\cdot\partial_i^2\boldsymbol{u}+\partial_i^2\partial_1\boldsymbol{u}\cdot\partial_i^2\boldsymbol{b}\,d\boldsymbol{x},\\I_2&=-\sum_i^2\int\partial_i^2(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\partial_i^2\boldsymbol{u}\,d\boldsymbol{x},\\I_3&=\sum_i^2\int(\partial_i^2(\boldsymbol{b}\cdot\nabla\boldsymbol{b})-\boldsymbol{b}\cdot\nabla\partial_i^2\boldsymbol{b})\cdot\partial_i^2\boldsymbol{u}\,d\boldsymbol{x},\\I_4&=-\sum_i^2\int\partial_i^2(\boldsymbol{u}\cdot\nabla\boldsymbol{b})\cdot\partial_i^2\boldsymbol{b}\,d\boldsymbol{x},\\I_5&=\sum_i^2\int(\partial_i^2(\boldsymbol{b}\cdot\nabla\boldsymbol{u})-\boldsymbol{b}\cdot\nabla\partial_i^2\boldsymbol{u})\cdot\partial_i^2\boldsymbol{b}\,d\boldsymbol{x}.\end{aligned}$$

Here, we use that

$$\begin{aligned}&\sum_i^2\int(-\boldsymbol{b}\cdot\nabla\partial_i^2\boldsymbol{b}\cdot\partial_i^2\boldsymbol{u}-\boldsymbol{b}\cdot\nabla\partial_i^2\boldsymbol{u}\cdot\partial_i^2\boldsymbol{b})\,d\boldsymbol{x}\\=&\sum_i^2\int(\boldsymbol{b}\cdot\nabla\partial_i^2\boldsymbol{u}\cdot\partial_i^2\boldsymbol{b}-\boldsymbol{b}\cdot\nabla\partial_i^2\boldsymbol{u}\cdot\partial_i^2\boldsymbol{b})\,d\boldsymbol{x}=0.\end{aligned}$$

Through integrating by parts, we obtain the following:

$$I_1=0. \tag{3.3}$$

For the estimate of I_2, we split it into four pieces as follows:

$$I_2=-\int\partial_1^2(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\partial_1^2\boldsymbol{u}\,d\boldsymbol{x}-\int\partial_2^2(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\partial_2^2\boldsymbol{u}\,d\boldsymbol{x}$$

$$
\begin{aligned}
&= -\int \partial_1^2 \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \partial_1^2 \boldsymbol{u}\, d\boldsymbol{x} - 2\int \partial_1 \boldsymbol{u} \cdot \nabla \partial_1 \boldsymbol{u} \cdot \partial_1^2 \boldsymbol{u}\, d\boldsymbol{x} \\
&\quad - \int \partial_2^2 \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \partial_2^2 \boldsymbol{u}\, d\boldsymbol{x} - 2\int \partial_2 \boldsymbol{u} \cdot \nabla \partial_2 \boldsymbol{u} \cdot \partial_2^2 \boldsymbol{u}\, d\boldsymbol{x} \\
&=: I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}
$$

Next, we estimate each of these terms separately. First, by the Hölder inequality and (2.7), we have the following:

$$
\begin{aligned}
I_{21} &= -\int \partial_1^2 \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \partial_1^2 \boldsymbol{u}\, d\boldsymbol{x} \lesssim \|\partial_1^2 \boldsymbol{u}\|_{L^2} \|\nabla \boldsymbol{u}\|_{L^\infty} \|\partial_1^2 \boldsymbol{u}\|_{L^2} \\
&\lesssim \|\partial_1^2 \boldsymbol{u}\|_{L^2}^2 \left(\|\nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} \right) \\
&\lesssim \|\partial_1 \boldsymbol{u}\|_{H^1}^2 \|\boldsymbol{u}\|_{H^2} + \|\partial_1 \boldsymbol{u}\|_{H^1} \|\boldsymbol{u}\|_{H^2} \|\Lambda_2^\alpha \boldsymbol{u}\|_{H^2};
\end{aligned}
$$

this step relies on the facts that

$$
\|\nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} \lesssim \|\boldsymbol{u}\|_{H^2}
$$

and

$$
\|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} \lesssim \|\Lambda_2^\alpha \boldsymbol{u}\|_{H^2}. \tag{3.4}
$$

Here, the derivation of (3.4) is based on the following reasons: if $\alpha \in (0, \frac{1}{2}]$, then one chooses ϵ sufficiently small to get

$$
\|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} = \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon-\alpha} \Lambda_2^\alpha \nabla \boldsymbol{u}\|_{L^2} \leq \|\Lambda_2^\alpha \boldsymbol{u}\|_{H^2}.
$$

Alternatively, if $\alpha \in (\frac{1}{2}, 1]$, then we set $\epsilon = \alpha - \frac{1}{2}$ and obtain that

$$
\|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} = \|\Lambda_1^\alpha \Lambda_2^\alpha \nabla \boldsymbol{u}\|_{L^2} \leq \|\Lambda_2^\alpha \boldsymbol{u}\|_{H^2}.
$$

Then, for any $\alpha \in (0, 1]$ and an appropriate ϵ, the estimate (3.4) holds true.

Similarly, for I_{22}, I_{23}, and I_{24}, one has the following:

$$
\begin{aligned}
I_{22} &= -2\int \partial_1 \boldsymbol{u} \cdot \nabla \partial_1 \boldsymbol{u} \cdot \partial_1^2 \boldsymbol{u}\, d\boldsymbol{x} \lesssim \|\partial_1^2 \boldsymbol{u}\|_{L^2} \|\partial_1 \boldsymbol{u}\|_{L^\infty} \|\nabla \partial_1 \boldsymbol{u}\|_{L^2} \\
&\lesssim \|\partial_1 \boldsymbol{u}\|_{H^1}^2 \left(\|\partial_1 \boldsymbol{u}\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \partial_1 \boldsymbol{u}\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \partial_1 \boldsymbol{u}\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \partial_1 \boldsymbol{u}\|_{L^2} \right) \\
&\lesssim \|\partial_1 \boldsymbol{u}\|_{H^1}^2 \|\boldsymbol{u}\|_{H^2} + \|\partial_1 \boldsymbol{u}\|_{H^1} \|\boldsymbol{u}\|_{H^2} \|\Lambda_2^\alpha \boldsymbol{u}\|_{H^2}.
\end{aligned}
$$

$$
\begin{aligned}
I_{23} &= -\int \partial_2^2 \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \partial_2^2 \boldsymbol{u}\, d\boldsymbol{x} \lesssim \|\partial_2^2 \boldsymbol{u}\|_{L^2}^2 \|\nabla \boldsymbol{u}\|_{L^\infty} \\
&\lesssim \|\partial_2^2 \boldsymbol{u}\|_{L^2}^2 \left(\|\nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} \right) \\
&\lesssim \|\Lambda_2^\alpha \boldsymbol{u}\|_{H^2}^2 \|\boldsymbol{u}\|_{H^2}.
\end{aligned}
$$

$$
\begin{aligned}
I_{24} &= -2\int \partial_2 \boldsymbol{u}\cdot\nabla\partial_2\boldsymbol{u}\cdot\partial_2^2\boldsymbol{u}\,d\boldsymbol{x} \lesssim \|\partial_2^2\boldsymbol{u}\|_{L^2}\|\partial_2\boldsymbol{u}\|_{L^\infty}\|\nabla\partial_2\boldsymbol{u}\|_{L^2}\\
&\lesssim \|\partial_2\boldsymbol{u}\|_{H^1}^2\left(\|\partial_2\boldsymbol{u}\|_{L^2}+\|\Lambda_1^{\frac12+\epsilon}\partial_2\boldsymbol{u}\|_{L^2}+\|\Lambda_2^{\frac12+\epsilon}\partial_2\boldsymbol{u}\|_{L^2}+\|\Lambda_1^{\frac12+\epsilon}\Lambda_2^{\frac12+\epsilon}\partial_2\boldsymbol{u}\|_{L^2}\right)\\
&\lesssim \|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2}^2\|\boldsymbol{u}\|_{H^2}.
\end{aligned}
$$

Thus, we obtain the following:

$$
I_2 \lesssim \|\boldsymbol{u}\|_{H^2}(\|\partial_1\boldsymbol{u}\|_{H^1}^2+\|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2}^2). \tag{3.5}
$$

For the term I_3, we first decompose it into two parts as follows:

$$
\begin{aligned}
I_3 &= 2\sum_i^2\int \partial_i\boldsymbol{b}\cdot\nabla\partial_i\boldsymbol{b}\cdot\partial_i^2\boldsymbol{u}\,d\boldsymbol{x}+\sum_i^2\int\partial_i^2\boldsymbol{b}\cdot\nabla\boldsymbol{b}\cdot\partial_i^2\boldsymbol{u}\,d\boldsymbol{x}\\
&=: I_{31}+I_{32}.
\end{aligned}
$$

We further split I_{31} into the following two sub-terms:

$$
\begin{aligned}
I_{31} &= 2\int\partial_1\boldsymbol{b}\cdot\nabla\partial_1\boldsymbol{b}\cdot\partial_1^2\boldsymbol{u}\,d\boldsymbol{x}+2\int\partial_2\boldsymbol{b}\cdot\nabla\partial_2\boldsymbol{b}\cdot\partial_2^2\boldsymbol{u}\,d\boldsymbol{x}\\
&=: I_{311}+I_{312}.
\end{aligned}
$$

In what follows, we will estimate terms I_{311} and I_{312}. For I_{311}, by using the Hölder inequality, (2.4), and (2.8), we have the following:

$$
\begin{aligned}
I_{311} &= 2\int\partial_1\boldsymbol{b}\cdot\nabla\partial_1\boldsymbol{b}\cdot\partial_1^2\boldsymbol{u}\,d\boldsymbol{x}\\
&\lesssim \|\nabla\partial_1\boldsymbol{b}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\boldsymbol{b}\|_{L^{\frac{2}{\alpha}}_{x_2}L^\infty_{x_1}}\|\partial_1^2\boldsymbol{u}\|_{L^{\frac{2}{1-\alpha}}_{x_2}L^2_{x_1}}\\
&\lesssim \|\boldsymbol{b}\|_{H^2}\left(\|\Lambda_2^{\frac{1-\alpha}{2}}\partial_1\boldsymbol{b}\|_{L^2}+\|\Lambda_2^{\frac{1-\alpha}{2}}\Lambda_1^{\frac12+\epsilon}\partial_1\boldsymbol{b}\|_{L^2}\right)\|\partial_1^2\boldsymbol{u}\|_{L^2}^{\frac12}\|\Lambda_2^{\alpha}\partial_1^2\boldsymbol{u}\|_{L^2}^{\frac12}\\
&\lesssim \|\boldsymbol{b}\|_{H^2}^2\|\boldsymbol{u}\|_{H^2}+\|\boldsymbol{b}\|_{H^2}^2\|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2}.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
I_{312} &= 2\int\partial_2\boldsymbol{b}\cdot\nabla\partial_2\boldsymbol{b}\cdot\partial_2^2\boldsymbol{u}\,d\boldsymbol{x}\\
&\lesssim \|\nabla\partial_2\boldsymbol{b}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_2\boldsymbol{b}\|_{L^{\frac{2}{\alpha}}_{x_2}L^\infty_{x_1}}\|\partial_2^2\boldsymbol{u}\|_{L^{\frac{2}{1-\alpha}}_{x_2}L^2_{x_1}}\\
&\lesssim \|\boldsymbol{b}\|_{H^2}\left(\|\Lambda_2^{\frac{1-\alpha}{2}}\partial_2\boldsymbol{b}\|_{L^2}+\|\Lambda_2^{\frac{1-\alpha}{2}}\Lambda_1^{\frac12+\epsilon}\partial_2\boldsymbol{b}\|_{L^2}\right)\|\partial_2^2\boldsymbol{u}\|_{L^2}^{\frac12}\|\Lambda_2^{\alpha}\partial_2^2\boldsymbol{u}\|_{L^2}^{\frac12}\\
&\lesssim \|\boldsymbol{b}\|_{H^2}^2\|\boldsymbol{u}\|_{H^2}+\|\boldsymbol{b}\|_{H^2}^2\|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2},
\end{aligned}
$$

and

$$
I_{32} = \int\partial_1^2\boldsymbol{b}\cdot\nabla\boldsymbol{b}\cdot\partial_1^2\boldsymbol{u}\,d\boldsymbol{x}+\int\partial_2^2\boldsymbol{b}\cdot\nabla\boldsymbol{b}\cdot\partial_2^2\boldsymbol{u}\,d\boldsymbol{x}
$$

$$
\begin{aligned}
&\lesssim \|\nabla \boldsymbol{b}\|_{L^{\frac{2}{\alpha}}_{x_2}L^{\infty}_{x_1}} \left(\|\partial_1^2 \boldsymbol{b}\|_{L^2_{x_2}L^2_{x_1}} \|\partial_1^2 \boldsymbol{u}\|_{L^{\frac{2}{1-\alpha}}_{x_2}L^2_{x_1}} + \|\partial_2^2 \boldsymbol{b}\|_{L^2_{x_2}L^2_{x_1}} \|\partial_2^2 \boldsymbol{u}\|_{L^{\frac{2}{1-\alpha}}_{x_2}L^2_{x_1}} \right) \\
&\lesssim \left(\|\Lambda_2^{\frac{1-\alpha}{2}} \nabla \boldsymbol{b}\|_{L^2} + \|\Lambda_2^{\frac{1-\alpha}{2}} \Lambda_1^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{b}\|_{L^2} \right) \|\boldsymbol{b}\|_{H^2} \|\nabla^2 \boldsymbol{u}\|_{L^2}^{\frac{1}{2}} \|\Lambda_2^{\alpha} \nabla^2 \boldsymbol{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\boldsymbol{b}\|_{H^2}^2 \|\boldsymbol{u}\|_{H^2} + \|\boldsymbol{b}\|_{H^2}^2 \|\Lambda_2^{\alpha} \boldsymbol{u}\|_{H^2}.
\end{aligned}
$$

Therefore, we deduce the following:

$$
\begin{aligned}
I_3 &\lesssim \|\boldsymbol{b}\|_{H^2}^2 \|\boldsymbol{u}\|_{H^2} + \|\boldsymbol{b}\|_{H^2}^2 \|\Lambda_2^{\alpha} \boldsymbol{u}\|_{H^2} \\
&\lesssim (\|\boldsymbol{u}\|_{H^2} + \|\boldsymbol{b}\|_{H^2})(\|\boldsymbol{b}\|_{H^2}^2 + \|\Lambda_2^{\alpha} \boldsymbol{u}\|_{H^2}^2).
\end{aligned} \tag{3.6}
$$

In what follows, we deal with I_4. First, we split I_4 as follows:

$$
\begin{aligned}
I_4 &= -2 \sum_{i}^{2} \int \partial_i \boldsymbol{u} \cdot \nabla \partial_i \boldsymbol{b} \cdot \partial_i^2 \boldsymbol{b} \, d\boldsymbol{x} - \sum_{i}^{2} \int \partial_i^2 \boldsymbol{u} \cdot \nabla \boldsymbol{b} \cdot \partial_i^2 \boldsymbol{b} \, d\boldsymbol{x} \\
&=: I_{41} + I_{42}.
\end{aligned}
$$

For I_{41}, by the Hölder inequality, (2.7), and (3.4), the following holds:

$$
\begin{aligned}
I_{41} &= -2 \sum_{i}^{2} \int \partial_i \boldsymbol{u} \cdot \nabla \partial_i \boldsymbol{b} \cdot \partial_i^2 \boldsymbol{b} \, d\boldsymbol{x} \\
&\lesssim \|\nabla^2 \boldsymbol{b}\|_{L^2}^2 \|\nabla \boldsymbol{u}\|_{L^\infty} \\
&\lesssim \|\boldsymbol{b}\|_{H^2}^2 \left(\|\nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} + \|\Lambda_1^{\frac{1}{2}+\epsilon} \Lambda_2^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{u}\|_{L^2} \right) \\
&\lesssim \|\boldsymbol{b}\|_{H^2}^2 \|\boldsymbol{u}\|_{H^2} + \|\boldsymbol{b}\|_{H^2}^2 \|\Lambda_2^{\alpha} \boldsymbol{u}\|_{H^2}.
\end{aligned}
$$

Regarding I_{42}, by the Hölder inequality, (2.4), and (2.8), one infers the following:

$$
\begin{aligned}
I_{42} &= -\sum_{i}^{2} \int \partial_i^2 \boldsymbol{u} \cdot \nabla \boldsymbol{b} \cdot \partial_i^2 \boldsymbol{b} \, d\boldsymbol{x} \\
&\lesssim \|\nabla^2 \boldsymbol{b}\|_{L^2_{x_2}L^2_{x_1}} \|\nabla \boldsymbol{b}\|_{L^{\frac{2}{\alpha}}_{x_2}L^{\infty}_{x_1}} \|\nabla^2 \boldsymbol{u}\|_{L^{\frac{2}{1-\alpha}}_{x_2}L^2_{x_1}} \\
&\lesssim \|\boldsymbol{b}\|_{H^2} \left(\|\Lambda_2^{\frac{1-\alpha}{2}} \nabla \boldsymbol{b}\|_{L^2} + \|\Lambda_2^{\frac{1-\alpha}{2}} \Lambda_1^{\frac{1}{2}+\epsilon} \nabla \boldsymbol{b}\|_{L^2} \right) \|\nabla^2 \boldsymbol{u}\|_{L^2}^{\frac{1}{2}} \|\Lambda_2^{\alpha} \nabla^2 \boldsymbol{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\boldsymbol{b}\|_{H^2}^2 \|\boldsymbol{u}\|_{H^2} + \|\boldsymbol{b}\|_{H^2}^2 \|\Lambda_2^{\alpha} \boldsymbol{u}\|_{H^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_4 &\lesssim \|\boldsymbol{b}\|_{H^2}^2 \|\boldsymbol{u}\|_{H^2} + \|\boldsymbol{b}\|_{H^2}^2 \|\Lambda_2^{\alpha} \boldsymbol{u}\|_{H^2} \\
&\lesssim (\|\boldsymbol{u}\|_{H^2} + \|\boldsymbol{b}\|_{H^2})(\|\boldsymbol{b}\|_{H^2}^2 + \|\Lambda_2^{\alpha} \boldsymbol{u}\|_{H^2}^2).
\end{aligned} \tag{3.7}
$$

Similar to I_4, we also have the following:

$$
I_5 \lesssim (\|\boldsymbol{u}\|_{H^2} + \|\boldsymbol{b}\|_{H^2})(\|\boldsymbol{b}\|_{H^2}^2 + \|\Lambda_2^{\alpha} \boldsymbol{u}\|_{H^2}^2). \tag{3.8}
$$

Now, plugging (3.3), (3.5), (3.6), (3.7), and (3.8) into (3.2), we arrive at the following:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla^2\boldsymbol{u}(t)\|_{L^2}^2+\|\nabla^2\boldsymbol{b}(t)\|_{L^2}^2)+\nu\|\nabla^2\Lambda_2^{\alpha}\boldsymbol{u}\|_{L^2}^2+\eta\|\nabla^2\boldsymbol{b}\|_{L^2}^2\\&\lesssim(\|\boldsymbol{u}\|_{H^2}+\|\boldsymbol{b}\|_{H^2})(\|\boldsymbol{b}\|_{H^2}^2+\|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2}^2+\|\partial_1\boldsymbol{u}\|_{H^1}^2).\end{aligned}\tag{3.9}$$

Therefore, by integrating (3.9) over time, one concludes the following:

$$\begin{aligned}&\|\nabla^2\boldsymbol{u}(t)\|_{L^2}^2+\|\nabla^2\boldsymbol{b}(t)\|_{L^2}^2+2\int_0^t(\nu\|\nabla^2\Lambda_2^{\alpha}\boldsymbol{u}\|_{L^2}^2+\eta\|\nabla^2\boldsymbol{b}\|_{L^2}^2)\,d\tau\\&\lesssim\|\boldsymbol{u}_0\|_{H^2}^2+\|\boldsymbol{b}_0\|_{H^2}^2+\int_0^t(\|\boldsymbol{u}\|_{H^2}+\|\boldsymbol{b}\|_{H^2})(\|\boldsymbol{b}\|_{H^2}^2+\|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2}^2+\|\partial_1\boldsymbol{u}\|_{H^1}^2)\,d\tau\\&\lesssim\|\boldsymbol{u}_0\|_{H^2}^2+\|\boldsymbol{b}_0\|_{H^2}^2+\sup_{0\le\tau\le t}(\|\boldsymbol{u}\|_{H^2}+\|\boldsymbol{b}\|_{H^2})\int_0^t(\|\boldsymbol{b}\|_{H^2}^2+\|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2}^2+\|\partial_1\boldsymbol{u}\|_{H^1}^2)\,d\tau,\end{aligned}\tag{3.10}$$

which, together with (3.1), yields the following:

$$\mathcal{E}_1(t)\lesssim\mathcal{E}_1(0)+\mathcal{E}_1(t)^{\frac{3}{2}}+\mathcal{E}_2(t)^{\frac{3}{2}}.\tag{3.11}$$

Next, we are in a position to bound $\|\partial_1\boldsymbol{u}\|_{H^1}$. To this end, we will present the estimations of $\|\partial_1\boldsymbol{u}\|_{L^2}$ and $\|\partial_1\boldsymbol{u}\|_{\dot{H}^1}$. From the second equation of (1.2), clearly,

$$\partial_1\boldsymbol{u}=\partial_t\boldsymbol{b}+\boldsymbol{u}\cdot\nabla\boldsymbol{b}+\eta\boldsymbol{b}-\boldsymbol{b}\cdot\nabla\boldsymbol{u}.\tag{3.12}$$

Dotting (3.12) with $\partial_1\boldsymbol{u}$ in L^2 and integrating over $\mathbb{R}^2$, one gets the following:

$$\begin{aligned}\|\partial_1\boldsymbol{u}\|_{L^2}^2=&\int\partial_1\boldsymbol{u}\cdot\partial_t\boldsymbol{b}\,d\boldsymbol{x}+\int\boldsymbol{u}\cdot\nabla\boldsymbol{b}\cdot\partial_1\boldsymbol{u}\,d\boldsymbol{x}\\&+\eta\int\boldsymbol{b}\cdot\partial_1\boldsymbol{u}\,d\boldsymbol{x}-\int\boldsymbol{b}\cdot\nabla\boldsymbol{u}\cdot\partial_1\boldsymbol{u}\,d\boldsymbol{x}=:N_1+N_2+N_3+N_4.\end{aligned}\tag{3.13}$$

For N_1, use the first equation of (1.2) to derive the following:

$$\begin{aligned}N_1=&\frac{d}{dt}\int\partial_1\boldsymbol{u}\cdot\boldsymbol{b}\,d\boldsymbol{x}-\int\boldsymbol{b}\cdot\partial_1(-\nu\Lambda_2^{2\alpha}\boldsymbol{u}+\boldsymbol{b}\cdot\nabla\boldsymbol{b}+\partial_1\boldsymbol{b}-\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nabla P)\,d\boldsymbol{x}\\=:&\,N_{11}+N_{12}+N_{13}+N_{14}+N_{15}+N_{16}.\end{aligned}$$

It is clear that $N_{16}=0$ in view of $\nabla\cdot\boldsymbol{b}=0$. By integrating by parts, the Hölder inequality, and the Young inequality,

$$\begin{aligned}N_{12}=&\,\nu\int\boldsymbol{b}\cdot\partial_1\Lambda_2^{2\alpha}\boldsymbol{u}\,d\boldsymbol{x}=-\nu\int\partial_1\boldsymbol{b}\cdot\Lambda_2^{2\alpha}\boldsymbol{u}\,d\boldsymbol{x}\\&\lesssim\|\partial_1\boldsymbol{b}\|_{L^2}\|\Lambda_2^{2\alpha}\boldsymbol{u}\|_{L^2}\lesssim\|\boldsymbol{b}\|_{H^2}\|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2}\lesssim\|\boldsymbol{b}\|_{H^2}^2+\|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2}^2.\end{aligned}$$

Similarly, with respect to N_{13} and N_{14},

$$N_{13}=-\int\boldsymbol{b}\cdot\partial_1(\boldsymbol{b}\cdot\nabla\boldsymbol{b})\,d\boldsymbol{x}=\int\partial_1\boldsymbol{b}\cdot(\boldsymbol{b}\cdot\nabla\boldsymbol{b})\,d\boldsymbol{x}$$

$$\lesssim \|\boldsymbol{b}\|_{L^\infty}\|\partial_1\boldsymbol{b}\|_{L^2}\|\nabla\boldsymbol{b}\|_{L^2} \lesssim \|\boldsymbol{b}\|_{H^2}^3,$$

and

$$N_{14} = -\int \boldsymbol{b}\cdot\partial_1^2\boldsymbol{b}\,d\boldsymbol{x} = \int \partial_1\boldsymbol{b}\cdot\partial_1\boldsymbol{b}\,d\boldsymbol{x} \lesssim \|\boldsymbol{b}\|_{H^2}^2.$$

By the Hölder inequality, integration by parts, and (2.5),

$$\begin{aligned}
N_{15} &= \int \boldsymbol{b}\cdot\partial_1(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\,d\boldsymbol{x} = -\int \partial_1\boldsymbol{b}\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\,d\boldsymbol{x}\\
&\lesssim \|\partial_1\boldsymbol{b}\|_{L^2_{x_2}L^2_{x_1}}\|\boldsymbol{u}\|_{L^\infty_{x_2}L^2_{x_1}}\|\nabla\boldsymbol{u}\|_{L^2_{x_2}L^\infty_{x_1}}\\
&\lesssim \|\partial_1\boldsymbol{b}\|_{L^2}\|\boldsymbol{u}\|_{L^2}^{\frac12}\|\partial_2\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac12}\|\partial_1\nabla\boldsymbol{u}\|_{L^2}^{\frac12}\\
&\lesssim \|\boldsymbol{b}\|_{H^2}\|\partial_1\boldsymbol{u}\|_{H^1}^{\frac12}\|\Lambda_2^\alpha\boldsymbol{u}\|_{H^2}^{\frac12}\|\boldsymbol{u}\|_{H^2},
\end{aligned}$$

where we have used the fact that

$$\|f\|_{L^2_{x_2}L^\infty_{x_1}(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac12}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac12}, \tag{3.14}$$

and its proof is similar to that of (2.5).

By a similar argument as the one for N_{15}, we can bound N_2 and N_4 as follows:

$$\begin{aligned}
N_2 &= \int \boldsymbol{u}\cdot\nabla\boldsymbol{b}\cdot\partial_1\boldsymbol{u}\,d\boldsymbol{x}\\
&\lesssim \|\partial_1\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{b}\|_{L^2}^{\frac12}\|\partial_2\nabla\boldsymbol{b}\|_{L^2}^{\frac12}\|\boldsymbol{u}\|_{L^2}^{\frac12}\|\partial_1\boldsymbol{u}\|_{L^2}^{\frac12}\\
&\lesssim \|\partial_1\boldsymbol{u}\|_{H^1}^{\frac32}\|\boldsymbol{b}\|_{H^2}^{\frac12}(\|\boldsymbol{b}\|_{H^2}+\|\boldsymbol{u}\|_{H^2}),
\end{aligned}$$

$$\begin{aligned}
N_4 &= -\int \boldsymbol{b}\cdot\nabla\boldsymbol{u}\cdot\partial_1\boldsymbol{u}\,d\boldsymbol{x}\\
&\lesssim \|\partial_1\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac12}\|\partial_1\nabla\boldsymbol{u}\|_{L^2}^{\frac12}\|\boldsymbol{b}\|_{L^2}^{\frac12}\|\partial_2\boldsymbol{b}\|_{L^2}^{\frac12}\\
&\lesssim \|\partial_1\boldsymbol{u}\|_{H^1}^{\frac32}\|\boldsymbol{b}\|_{H^2}^{\frac12}(\|\boldsymbol{b}\|_{H^2}+\|\boldsymbol{u}\|_{H^2}).
\end{aligned}$$

Additionally, it is obvious that

$$N_3 = \eta\int \boldsymbol{b}\cdot\partial_1\boldsymbol{u}\,d\boldsymbol{x} \;\lesssim \|\partial_1\boldsymbol{u}\|_{L^2}\|\boldsymbol{b}\|_{L^2} \lesssim \|\partial_1\boldsymbol{u}\|_{H^1}^2 + \|\boldsymbol{b}\|_{H^2}^2.$$

This completes the L^2 estimate of $\partial_1\boldsymbol{u}$. Next, we begin to establish the $\dot{H}^1$ estimate of $\partial_1\boldsymbol{u}$. From (3.12), we obtain the following:

$$\begin{aligned}
\|\nabla\partial_1\boldsymbol{u}\|_{L^2}^2 &= \int \nabla\partial_1\boldsymbol{u}\cdot\partial_t\nabla\boldsymbol{b}\,d\boldsymbol{x} + \int \nabla\partial_1\boldsymbol{u}\cdot\nabla(\boldsymbol{u}\cdot\nabla\boldsymbol{b})\,d\boldsymbol{x}\\
&\quad + \eta\int \nabla\partial_1\boldsymbol{u}\cdot\nabla\boldsymbol{b}\,d\boldsymbol{x} - \int \nabla\partial_1\boldsymbol{u}\cdot\nabla(\boldsymbol{b}\cdot\nabla\boldsymbol{u})\,d\boldsymbol{x}
\end{aligned}$$

$$=: M_1 + M_2 + M_3 + M_4. \tag{3.15}$$

For M_1, integrating by parts and the equation for $\boldsymbol{u}$ in (1.2) give the following:

$$\begin{aligned} M_1 &= \frac{d}{dt}\int \nabla\partial_1\boldsymbol{u}\cdot\nabla\boldsymbol{b}\,d\boldsymbol{x} - \int \nabla\boldsymbol{b}\cdot\nabla\partial_1(-\nu\Lambda_2^{2\alpha}\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{b} + \partial_1\boldsymbol{b} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla P)\,d\boldsymbol{x} \\ &=: M_{11} + M_{12} + M_{13} + M_{14} + M_{15} + M_{16}. \end{aligned}$$

Clearly, $M_{16} = 0$. Through integrating by parts, combined with the Hölder inequality and the Young inequality, one has the following:

$$\begin{aligned} M_{12} &= \nu\int \nabla\boldsymbol{b}\cdot\nabla\partial_1\Lambda_2^{2\alpha}\boldsymbol{u}\,d\boldsymbol{x} = -\nu\int \partial_1\nabla\boldsymbol{b}\cdot\Lambda_2^{2\alpha}\nabla\boldsymbol{u}\,d\boldsymbol{x} \\ &\lesssim \|\partial_1\nabla\boldsymbol{b}\|_{L^2}\|\Lambda_2^{2\alpha}\nabla\boldsymbol{u}\|_{L^2} \lesssim \|\boldsymbol{b}\|_{H^2}\|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2} \lesssim \|\boldsymbol{b}\|_{H^2}^2 + \|\Lambda_2^{\alpha}\boldsymbol{u}\|_{H^2}^2. \end{aligned}$$

By the anisotropic inequality (3.14), we arrive at the following:

$$\begin{aligned} M_{13} &= -\int \nabla\boldsymbol{b}\cdot\nabla\partial_1(\boldsymbol{b}\cdot\nabla\boldsymbol{b})\,d\boldsymbol{x} = \int \partial_1\nabla\boldsymbol{b}\cdot\nabla(\boldsymbol{b}\cdot\nabla\boldsymbol{b})\,d\boldsymbol{x} \\ &= \int \partial_1\nabla\boldsymbol{b}\cdot\nabla\boldsymbol{b}\cdot\nabla\boldsymbol{b}\,d\boldsymbol{x} + \int \partial_1\nabla\boldsymbol{b}\cdot\boldsymbol{b}\cdot\nabla^2\boldsymbol{b}\,d\boldsymbol{x} \\ &\lesssim \|\partial_1\nabla\boldsymbol{b}\|_{L^2}\|\nabla\boldsymbol{b}\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\boldsymbol{b}\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{b}\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\boldsymbol{b}\|_{L^2}^{\frac{1}{2}} \\ &\quad + \|\partial_1\nabla\boldsymbol{b}\|_{L^2}\|\nabla^2\boldsymbol{b}\|_{L^2}\|\boldsymbol{b}\|_{L^\infty} \\ &\lesssim \|\boldsymbol{b}\|_{H^2}^3, \end{aligned}$$

$$\begin{aligned} M_{15} &= \int \nabla\boldsymbol{b}\cdot\nabla\partial_1(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\,d\boldsymbol{x} = -\int \Delta\boldsymbol{b}\cdot\partial_1(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\,d\boldsymbol{x} \\ &= -\int \Delta\boldsymbol{b}\cdot\partial_1\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,d\boldsymbol{x} - \int \Delta\boldsymbol{b}\cdot\boldsymbol{u}\cdot\partial_1\nabla\boldsymbol{u}\,d\boldsymbol{x} \\ &\lesssim \|\Delta\boldsymbol{b}\|_{L^2}\|\partial_1\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}} \\ &\quad + \|\Delta\boldsymbol{b}\|_{L^2}\|\boldsymbol{u}\|_{L^\infty}\|\partial_1\nabla\boldsymbol{u}\|_{L^2} \\ &\lesssim \|\boldsymbol{b}\|_{H^2}\|\partial_1\boldsymbol{u}\|_{H^1}\|\boldsymbol{u}\|_{H^2}. \end{aligned}$$

Clearly,

$$M_{14} = -\int \nabla\boldsymbol{b}\cdot\nabla\partial_1^2\boldsymbol{b}\,d\boldsymbol{x} = \int \partial_1\nabla\boldsymbol{b}\cdot\partial_1\nabla\boldsymbol{b}\,d\boldsymbol{x} \lesssim \|\partial_1\nabla\boldsymbol{b}\|_{L^2}^2 \lesssim \|\boldsymbol{b}\|_{H^2}^2.$$

M_2 and M_4 can be bounded similarly as for M_{15}:

$$\begin{aligned} M_2 &= \int \nabla\partial_1\boldsymbol{u}\cdot\nabla(\boldsymbol{u}\cdot\nabla\boldsymbol{b})\,d\boldsymbol{x} \\ &= \int \nabla\partial_1\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{b}\,d\boldsymbol{x} + \int \nabla\partial_1\boldsymbol{u}\cdot\boldsymbol{u}\cdot\nabla^2\boldsymbol{b}\,d\boldsymbol{x} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\nabla\partial_1\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{b}\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\boldsymbol{b}\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\nabla\partial_1\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{L^\infty}\|\nabla^2\boldsymbol{b}\|_{L^2} \\
&\lesssim \|\partial_1\boldsymbol{u}\|_{H^1}\|\boldsymbol{u}\|_{H^2}\|\boldsymbol{b}\|_{H^2},
\end{aligned}
$$

$$
\begin{aligned}
M_4 &= -\int \nabla\partial_1\boldsymbol{u}\cdot\nabla(\boldsymbol{b}\cdot\nabla\boldsymbol{u})\,d\boldsymbol{x} \\
&= -\int \nabla\partial_1\boldsymbol{u}\cdot\nabla\boldsymbol{b}\cdot\nabla\boldsymbol{u}\,d\boldsymbol{x} + \int \nabla\partial_1\boldsymbol{u}\cdot\boldsymbol{b}\cdot\nabla^2\boldsymbol{u}\,d\boldsymbol{x} \\
&\lesssim \|\nabla\partial_1\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{b}\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\boldsymbol{b}\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\nabla\partial_1\boldsymbol{u}\|_{L^2}\|\boldsymbol{b}\|_{L^\infty}\|\nabla^2\boldsymbol{u}\|_{L^2} \\
&\lesssim \|\partial_1\boldsymbol{u}\|_{H^1}\|\boldsymbol{u}\|_{H^2}\|\boldsymbol{b}\|_{H^2}.
\end{aligned}
$$

Obviously,

$$
M_3 = \eta\int \nabla\partial_1\boldsymbol{u}\cdot\nabla\boldsymbol{b}\,d\boldsymbol{x} \lesssim \|\nabla\partial_1\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{b}\|_{L^2} \lesssim \|\partial_1\boldsymbol{u}\|_{H^1}^2 + \|\boldsymbol{b}\|_{H^2}^2.
$$

Adding (3.13) to (3.15), integrating on $[0,t]$, taking the bounds above, and using the Hölder inequality to the time integrals, we achieve the following:

$$
\mathcal{E}_2(t) \lesssim \mathcal{E}_1(0) + \mathcal{E}_1(t) + \mathcal{E}_1(t)^{\frac{3}{2}} + \mathcal{E}_2(t)^{\frac{3}{2}}. \tag{3.16}
$$

More precisely, the time integrals of N_{11} and M_{11} are bounded by the following:

$$
\begin{aligned}
\int_0^t N_{11}\,d\tau &= \int \boldsymbol{b}(\boldsymbol{x},t)\cdot\partial_1\boldsymbol{u}(\boldsymbol{x},t)\,d\boldsymbol{x} - \int \boldsymbol{b}(\boldsymbol{x},0)\cdot\partial_1\boldsymbol{u}(\boldsymbol{x},0)\,d\boldsymbol{x} \\
&\lesssim \|\boldsymbol{b}_0\|_{H^2}\|\boldsymbol{u}_0\|_{H^2} + \|\boldsymbol{b}\|_{H^2}\|\boldsymbol{u}\|_{H^2} \\
&\lesssim \mathcal{E}_1(0) + \mathcal{E}_1(t),
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t M_{11}\,d\tau &= \int \nabla\partial_1\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{b}(\boldsymbol{x},t)\,d\boldsymbol{x} - \int \nabla\partial_1\boldsymbol{u}(\boldsymbol{x},0)\cdot\nabla\boldsymbol{b}(\boldsymbol{x},0)\,d\boldsymbol{x} \\
&\lesssim \|\boldsymbol{b}_0\|_{H^2}\|\boldsymbol{u}_0\|_{H^2} + \|\boldsymbol{b}\|_{H^2}\|\boldsymbol{u}\|_{H^2} \\
&\lesssim \mathcal{E}_1(0) + \mathcal{E}_1(t).
\end{aligned}
$$

Clearly,

$$
\int_0^t N_{12} + N_{14}\,d\tau \lesssim \int_0^t \|\boldsymbol{b}\|_{H^2}^2 + \|\Lambda_2^\alpha\boldsymbol{u}\|_{H^2}^2\,d\tau \lesssim \mathcal{E}_1(t),
$$

and

$$
\int_0^t N_3\,d\tau \lesssim \int_0^t \|\partial_1\boldsymbol{u}\|_{H^1}^2 + \|\boldsymbol{b}\|_{H^2}^2\,d\tau \lesssim \mathcal{E}_1(t) + \mathcal{E}_2(t).
$$

By the Hölder inequality, we obtain the following:

$$\int_0^t N_{13}\,d\tau \lesssim \int_0^t \|\boldsymbol{b}\|_{H^2}^3\,d\tau \lesssim \sup_{0\le\tau\le t}\|\boldsymbol{b}\|_{H^2}\int_0^t \|\boldsymbol{b}\|_{H^2}^2\,d\tau \lesssim \mathcal{E}_1(t)^{\frac{3}{2}}.$$

By the Hölder inequality and the Young inequality, we infer the following:

$$\begin{aligned}\int_0^t N_{15}\,d\tau &\lesssim \int_0^t \|\boldsymbol{b}\|_{H^2}\|\partial_1\boldsymbol{u}\|_{H^1}^{\frac{1}{2}}\|\Lambda_2^\alpha\boldsymbol{u}\|_{H^2}^{\frac{1}{2}}\|\boldsymbol{u}\|_{H^2}\,d\tau\\ &\lesssim \sup_{0\le\tau\le t}\|\boldsymbol{u}\|_{H^2}\int_0^t(\|\boldsymbol{b}\|_{H^2}^2+\|\partial_1\boldsymbol{u}\|_{H^1}^2+\|\Lambda_2^\alpha\boldsymbol{u}\|_{H^2}^2)\,d\tau\\ &\lesssim \mathcal{E}_1(t)^{\frac{1}{2}}(\mathcal{E}_1(t)+\mathcal{E}_2(t))\\ &\lesssim \mathcal{E}_1(t)^{\frac{3}{2}}+\mathcal{E}_2(t)^{\frac{3}{2}},\end{aligned}$$

$$\begin{aligned}\int_0^t N_2\,d\tau &\lesssim \int_0^t \|\partial_1\boldsymbol{u}\|_{H^1}^{\frac{3}{2}}\|\boldsymbol{b}\|_{H^2}^{\frac{1}{2}}(\|\boldsymbol{b}\|_{H^2}+\|\boldsymbol{u}\|_{H^2})\,d\tau\\ &\lesssim \sup_{0\le\tau\le t}(\|\boldsymbol{u}\|_{H^2}+\|\boldsymbol{b}\|_{H^2})\int_0^t(\|\partial_1\boldsymbol{u}\|_{H^1}^2+\|\boldsymbol{b}\|_{H^2}^2)\,d\tau\\ &\lesssim \mathcal{E}_1(t)^{\frac{1}{2}}(\mathcal{E}_1(t)+\mathcal{E}_2(t))\\ &\lesssim \mathcal{E}_1(t)^{\frac{3}{2}}+\mathcal{E}_2(t)^{\frac{3}{2}}.\end{aligned}$$

Similar to N_2,

$$\int_0^t N_4\,d\tau \lesssim \mathcal{E}_1(t)^{\frac{3}{2}}+\mathcal{E}_2(t)^{\frac{3}{2}}.$$

Furthermore, by the same estimates as above, we obtain the following:

$$\int_0^t M_{12}+M_{14}\,d\tau \lesssim \mathcal{E}_1(t),\quad \int_0^t M_{13}\,d\tau \lesssim \mathcal{E}_1(t)^{\frac{3}{2}},\quad \int_0^t M_{15}\,d\tau \lesssim \mathcal{E}_1(t)^{\frac{3}{2}}+\mathcal{E}_2(t)^{\frac{3}{2}},$$
$$\int_0^t M_2+M_4\,d\tau \lesssim \mathcal{E}_1(t)^{\frac{3}{2}}+\mathcal{E}_2(t)^{\frac{3}{2}},\quad \int_0^t M_3\,d\tau \lesssim \mathcal{E}_1(t)+\mathcal{E}_2(t).$$

Combining all the bounds above gives

$$\mathcal{E}_2(t) \lesssim \mathcal{E}_1(0)+\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_1(t)^{\frac{3}{2}}+\mathcal{E}_2(t)^{\frac{3}{2}},$$

which yields (3.16). Adding (3.11) to (3.16), and according to the definition (1.5), we can obtain the following:

$$\mathcal{E}(t) \le \mathcal{E}(0)+C\mathcal{E}(t)^{\frac{3}{2}}.$$

The bootstrapping argument shows that if $\mathcal{E}(0) < \epsilon^2$, then there exists a $\epsilon > 0$ such that

$$\mathcal{E}(t) \le C_0^2\epsilon^2,$$

for $C_0 > 0$ and any $t > 0$. The proof of Theorem 1.1 is now completed. □

Author contributions

Qunyi Bie: Writing-review & editing, methodology, funding acquisition; Qiaoqi Yang: Writing-original draft, formal analysis; Xianbao Yuan: Writing-review & editing, methodology, conceptualization.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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