Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 18(3): 553–592.
DOI: 10.3934/cam.2026023
Received: 19 September 2025
Revised: 09 April 2026
Accepted: 06 May 2026
Published: 15 July 2026

Research article

The vacuum free boundary problem for the spherically symmetric non-isentropic compressible Euler equations

La-Su Mai* **and Jiarong Dong**

School of Mathematical Science, Inner Mongolia University, Hohhot 010021, P.R. China

* **Correspondence:** Email: 111983046@imu.edu.cn; Tel:0471-4991650; Fax: 0471-4991650.

Abstract: This paper investigates the well-posedness of local smooth solutions for the vacuum free boundary problem of the spherically symmetric non-isentropic compressible Euler equations. At the free boundary, the system degenerates into a hyperbolic system while exhibiting a singularity at the spherical center. To overcome these difficulties, a Lagrangian coordinate transformation is introduced, converting the problem into an initial-boundary value problem with fixed boundaries. Within this framework, the degeneracy of the system can be explicitly expressed by the function $\rho_0^{\gamma-1}(\gamma > 1)$ of the initial density, whose value equals the distance function near the boundaries. Consequently, by employing the distance-dependent Hardy's inequality and weighted Sobolev spaces, the degeneracy and singularity of the system are successfully resolved, and a priori estimates for the local smooth solutions are established. Specifically, by building upon high-order time derivative estimates, the high-order spatial derivative estimates are rigorously proved via mathematical induction. Crucially, the influence of entropy introduces additional higher-order terms, necessitating boundedness of initial entropy derivatives up to a certain order. Finally, we establish the existence and uniqueness of local smooth solutions.

Keywords: Non-isentropic Euler equations; spherical symmetry; local well-posedness; degeneracy and singularity
Mathematics Subject Classification: 35Q35, 35A01

1. Introduction

This paper studies the free boundary problem associated with the spherically symmetric, non-isentropic compressible Euler equations in the presence of a vacuum:

$$\begin{cases} \partial_t\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0, \\ \rho\left(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p = 0, \\ \partial_t S + \mathbf{u}\cdot\nabla S = 0. \end{cases} \tag{1.1}$$

Let ρ, $\mathbf{u}$, and S represent the electron density, velocity, and entropy, respectively. The pressure is defined by the relation $p(\rho) = \rho^\gamma e^S$, where $\gamma > 1$ denotes the adiabatic index (characteristic of a polytropic gas). Our analysis concentrates on configurations where ρ approaches zero at the boundary, corresponding to a vacuum condition at the domain edge.

By introducing the spherical symmetry transformation

$$\begin{cases} \rho(\overrightarrow{x}, t) = \rho(r, t), \ r = |\overrightarrow{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \\ \mathbf{u}(\overrightarrow{x}, t) = \dfrac{\overrightarrow{x}}{r}\omega(r, t), \ t > 0, \end{cases} \tag{1.2}$$

system (1.1) admits the following restructured formulation:

$$\begin{cases} \partial_t \left(r^2 \rho\right) + \partial_r \left(r^2 \rho \omega\right) = 0, \\ \rho \omega_t + \rho \omega \omega_r + \left(\rho^\gamma e^S\right)_r = 0, \\ S_t + \omega S_r = 0. \end{cases} \tag{1.3}$$

We analyze the free boundary problem associated with system (1.3) on the time-evolving domain $(0, R(t)) \times (0, T]$, supplemented by the prescribed boundary conditions and initial data

$$\begin{cases} \rho > 0, \ \text{in } (0, R(t)), \\ \rho(R(t), t) = 0, \ \omega(0, t) = 0, \\ \dfrac{dR(t)}{dt} = \omega(R(t), t), \ R(0) = 1, \\ (\rho, \omega, S)(x, 0) = (\rho_0(x), \omega_0(x), S_0(x)), \rho_0(x) > 0, \ \text{in } (0, 1), \\ -\infty < \dfrac{\partial \rho_0^{\gamma-1}}{\partial x} < 0, \ \text{on } x = 1. \end{cases} \tag{1.4}$$

Condition $(1.4)_2$ necessitates the existence of a vacuum state in the electron density field, causing system (1.1) to become degenerate due to the vanishing of density. Correspondingly, condition $(1.4)_5$ dictates the asymptotic behavior $\rho_0^{\gamma-1} \sim d(x)$ in the vicinity of $x = 1$, which is called the physical vacuum condition in compressible fluid systems (cf. [1–7]).

For the vacuum free boundary problem, breakthrough results have been established concerning both local and global smooth solutions to the compressible Euler equations [1, 8–16]. While the proof of local smooth solutions for the non-isentropic compressible Euler equations with vacuum free boundaries follows similar arguments to the isentropic case, the careful analysis of entropy dependence becomes crucial when establishing a priori estimates (see [15] for results on spherically symmetric compressible Euler equations with vacuum free boundaries). A fundamental distinction emerges between isentropic and non-isentropic compressible Euler equations when constructing global a priori estimates for vacuum free boundary problems. Recently, Rickard, Hadžić, and Jang [17] made significant progress by overcoming key technical challenges; they proved the existence of global smooth solutions for all adiabatic exponents $\gamma > 1$ through perturbations around a broad class of non-isentropic affine solutions. However, existing research on local smooth solutions for the non-isentropic compressible Euler equations remains relatively limited. Recently, Liu and Luo [18], combining Eulerian and Lagrangian coordinates, proved the local well-posedness in the sense of Hadamard for the vacuum free boundary problem of the three-dimensional non-isentropic compressible Euler equations in low-regularity weighted Sobolev spaces.

This paper develops the local well-posedness theory for smooth solutions to the free boundary problem (1.3) and (1.4). Employing the Lagrangian transformation (2.1), we reformulate the system as the initial boundary value problem (2.8) and (2.9). For $\gamma > 1$, we derive a priori estimates using Hardy-type inequalities within weighted Sobolev spaces. Existence and uniqueness are then established via a systematic degenerate parabolic regularization framework, supported by precise energy estimates, which is formulated in Theorem 2.1.

The main difficulties handled in this paper arise from the following three aspects: (1) If the density vanishes at the free boundary, then when system (1.1) is written as a symmetric hyperbolic system, the coefficient matrix becomes degenerate. Consequently, the standard Friedrichs-Lax-Kato theory of symmetric hyperbolic systems, which critically relies on a positive lower bound of the density, is no longer applicable [6, 19]. To overcome this obstacle, we employ the physical vacuum condition $(1.4)_5$ to equivalently characterize the initial density function $\rho_0^{\gamma-1}$ near the free boundary as the distance function $d(x)$. This function serves as a weight in the Sobolev embedding inequality, establishing a connection between the L^2-norm and weighted Sobolev spaces. In Lagrangian coordinates, the coefficients of the equation are exactly expressed by $\rho_0^{\gamma-1}$ (see equation (2.5)). Therefore, we establish the corresponding a priori estimates by using Hardy's inequality and weighted Sobolev spaces. Building upon this structure, we apply Hardy's inequality within the weighted Sobolev space framework to resolve this difficulty. (2) Given the singularity of system (1.1) at the center $x = 0$, we establish key a priori estimates at this singular point by enhancing the spatial regularity of the velocity field and incorporating Hardy's inequality. (3) The entropy effect introduces additional higher-order derivative terms, necessitating boundedness of specific derivatives of the initial entropy. More precisely, under isentropic conditions, e^{S_0} is constant and can be factored out from the second term of equation (2.7); under non-isentropic conditions, S_0 depends on the spatial variable x and cannot be factored out, which leads to additional terms such as terms 6, 12, and 13 in (5.3) in the higher-order spatial derivatives arising in the elliptic estimates, all of which vanish in the isentropic case.

The subsequent sections of this paper are organized as follows. In Section 2, we introduce the Lagrangian transformation, which reformulates the free boundary problem as an initial-boundary value problem, and the main theorem is presented. Sections 3–5 are devoted to establishing higher-order energy estimates for temporal and spatial derivatives, respectively.

Notations and weighted Sobolev spaces. For any positive integers $i \leq k$, we denote $C_k^i = \frac{k!}{i!(k-i)!}$. Set $I = (0, 1)$, and let $H^k(I)$ denote the usual Sobolev spaces with the norm $\|\cdot\|_k$, especially $\|\cdot\|_{L^2(I)} = \|\cdot\|_0$. For a real number l, the Sobolev spaces $H^l(I)$ and the norm $\|\cdot\|_l$ are defined by interpolation. The function space $L^\infty(I)$ is simplified as L^∞.

Let $d(x)$ be the distance function to boundary ∂I,$d(x) = dist(x, \partial I)$. For any $a > 0$ and nonnegative b, the weighted Sobolev space $H^{a,b}(I)$ is given by $H^{a,b}(I) := \{d^{\frac{a}{2}} F \in L^2(I) : \int_I d^a |\partial_x^k F|^2 dx < \infty,\ 0 \leq k \leq b\}$ with the norm

$$\|F\|^2_{H^{a,b}(I)} := \sum_0^b \int_I d^a |\partial_x^k F|^2 dx. \tag{1.5}$$

Then, the following embedding holds: $H^{a,b}(I) \hookrightarrow H^{b-a/2}(I)$, with the estimate

$$\|F\|_{H^{b-a/2}(I)} \leq C_0\|F\|_{H^{a,b}(I)}. \tag{1.6}$$

In addition, for any $\delta > 0$, we also have the following embedding: $H^{1/2+\delta}(I) \hookrightarrow L^\infty(I)$, with the estimate

$$\|F\|_{L^\infty(I)} \leq C(\delta)\|F\|_{H^{1/2+\delta}(I)}. \tag{1.7}$$

Hardy's type inequality. [15] Consider a fixed real parameter $k > 1$. Suppose F is a function defined in a neighborhood of the origin, and assume there exists some $\delta > 0$ such that $\int_0^\delta x^k(F^2 + F_x^2)\, dx < \infty$. Then F obeys the following bound:

$$\int_0^\delta x^{k-2}F^2\, dx \leq C(\delta, k)\int_0^\delta x^k(F^2 + F_x^2)\, dx, \tag{1.8}$$

where $C(\delta, k)$ denotes a constant depending only on δ and k.

The function F satisfies the prescribed condition in a neighborhood of the boundary

$$\int_{\frac{1}{2}}^1 (1-x)^{k-2}F^2\, dx \leq C_0\int_{\frac{1}{2}}^1 (1-x)^k(F^2 + F_x^2)\, dx, \tag{1.9}$$

where C_0 denotes a constant depending only on k.

It follows from (1.8) and (1.9) that, for $k > 1$, the following estimate holds:

$$\int_I d^{k-2}(x)F^2\, dx \leq C_0\int_I d^k(x)(F^2 + F_x^2)\, dx. \tag{1.10}$$

2. Main results

We develop a Lagrangian transformation that converts the time-evolving free boundary problem on $(0, R(t))$ to a fixed-domain initial-boundary value problem on (1.1) as follows:

$$\partial_t r(x,t) = \omega\left(r(x,t), t\right),\ t > 0,\ r(x,0) = x,\ x \in (0,1)\,. \tag{2.1}$$

Defining by

$$\begin{cases} v(x,t) = \omega\left(r(x,t), t\right), \\ f(x,t) = \rho\left(r(x,t), t\right), \end{cases}$$

the first equation of (1.3) can be expressed as follows:

$$f = \left(\frac{x}{r}\right)^2 \frac{1}{r_x}\rho_0. \tag{2.2}$$

Consequently, combining this result with (1.3) yields

$$\begin{cases} \left(\dfrac{x}{r}\right)^2 \rho_0\partial_t v + \partial_x\left[\left(\dfrac{x}{r}\right)^{2\gamma}\left(\dfrac{1}{r_x}\right)^\gamma \rho_0^\gamma e^{S_0}\right] = 0, \\ S = S_0. \end{cases} \tag{2.3}$$

Substitute the results of (2.2) into $(2.3)_1$, and introduce a new variable

$$\sigma(x) = \rho_0^{\gamma-1}. \tag{2.4}$$

Through straightforward algebraic substitution and simplification, we obtain

$$\sigma^{\alpha}\partial_t v + \left(\frac{r}{x}\right)^2 \partial_x \left[\sigma^{\alpha+1}\left(\frac{x}{r}\right)^{2\gamma}\left(\frac{1}{r_x}\right)^{\gamma} e^{S_0}\right] = 0. \tag{2.5}$$

Let ξ be a newly defined variable where

$$\xi = \frac{r}{x}. \tag{2.6}$$

Consequently, (2.5) transforms into the following system:

$$x\sigma^{\alpha}\xi_{tt} + \xi^2\partial_x\left\{\sigma^{\alpha+1}\xi^{-2\gamma}(\xi + x\xi_x)^{-\gamma}e^{S_0}\right\} = 0. \tag{2.7}$$

With the identification $u = \xi_t$, differentiating (2.7) with respect to time yields

$$\begin{aligned} &x\sigma^{\alpha}u_{tt} + 2\xi u\partial_x\left\{\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma}(\xi + x\xi_x)^{-\gamma}\right\} - 2\gamma\xi^2\partial_x\left\{\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma-1}(\xi + x\xi_x)^{-\gamma}u\right\} \\ &- \gamma\xi^2\partial_x\left\{\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma}(\xi + x\xi_x)^{-\gamma-1}(u + xu_x)\right\} = 0. \end{aligned} \tag{2.8}$$

The initial and boundary conditions (1.4) transform as follows:

$$\begin{cases} \rho_0 > 0, \text{ in } (0,1), \\ \rho(1,t) = 0, \ u(0,t) = 0, \\ (r,u,S)(x,0) = (x, u_0(x), S_0(x)), \ x \in (0,1), \\ -\infty < \dfrac{\partial\rho_0^{\gamma-1}}{\partial x} < 0, \text{ at } x = 1. \end{cases} \tag{2.9}$$

Let $I = (0,1)$, with $\sigma(x) = \rho_0^{\gamma-1}$ and $\alpha = \frac{1}{\gamma-1}$. Denote the integer part of α by $[\alpha]$. We define the high-order energy functional $E(t)$ as follows:

$$E(t) = \sum_{j=0}^{3+[\alpha]}\left(E_j(t) + \sum_{i=1}^{3+[\alpha]-j} E_{j,i}(t)\right), \tag{2.10}$$

and

$$\begin{aligned} E_j(t) &= \int_I \left[x^4\sigma^{\alpha}(\partial_t^{j+1}u)^2 + x^2\sigma^{\alpha+1}(\partial_t^j u)^2 + x^4\sigma^{\alpha+1}(\partial_t^j u_x)^2\right](x,t)dx, \\ E_{j,i}(t) &= \int_I \left[x^2\sigma^{\alpha+i-1}(\partial_t^j\partial_x^i u)^2 + x^4\sigma^{\alpha+i+1}(\partial_t^j\partial_x^{i+1}u)^2\right](x,t)dx, \end{aligned} \tag{2.11}$$

with the following compatibility conditions for $k = 1, 2, \ldots, 3 + [\alpha]$:

$$\partial_t^k u(x,0) := \partial_t^{k-1}\left[-\frac{(\alpha+1)(\rho_0^{\gamma-1})_x e^{S_0} + \rho_0^{\gamma-1}\partial_x(e^{S_0})}{x}\right]|_{t=0}, \tag{2.12}$$

where S_0 is given by (2.3). In this paper, we denote P as a generic polynomial function of its argument and $P(0) = P(E(0))$.

Theorem 2.1. *Let the initial density* $\rho_0 \in C^{3+[\alpha]}[0,1]$, *the initial boundary condition* (2.9)*, and the compatibility conditions* (2.12) *hold. If the following estimates hold:*

$$E(0) < +\infty\,,\ \ \|\rho_0\|_{C^{3+[\alpha]}[0,1]} \leq M_0, \tag{2.13}$$

and

$$\sup_{x\in I} \|\partial_x^k S_0\|^2_{L^\infty(I)} \leq M_0\ ,\ for\ \ 0 \leq k \leq [\alpha]+3,$$

with M_0 *being a positive constant, then there is a positive constant* T *such that the initial boundary value problem* (2.8)–(2.9) *admits a unique smooth solution* (ξ, u) *in* $(0,1)\times(0,T]$ *satisfying*

$$\sup_{t\in[0,T]} E(t) \leq 2P(0). \tag{2.14}$$

Proof. Employing analytical techniques from [6, 8, 15, 20], we first obtain the a priori estimate (2.14) (using Lemmas 4.1 and 5.1). This estimate then facilitates a proof of local smooth solution existence for the initial boundary value problem (2.8)–(2.9) via degenerate parabolic regularization and the fixed-point theorem. Energy estimates subsequently verify uniqueness. Detailed proofs are omitted here. □

Remark 2.2. *Under the physical vacuum condition* $(1.4)_5$*, we can take the initial density in the simplified form* $\rho_0(x) = (1-x^2)^{\frac{1}{\gamma-1}}$*, which implies that as the adiabatic index* γ *approaches* 1*, the density degeneracy near the free boundary becomes increasingly pronounced.*

3. Some preliminaries

Let (ξ, u) be a smooth solution to the system (2.8)–(2.9) defined on the spacetime domain $(0,1)\times(0,T]$. Under the a priori assumption (3.2), the fundamental theorem of calculus guarantees the existence of a temporal parameter $T \in (0,1)$ such that for all $(x,t) \in (0,1)\times(0,T]$, the following estimates hold:

$$\frac{1}{2} \leq \xi_x(x,t) \leq \frac{3}{2}\ . \tag{3.1}$$

Within the framework of a priori estimates, the crucial estimate (3.2) will be used repeatedly.

Lemma 3.1. *Assume* $T > 0$*, and let* (ξ, u) *be a smooth solution to the initial boundary value problem* (2.8)–(2.9) *on the domain* $(0,1)\times(0,T]$. *If the energy function* $E(t)$ *remains finite, then the following holds:*

$$\begin{aligned}&\sum_{j=0}^{1}\left\|\partial_t^j u\right\|^2_{L^\infty(I)} + \|u_x\|^2_{L^\infty(I)} + \sum_{i+j\leq 1+[\alpha],2i+j\geq 2}\left\|\sigma^{\frac{2i+j-2}{2}}\partial_t^j\partial_x^i u\right\|^2_{L^\infty(I)}\\ &+\sum_{i+j=2+[\alpha]}\left\|x\sigma^{\frac{2i+j-2}{2}}\partial_t^j\partial_x^i u\right\|^2_{L^\infty(I)} + \sum_{i+j=3+[\alpha]}\left\|x^2\sigma^{\frac{2i+j-2}{2}}\partial_t^j\partial_x^i u\right\|^2_{L^\infty(I)} \lesssim E(t).\end{aligned} \tag{3.2}$$

Proof. The proof consists of two parts, namely the estimate near the origin (3.7) and the boundary estimate (3.11) and (3.12). For this goal, we define

$$I = I_0 \cup I_b, \tag{3.3}$$

$$I_0 = (0, 1/2)\ ,\ I_b = [1/2, 1).$$

To establish the estimate $E_{j,0}(t) \lesssim E_j(t)$ for $j = 0, \ldots, 3 + [\alpha]$, we apply Hardy's inequality to deduce

$$\int_{I_b} \sigma^{\alpha-1}(\partial_t^j u)^2 dx \lesssim \int_{I_b} x^2\sigma^{\alpha+1}(\partial_t^j u)^2 dx + \int_{I_b} x^4\sigma^{\alpha+1}(\partial_t^j u_x)^2 dx \lesssim E_j(t). \tag{3.4}$$

Therefore, it follows that

$$\begin{aligned} E_{j,0}(t) &= \int_I \left[x^2\sigma^{\alpha-1}(\partial_t^j u)^2 + x^4\sigma^{\alpha+1}(\partial_t^j u_x)^2\right] dx \\ &\lesssim \int_{I_0} x^2\sigma^{\alpha-1}(\partial_t^j u)^2 dx + \int_{I_b} \sigma^{\alpha-1}(\partial_t^j u)^2 dx + \int_I x^4\sigma^{\alpha+1}(\partial_t^j u_x)^2 dx \\ &\lesssim E_j(t),\ \text{for } j = 0, \ldots, 3 + [\alpha]. \end{aligned} \tag{3.5}$$

Thus, we have

$$\sum_{j=0}^{3+[\alpha]} \left(E_j(t) + \sum_{i=0}^{3+[\alpha]-j} E_{j,i}(t) \right) \lesssim E(t). \tag{3.6}$$

Step 1: (Near the origin)

$$\begin{aligned} &\sum_{i+j\le 1+[\alpha]} \left\|\partial_t^j\partial_x^i u\right\|^2_{L^\infty(I_0)} + \sum_{i+j=2+[\alpha]} \left\|x\partial_t^j\partial_x^i u\right\|^2_{L^\infty(I_0)} \\ &+ \sum_{i+j=3+[\alpha]} \left\|x^2\partial_t^j\partial_x^i u\right\|^2_{L^\infty(I_0)} \lesssim E(t). \end{aligned} \tag{3.7}$$

To control the first term in (3.7) for indices $j = 0, 1, \ldots, 1 + [\alpha]$ and i satisfying $i + j = 1 + [\alpha]$,

$$\begin{aligned} &\sum_{i=0}^{1+[\alpha]-j} \left\|\partial_t^j\partial_x^i u\right\|^2_{L^\infty(I_0)} \lesssim \sum_{i=0}^{1+[\alpha]-j} \left\|\partial_t^j\partial_x^i u\right\|^2_{H^1(I_0)} \lesssim \left\|\partial_t^j u\right\|_{H^{3+[\alpha]-j-1}(I_0)} \\ &\lesssim \left\|\partial_t^j u\right\|_{H^{2,3+[\alpha]-j}(I_0)} \lesssim \sum_{i=0}^{3+[\alpha]-j} \int_{I_0} x^2(\partial_t^j\partial_x^i u)^2 dx \\ &\lesssim \sum_{i=0}^{3+[\alpha]-j} \int_I \left[x^2\sigma^{\alpha+i-1}(\partial_t^j\partial_x^i u)^2 + x^4\sigma^{\alpha+i+1}(\partial_t^j\partial_x^{i+1} u)^2\right](x,t) dx \\ &\lesssim \sum_{i=0}^{3+[\alpha]-j} E_{j,i}(t) \lesssim E(t). \end{aligned} \tag{3.8}$$

Regarding the second term in (3.7), for indices $j = 0, 1, \ldots, 2 + [\alpha]$ and i satisfying $i + j = 2 + [\alpha]$,

$$\begin{aligned} \left\|x\partial_t^j\partial_x^{2+[\alpha]-j} u\right\|^2_{L^\infty(I_0)} &\lesssim \left\|x\partial_t^j\partial_x^{2+[\alpha]-j} u\right\|^2_{H^1(I_0)} \\ &\lesssim \int_{I_0} x^2\left(\partial_t^j\partial_x^{2+[\alpha]-j} u\right)^2 dx + \int_{I_0} x^4\left(\partial_t^j\partial_x^{2+[\alpha]-j+1} u\right)^2 dx \end{aligned} \tag{3.9}$$

$$+\int_{I_0} x^4 \left(\partial_t^j \partial_x^{2+[\alpha]-j+2} u\right)^2 dx \lesssim E_{j,2+[\alpha]-j}(t) + E_{j,3+[\alpha]-j}(t)$$
$$\lesssim E(t).$$

For the third term in (3.7), where j ranges over $0, 1, \ldots, k$ and $i = k - j$ with $k = 3 + [\alpha]$,

$$\begin{aligned}\left\|x^2 \partial_t^j \partial_x^{3+[\alpha]-j} u\right\|_{L^\infty(I_0)}^2 &\lesssim \left\|x^2 \partial_t^j \partial_x^{3+[\alpha]-j} u\right\|_{H^1(I_0)}^2 \\ &\lesssim \int_{I_0} x^2 \left(\partial_t^j \partial_x^{3+[\alpha]-j} u\right)^2 dx + \int_{I_0} x^4 \left(\partial_t^j \partial_x^{3+[\alpha]-j+1} u\right)^2 dx \\ &\lesssim E_{j,3+[\alpha]-j}(t) \lesssim E(t).\end{aligned} \tag{3.10}$$

Step 2: (Near the boundary)

$$\sum_{j=0}^{2} \left\|\partial_t^j u\right\|_{L^\infty(I_b)}^2 + \|u_x\|_{L^\infty(I_b)}^2 \lesssim E(t), \tag{3.11}$$

$$\left\|\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u\right\|_{L^\infty(I_b)}^2 \lesssim E(t), \text{ for } 2i + j \geq 3. \tag{3.12}$$

We define $d_b(x) = \operatorname{dist}(x, \partial I_b)$, satisfying the bound

$$d_b(x) \lesssim 1 - x \lesssim \sigma(x), \text{ for } x \in I_b,$$

and consider indices j with $0 \leq j \leq 2$:

$$\begin{aligned}\left\|\partial_t^j u\right\|_{L^\infty(I_b)}^2 &\lesssim \left\|\partial_t^j u\right\|_{H^{\frac{4+[\alpha]-\alpha-j}{2}}(I_b)}^2 \lesssim \left\|\partial_t^j u\right\|_{H^{4+[\alpha]-j+\alpha, 4+[\alpha]-j}(I_b)}^2 \\ &= \sum_{k=0}^{4+[\alpha]-j} \int_{I_b} d_b^{\alpha+4+[\alpha]-j} \left|\partial_x^k \partial_t^j u\right|^2 dx \lesssim \sum_{k=0}^{4+[\alpha]-j} \int_{I_b} \sigma^{\alpha+4+[\alpha]-j} \left|\partial_x^k \partial_t^j u\right|^2 dx \\ &\lesssim \sum_{k=0}^{4+[\alpha]-j} \int_{I_b} \sigma^{\alpha+k} \left|\partial_x^k \partial_t^j u\right|^2 dx \lesssim E_j(t) + \sum_{i=1}^{3+[\alpha]-j} E_{j,i}(t) \lesssim E(t).\end{aligned} \tag{3.13}$$

To establish estimate (3.12) via inequalities (1.6), (1.7), and (1.10), we divide the analysis into distinct cases based on whether the order is integer or non-integer.

Case 1: If $\alpha \neq [\alpha]$, we define $\delta = \frac{1}{2} + \frac{[\alpha]-\alpha}{2}$. For all indices i, with j satisfying $i + j \leq 3 + [\alpha]$ and $2i + j \geq 3$,

$$\begin{aligned}&\|\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u\|_{L^\infty(I_b)}^2 \lesssim \|\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u\|_{H^{\frac{2+[\alpha]-\alpha}{2}}(I_b)}^2 \\ =&\|\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u\|_{H^{4+[\alpha]-i-j-\frac{6+[\alpha]+\alpha-2i-2j}{2}}(I_b)}^2 \\ \lesssim&\|\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u\|_{H^{6+[\alpha]+\alpha-2i-2j, 4+[\alpha]-i-j}(I_b)}^2 \\ \lesssim& \sum_{k=0}^{4+[\alpha]-i-j} \int_{I_b} \sigma^{6+[\alpha]+\alpha-2i-2j} \left|\partial_x^k (\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u)\right|^2 dx\end{aligned} \tag{3.14}$$

$$\lesssim E_{j,i}(t) + \sum_{k=1}^{4+[\alpha]-i-j} \int_{I_b} \sigma^{6+[\alpha]+\alpha-2i-2j} \left|\partial_x^k(\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u)\right|^2 dx.$$

For $1 \le k \le 4 + \lfloor\alpha\rfloor - i - j$, applying the Leibniz formula yields

$$\left|\partial_x^k(\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u)\right| \lesssim \sum_{p=0}^{k} \left|\sigma^{\frac{2i+j-2}{2}-p} \partial_t^j \partial_x^{i+k-p} u\right|. \tag{3.15}$$

The following results are established:

$$\begin{aligned}
&\int_{I_b} \sigma^{6+[\alpha]+\alpha-2i-2j} \left|\partial_x^k(\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u)\right|^2 dx \lesssim \int_{I_b} \sum_{p=0}^{k} \sigma^{4+[\alpha]+\alpha-j-2p} \left|\partial_t^j \partial_x^{i+k-p} u\right|^2 dx \\
\lesssim &\int_{I_b} \sigma^{4+[\alpha]-i-j-k} \sum_{p=0}^{1} \sigma^{\alpha+i+k-2p} \left|\partial_t^j \partial_x^{i+k-p} u\right|^2 dx + \sum_{p=2}^{k} \int_{I_b} \sigma^{4+[\alpha]+\alpha-j-2p} \left|\partial_t^j \partial_x^{i+k-p} u\right|^2 dx \\
\lesssim &E_{j,i+k-1}(t) + \sum_{p=2}^{k} \int_{I_b} \sigma^{4+[\alpha]+\alpha-j-2p} \left|\partial_t^j \partial_x^{i+k-p} u\right|^2 dx.
\end{aligned} \tag{3.16}$$

To apply Hardy's inequality, we consider $p \in [2, k]$ satisfying

$$\begin{aligned}
&4 + [\alpha] + \alpha - j - 2p \\
=&2(4 + [\alpha] - i - j - k) + 2(k - p) + (\alpha - [\alpha]) + (2i + j - 3) - 1 > -1.
\end{aligned} \tag{3.17}$$

Subsequently, we derive the estimate

$$\begin{aligned}
&\int_{I_b} \sigma^{4+[\alpha]+\alpha-j-2p} \left|\partial_t^j \partial_x^{i+k-p} u)\right|^2 dx \lesssim \sum_{l=0}^{p} \int_{I_b} d_b^{4+[\alpha]+\alpha-j} \left|\partial_t^j \partial_x^{i+k-p+l} u\right|^2 dx \\
=&\sum_{l=0}^{p} \int_{I_b} \sigma^{4+[\alpha]-i-j-k+(p-l)} \sigma^{\alpha+i+k-p+l} \left|\partial_t^j \partial_x^{i+k-p+l} u\right|^2 dx \\
\lesssim &\int_{I_b} \sigma^{\alpha+i+k} \left|\partial_t^j \partial_x^{i+k} u)\right|^2 dx + \sum_{l=0}^{p-1} \int_{I_b} \sigma^{\alpha+i+k-p+l-1} \left|\partial_t^j \partial_x^{i+k-p+l} u\right|^2 dx \\
\lesssim &\sum_{l=i+k-p}^{i+k-1} E_{j,l}(t).
\end{aligned} \tag{3.18}$$

Therefore, combining (3.14), (3.16), and (3.18), for indices i, with j satisfying $i + j \le 3 + [\alpha]$ and $2i + j \ge 3$,

$$\|\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u\|_{L^\infty(I_b)}^2 \lesssim \sum_{l=i}^{3+[\alpha]-j} E_{j,l}(t) \lesssim E(t). \tag{3.19}$$

Case 2: If $\alpha = [\alpha]$, we set $\delta = \frac{1}{4}$. By analogy to (3.18), it follows that for $p \in [2, k]$,

$$2[\alpha] - j - 2p + \frac{9}{2} = 2(4 + [\alpha] - i - j - k) + 2(k - p) + (2i + j - 3) - \frac{1}{2} \ge -\frac{1}{2}.$$

Therefore,

$$\int_{I_b} \sigma^{2[\alpha]-j-2p+\frac{9}{2}} \left|\partial_t^j \partial_x^{i+k-p} u\right|^2 dx \lesssim \sum_{l=0}^{p} \int_{I_b} \sigma^{2[\alpha]-j+\frac{9}{2}} \left|\partial_t^j \partial_x^{i+k-p+l} u\right|^2 dx$$
$$\lesssim \sum_{l=i+k-p}^{i+k-1} E_{j,l}(t). \tag{3.20}$$

For $1 \leq k \leq 4 + [\alpha] - i - j$, similar to (3.16), we obtain the following estimate:

$$\begin{aligned}
&\int_{I_b} \sigma^{2[\alpha]-2i-2j+\frac{13}{2}} \left|\partial_x^k (\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u)\right|^2 dx \\
\lesssim& E_{j,i+k-1}(t) + \sum_{p=2}^{k} \sigma^{2[\alpha]-j-2p+\frac{9}{2}} \int_{I_b} \sigma^{2[\alpha]-j-2p+\frac{9}{2}} \left|\partial_t^j \partial_x^{i+k-p} u\right|^2 dx \\
\lesssim& \sum_{l=i}^{i+k-1} E_{j,l}(t).
\end{aligned} \tag{3.21}$$

Hence, analogous to (3.13), for indices i, with j satisfying $i + j \leq 3 + \lfloor\alpha\rfloor$ and $2i + j \geq 3$, we derive

$$\begin{aligned}
&\|\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u\|^2_{L^\infty(I_b)} \lesssim \|\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u\|^2_{H^{\frac{3}{4}}(I_b)} \\
\lesssim& \|\sigma^{\frac{2i+j-2}{2}} \partial_t^j \partial_x^i u\|^2_{H^{2[\alpha]-2i-2j+\frac{13}{2},4+[\alpha]-i-j}(I_b)} \lesssim E(t).
\end{aligned} \tag{3.22}$$

□

4. Higher-order temporal derivative energy estimates

This section aims to derive the higher-order temporal derivative estimates required in (4.1).

Lemma 4.1. *Let $T > 0$, and assume (ξ, u) is a smooth solution to* (2.8)*, subject to* (3.1)*, over the space-time domain in* $(0,1) \times (0,T]$. *The following estimates hold for all $t \in (0,T]$:*

$$\|x^2 \sigma^{\frac{\alpha}{2}} \partial_t^{4+[\alpha]} u\|_0^2 + \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_t^{3+[\alpha]} u_x\|_0^2 + \|x \sigma^{\frac{\alpha+1}{2}} \partial_t^{3+[\alpha]} u\|_0^2 \lesssim P(0) + CtP\left(\sup_{0\leq\tau\leq t} E(\tau)\right). \tag{4.1}$$

Proof. Taking the time derivative of order $3 + \lfloor\alpha\rfloor$ in (2.8), we obtain

$$\begin{aligned}
&x\sigma^\alpha \partial_t^{5+[\alpha]} u - \gamma\xi^2 \partial_x \left\{\left(\partial_t^{3+[\alpha]} u + x\partial_t^{3+[\alpha]} u_x\right) \sigma^{\alpha+1} e^{S_0} \xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma-1}\right\} \\
&- 2\gamma\xi^2 \partial_x \left\{\partial_t^{3+[\alpha]} u \left[\sigma^{\alpha+1} e^{S_0} \xi^{-2\gamma-1} (\xi + x\xi_x)^{-\gamma}\right]\right\} = \sum_{i=1}^{5} D_i,
\end{aligned} \tag{4.2}$$

where

$$D_1 = \gamma\xi^2 \partial_x \left\{\sum_{p=1}^{3+[\alpha]} C_{3+[\alpha]}^p \partial_t^{3+[\alpha]-p} (u + u_x) \partial_t^p [\sigma^{\alpha+1} e^{S_0} \xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma-1}]\right\},$$

$$
\begin{aligned}
D_2 &= 2\gamma\xi^2\partial_x \left\{ \sum_{p=1}^{3+[\alpha]} C_{3+[\alpha]}^p \partial_t^{3+[\alpha]-p} u \partial_t^p [\sigma^{\alpha+1} e^{S_0} \xi^{-2\gamma-1}(\xi + x\xi_x)^{-\gamma-1}] \right\}, \\
D_3 &= \gamma \sum_{k=1}^{3+[\alpha]} C_{3+[\alpha]}^k \partial_t^k \xi^2 \partial_t^{3+[\alpha]-k} \left\{ \sigma^{\alpha+1} e^{S_0} \xi^{-2\gamma}(\xi + x\xi_x)^{-\gamma-1}(u + xu_x) \right\}_x, \\
D_4 &= 2\gamma \sum_{k=1}^{3+[\alpha]} C_{3+[\alpha]}^k \partial_t^k \xi^2 \partial_t^{3+[\alpha]-k} \left\{ \sigma^{\alpha+1} e^{S_0} \xi^{-2\gamma-1}(\xi + x\xi_x)^{-\gamma} u \right\}_x, \\
D_5 &= -2\partial_t^{3+[\alpha]} \left\{ \xi u \partial_x [\sigma^{\alpha+1} e^{S_0} \xi^{-2\gamma}(\xi + x\xi_x)^{-\gamma}] \right\}.
\end{aligned}
\tag{4.3}
$$

Multiplying (4.2) by $x^3\partial_t^{4+[\alpha]}u$ and integrating both sides over $(0,t)\times(0,1)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\| x^2\sigma^{\frac{\alpha}{2}}\partial_t^{4+[\alpha]}u\|_0^2 - \frac{1}{2}\int_I x^4\sigma^\alpha \left(\partial_t^{4+[\alpha]}u(0)\right)^2 dx \\
&-\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\xi^2\partial_\tau^{4+[\alpha]}u\partial_x\left\{\left(\partial_\tau^{3+[\alpha]}u + x\partial_\tau^{3+[\alpha]}u_x\right)\xi^{-2\gamma}(\xi + x\xi_x)^{-\gamma-1}\right\}dxd\tau \\
&-2\gamma\int_0^t\int_I x^3\xi^2\partial_\tau^{4+[\alpha]}u\partial_x\left\{\partial_\tau^{3+[\alpha]}u\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma-1}(\xi + x\xi_x)^{-\gamma}\right\}dxd\tau \\
&= \sum_{i=1}^5\int_0^t\int_I x^3\partial_\tau^{4+[\alpha]}uD_i dxd\tau.
\end{aligned}
\tag{4.4}
$$

We now consider the third term on the left-hand side of (4.4) by applying integration by parts:

$$
\begin{aligned}
&-\gamma\int_0^t\int_I x^3\xi^2\partial_\tau^{4+[\alpha]}u\partial_x\left\{\left(\partial_\tau^{3+[\alpha]}u + x\partial_\tau^{3+[\alpha]}u_x\right)\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma}(\xi + x\xi_x)^{-\gamma-1}\right\}dxd\tau \\
=&\gamma\int_0^t\int_I \partial_x\left\{x^3\xi^2\partial_\tau^{4+[\alpha]}u\right\}\left(\partial_\tau^{3+[\alpha]}u + x\partial_\tau^{3+[\alpha]}u_x\right)\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma}(\xi + x\xi_x)^{-\gamma-1}dxd\tau \\
=&\sum_{i=1}^3 F_i,
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
F_1 &= \gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u_x\left(\partial_\tau^{3+[\alpha]}u + x\partial_\tau^{3+[\alpha]}u_x\right)\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-1}dxd\tau, \\
F_2 &= 3\gamma\int_0^t\int_I x^2\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\left(\partial_\tau^{3+[\alpha]}u + x\partial_\tau^{3+[\alpha]}u_x\right)\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-1}dxd\tau, \\
F_3 &= 2\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\left(\partial_\tau^{3+[\alpha]}u + x\partial_\tau^{3+[\alpha]}u_x\right)\xi^{1-2\gamma}(\xi + x\xi_x)^{-\gamma-1}\xi_x dxd\tau.
\end{aligned}
\tag{4.6}
$$

First, we write F_3 in the following form:

$$
F_3 = 2\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u_x\partial_\tau^{3+[\alpha]}u\xi^{1-2\gamma}(\xi + x\xi_x)^{-\gamma-1}dxd\tau
$$

$$
\begin{aligned}
&+2\gamma\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dxd\tau\\
&=F_{31}+F_{32}.
\end{aligned}
$$

We have by the Cauchy inequality

$$
\begin{aligned}
|F_{31}|&\lesssim\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\left(\partial_\tau^{4+[\alpha]}u\right)^2dxd\tau+\int_0^t\int_I x^2\sigma^{\alpha+1}e^{S_0}\left(\partial_\tau^{3+[\alpha]}u\right)^2dxd\tau\\
&\lesssim CtP\left(\sup_{0\le\tau\le t}E(\tau)\right),\\
|F_{32}|&\lesssim\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\left(\partial_\tau^{4+[\alpha]}u\right)^2dxd\tau+\int_0^t\int_I x^2\sigma^{\alpha+1}e^{S_0}\left(\partial_\tau^{3+[\alpha]}u_x\right)^2dxd\tau\\
&\lesssim CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}
$$

Then, it follows that

$$
|F_3|\lesssim CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
$$

We define $F_1=F_{11}+F_{12}$ and $F_2=F_{21}+F_{22}$, where

$$
\begin{aligned}
F_{11}&=\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u_x\partial_\tau^{3+[\alpha]}u\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dxd\tau,\\
F_{12}&=\gamma\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u_x\partial_\tau^{3+[\alpha]}u_x\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dxd\tau,\\
F_{21}&=3\gamma\int_0^t\int_I x^2\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dxd\tau,\\
F_{22}&=3\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u_x\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dxd\tau.
\end{aligned}
\tag{4.7}
$$

For F_{12} and F_{21}, integration by parts yields

$$
\begin{aligned}
F_{12}\gtrsim&\frac{\gamma}{2}\int_I x^4\sigma^{\alpha+1}e^{S_0}\left(\partial_t^{3+[\alpha]}u_x\right)^2\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dx\\
&-P(0)-CtP\left(\sup_{0\le\tau\le t}E(\tau)\right),\\
F_{21}\gtrsim&\frac{3\gamma}{2}\int_I x^2\sigma^{\alpha+1}e^{S_0}\left(\partial_t^{3+[\alpha]}u\right)^2\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dx\\
&-P(0)-CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}
\tag{4.8}
$$

To bound F_{22}, we examine the fourth term on the left-hand side of (4.4):

$$
-2\gamma\int_0^t\int_I x^3\xi^2\partial_\tau^{4+[\alpha]}u\partial_x\left\{\partial_\tau^{3+[\alpha]}u\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma-1}(\xi+x\xi_x)^{-\gamma}\right\}dxd\tau
\tag{4.9}
$$

$$=K_1+K_2,$$

where

$$K_1=-2\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u_x\xi^{-2\gamma+1}(\xi+x\xi_x)^{-\gamma}dxd\tau, \tag{4.10}$$
$$K_2=-2\gamma\int_0^t\int_I x^3\xi^2\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u\partial_x\left\{\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma-1}(\xi+x\xi_x)^{-\gamma}\right\}dxd\tau.$$

Then,

$$\begin{aligned}K_1+F_{22}=&3\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma}\frac{\xi+x\xi_x-x\xi_x}{\xi+x\xi_x}dxd\tau\\&-2\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma}dxd\tau\\&=M_1+M_2.\end{aligned} \tag{4.11}$$

Here,

$$M_1=\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma}dxd\tau, \tag{4.12}$$
$$M_2=-3\gamma\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\xi_xdxd\tau.$$

To bound F_{11}, we decompose it as $F_{11}=N_1+N_2+N_3$, where

$$\begin{aligned}N_1=&-\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{3+[\alpha]}u_x\partial_\tau^{4+[\alpha]}u\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dxd\tau,\\N_2=&-\gamma\int_I x^3\sigma^{\alpha+1}e^{S_0}\left[\partial_t^{3+[\alpha]}u_x\partial_t^{3+[\alpha]}u\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right](0)dx\\&-\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{3+[\alpha]}u_x\partial_\tau^{3+[\alpha]}u\partial_\tau\left[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right]dxd\tau,\\N_3=&\gamma\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_t^{3+[\alpha]}u_x\partial_t^{3+[\alpha]}u\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dx,\end{aligned} \tag{4.13}$$

where

$$N_1+M_1=\gamma\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\partial_\tau^{3+[\alpha]}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\xi_xdxd\tau, \tag{4.14}$$

which can be bounded by $CtP\left(\sup_{0\le\tau\le t}E(\tau)\right)$. For N_3 and any $\varepsilon>0$, applying the Cauchy inequality yields

$$\begin{aligned}N_3=&\gamma\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_t^{3+[\alpha]}u_x\partial_t^{3+[\alpha]}u\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dx\\\ge&-\frac{\gamma}{2\varepsilon}\int_I x^2\sigma^{\alpha+1}e^{S_0}\left(\partial_t^{3+[\alpha]}u\right)^2\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dx\end{aligned} \tag{4.15}$$

$$-\frac{\gamma\varepsilon}{2}\int_I x^4\sigma^{\alpha+1}e^{S_0}\left(\partial_t^{3+[\alpha]}u_x\right)^2\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dx.$$

From estimates (4.5)–(4.15), taking $\varepsilon=\frac{1}{2}$ satisfies the inequalities

$$\begin{aligned}
&\frac{1}{2}\|x^2\sigma^{\frac{\alpha}{2}}\partial_t^{4+[\alpha]}u\|_0^2-\frac{1}{2}\int_I x^4\sigma^{\alpha}\left(\partial_t^{4+[\alpha]}u(0)\right)^2dx\\
&+\left(\frac{3\gamma}{2}-\frac{\gamma}{2\varepsilon}\right)\int_I x^2\sigma^{\alpha+1}e^{S_0}\left(\partial_t^{3+[\alpha]}u\right)^2\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dx\\
&+\left(\frac{\gamma}{2}-\frac{\gamma\varepsilon}{2}\right)\int_I x^4\sigma^{\alpha+1}e^{S_0}\left(\partial_t^{3+[\alpha]}u_x\right)^2\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}dx-P(0)-CtP\left(\sup_{0\le\tau\le t}E(\tau)\right)\\
&\gtrsim\|x^2\sigma^{\frac{\alpha}{2}}\partial_t^{4+[\alpha]}u\|_0^2+\|x^2\sigma^{\frac{\alpha+1}{2}}\partial_t^{3+[\alpha]}u_x\|_0^2+\|x\sigma^{\frac{\alpha+1}{2}}\partial_t^{3+[\alpha]}u\|_0^2-P(0)-CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}\tag{4.16}$$

The terms K_2, M_2, and N_2 are all bounded by $P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right)$.

We now deal with the right-hand side of (4.4). We only provide a detailed proof for the two difficult terms, namely those involving D_1 and D_5. For D_1, we have from (4.3)

$$\begin{aligned}
D_1=&\gamma\xi^2\partial_x\left\{\sum_{p=1}^{3+[\alpha]}C_{3+[\alpha]}^p\partial_t^{3+[\alpha]-p}(u+xu_x)\partial_t^p[\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\right\}\\
=&\gamma\xi^2\partial_x\left\{\sum_{p=1}^{3+[\alpha]}C_{3+[\alpha]}^p\partial_t^{3+[\alpha]-p}u\partial_t^p[\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\right\}\\
&+\gamma\xi^2\partial_x\left\{\sum_{p=1}^{3+[\alpha]}C_{3+[\alpha]}^p\partial_t^{3+[\alpha]-p}(xu_x)\partial_t^p[\sigma^{\alpha+1}e^{S_0}\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\right\}.
\end{aligned}\tag{4.17}$$

We denote the second term on the right-hand side of (4.17) by

$$D_{12}=\gamma\xi^2\partial_x\left\{x\sigma^{\alpha+1}e^{S_0}H_{12}\right\},\tag{4.18}$$

where

$$H_{12}=\sum_{p=1}^{3+[\alpha]}H_{12}^p=C_{3+[\alpha]}^p\partial_t^{3+[\alpha]-p}u_x\partial_t^p\left\{\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right\}.\tag{4.19}$$

Thus, it holds that

$$\begin{aligned}
&\gamma\int_0^t\int_I x^3\partial_\tau^{4+[\alpha]}u\xi^2\partial_x\left\{x\sigma^{\alpha+1}e^{S_0}H_{12}\right\}dxd\tau\\
=&-\gamma\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u_x\xi^2H_{12}dxd\tau-3\gamma\int_0^t\int_I x^3\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\xi^2H_{12}dxd\tau\\
&-2\gamma\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\partial_\tau^{4+[\alpha]}u\xi\xi_xH_{12}dxd\tau.
\end{aligned}\tag{4.20}$$

We only consider the first term on the right-hand side of (4.20). For $p=1$, it derives from (4.19) that

$$H_{12}^1=(3+[\alpha])\partial_t^{2+[\alpha]}u_x\partial_t\left\{\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right\},\tag{4.21}$$

and

$$
\begin{aligned}
&-\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{4+[\alpha]} u_x \xi^2 H_{12}^1 dx d\tau \\
=&-\gamma \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_t^{3+[\alpha]} u_x \xi^2 H_{12}^1 dx + \gamma \int_I x^4 \sigma^{\alpha+1} e^{S_0} \left[\partial_t^{3+[\alpha]} u_x \xi^2 H_{12}^1\right](0) dx \qquad (4.22)\\
&+\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \partial_\tau \left(\xi^2 H_{12}^1\right) dx d\tau.
\end{aligned}
$$

For any $\varepsilon > 0$, we apply Young's inequality to bound the first term on the right-hand side of (4.22) as

$$
\begin{aligned}
&-\gamma \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_t^{3+[\alpha]} u_x \xi^2 H_{12}^1 dx \\
\lesssim&\varepsilon \int_I x^4 \sigma^{\alpha+1} e^{S_0} \left(\partial_t^{3+[\alpha]} u_x\right)^2 dx + C(\varepsilon) \int_I x^4 \sigma^{\alpha+1} e^{S_0} (H_{12}^1)^2 dx. \qquad (4.23)
\end{aligned}
$$

From (4.23), the latter integral admits the estimate

$$
\begin{aligned}
&\int_I x^4 \sigma^{\alpha+1} e^{S_0} (H_{12}^1)^2 dx \\
\lesssim&\|e^{\frac{S_0}{2}}\|_{L^\infty}^2 \int_0^t \left(\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2\right) \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_\tau^{3+[\alpha]} u_x\|_0^2 d\tau \\
&+\|e^{\frac{S_0}{2}}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_\tau^{2+[\alpha]} u_x(0) \left(\int_0^t \partial_\tau u d\tau + u(0)\right)\|_0^2 \qquad (4.24)\\
&+\|e^{\frac{S_0}{2}}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_\tau^{2+[\alpha]} u_x(0) \left(\int_0^t \partial_\tau u_x d\tau + u_x(0)\right)\|_0^2 \\
\lesssim&P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).
\end{aligned}
$$

Our attention now turns to the third term in the right-hand side of (4.22):

$$
\begin{aligned}
&\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \partial_\tau \left[\xi^2 H_{12}^1\right] dx d\tau \qquad (4.25)\\
=&2\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \xi u H_{12}^1 dx d\tau + \gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \xi^2 \partial_\tau H_{12}^1 dx d\tau.
\end{aligned}
$$

For the second term on the right-hand side of (4.25), we have

$$
\begin{aligned}
&\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \xi^2 \partial_\tau H_{12}^1 dx d\tau \\
=&(3+[\alpha])\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \partial_\tau^{2+[\alpha]} u_x \xi^2 \partial_\tau^2 \left\{\xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma-1}\right\} dx d\tau \qquad (4.26)\\
&+(3+[\alpha])\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \left(\partial_\tau^{3+[\alpha]} u_x\right)^2 \xi^2 \partial_\tau \left\{\xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma-1}\right\} dx d\tau.
\end{aligned}
$$

From the first term in (4.26), it follows that

$$\begin{aligned}
&\left|-(3+[\alpha])(\gamma+1)\gamma\int_0^t\int_I x^5\sigma^{\alpha+1}e^{S_0}\partial_\tau^{3+[\alpha]}u_x\partial_\tau^{2+[\alpha]}u_x\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\partial_\tau u_x dxd\tau\right|\\
\lesssim&\|e^{\frac{S_0}{2}}\|_{L^\infty}^2\int_0^t\|x^2\sigma^{\frac{\alpha}{2}}\partial_\tau^{3+[\alpha]}u_x\|_0^2d\tau+\|e^{\frac{S_0}{2}}\|_{L^\infty}^2\int_0^t\|\sigma^{\frac{1}{2}}\partial_\tau u_x\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha}{2}}\partial_\tau^{2+[\alpha]}u_x\|_0^2d\tau \qquad (4.27)\\
\lesssim&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}$$

Without loss of generality, we assume that $[\alpha]$ is an odd integer, hence $3+[\alpha]$ is an even integer. For $p=\frac{3+[\alpha]}{2}$, we consider $H_{12}^{\frac{3+[\alpha]}{2}}$ from (4.19):

$$H_{12}^{\frac{3+[\alpha]}{2}}=-(\gamma+1)C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}}x\partial_t^{\frac{3+[\alpha]}{2}}u_x\partial_t^{\frac{1+[\alpha]}{2}}u_x\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-2}. \qquad (4.28)$$

Similar to (4.24), we estimate the following term:

$$\begin{aligned}
&\int_I x^4\sigma^{\alpha+1}e^{S_0}\left(H_{12}^{\frac{3+[\alpha]}{2}}\right)^2dx\\
\lesssim&\|e^{\frac{S_0}{2}}\|_{L^\infty}^2\int_0^t\|x^2\sigma^{\frac{2\alpha-[\alpha]-1}{4}}\partial_\tau^{\frac{3+[\alpha]}{2}}u_x\|_0^2\|\sigma^{\frac{3+[\alpha]}{4}}\partial_\tau^{\frac{3+[\alpha]}{2}}u_x\|_{L^\infty}^2d\tau \qquad (4.29)\\
&+\|e^{\frac{S_0}{2}}\|_{L^\infty}^2\|x^2\sigma^{\frac{1+\alpha}{2}}\partial_t^{\frac{3+[\alpha]}{2}}u_x\partial_t^{\frac{1+[\alpha]}{2}}u_x(0)\|_0^2.
\end{aligned}$$

The condition $2\alpha-[\alpha]-1>-1$ allows us to apply Hardy's inequality to obtain

$$\begin{aligned}
\|x^2\sigma^{\frac{2\alpha-[\alpha]-1}{4}}\partial_t^{\frac{3+[\alpha]}{2}}u_x\|_0^2\lesssim&\sum_{l=0}^{\frac{3+[\alpha]}{2}}\int_I x^4\sigma^{\frac{2\alpha+[\alpha]+5}{2}}\left(\partial_t^{\frac{3+[\alpha]}{2}}\partial_x^{1+l}u\right)^2dx\\
\lesssim&\sum_{l=0}^{\frac{3+[\alpha]}{2}}\int_I x^4\sigma^{\alpha+1+l}\sigma^{\frac{[\alpha]+3-2l}{2}}\left(\partial_t^{\frac{3+[\alpha]}{2}}\partial_x^{1+l}u\right)^2dx \qquad (4.30)\\
\lesssim&\sum_{l=0}^{\frac{3+[\alpha]}{2}}\int_I x^4\sigma^{\alpha+1+l}\left(\partial_t^{\frac{3+[\alpha]}{2}}\partial_x^{1+l}u\right)^2dx\\
\lesssim&E(t).
\end{aligned}$$

We handle the right-hand side of (4.29):

$$\begin{aligned}
&\|x^2\sigma^{\frac{1+\alpha}{2}}\partial_t^{\frac{3+[\alpha]}{2}}u_x\partial_t^{\frac{1+[\alpha]}{2}}u_x(0)\|_0^2 \qquad (4.31)\\
\lesssim&\int_0^t\|x^2\sigma^{\frac{2\alpha-[\alpha]-1}{4}}\partial_\tau^{\frac{5+[\alpha]}{2}}u_x\|_0^2\|\sigma^{\frac{1+[\alpha]}{4}}\partial_\tau^{\frac{1+[\alpha]}{2}}u_x(0)\|_{L^\infty}^2d\tau+\|x^2\sigma^{\frac{1+\alpha}{2}}\partial_t^{\frac{3+[\alpha]}{2}}u_x(0)\partial_t^{\frac{1+[\alpha]}{2}}u_x(0)\|_0^2.
\end{aligned}$$

Since $2\alpha-[\alpha]-1>-1$, by Hardy's inequality, we obtain

$$\|x^2\sigma^{\frac{2\alpha-[\alpha]-1}{4}}\partial_t^{\frac{5+[\alpha]}{2}}u_x\|_0^2\lesssim\sum_{l=0}^{\frac{1+[\alpha]}{2}}\int_I x^4\sigma^{\frac{2\alpha+[\alpha]+3}{2}}\left(\partial_t^{\frac{5+[\alpha]}{2}}\partial_x^{1+l}u\right)^2dx$$

$$
\begin{aligned}
&\lesssim \sum_{l=0}^{\frac{1+[\alpha]}{2}} \int_I x^4 \sigma^{\alpha+1+l} \sigma^{\frac{[\alpha]-2l+1}{2}} \left(\partial_t^{\frac{5+[\alpha]}{2}} \partial_x^{1+l} u\right)^2 dx \qquad (4.32)\\
&\lesssim \sum_{l=0}^{\frac{1+[\alpha]}{2}} \int_I x^4 \sigma^{\alpha+1+l} \left(\partial_t^{\frac{5+[\alpha]}{2}} \partial_x^{1+l} u\right)^2 dx\\
&\lesssim E(t).
\end{aligned}
$$

Analogous to the treatment of (4.25), we bound the following expression:

$$
\begin{aligned}
&\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \partial_\tau \left\{ \xi^2 H_{12}^{\frac{3+[\alpha]}{2}} \right\} dx d\tau \qquad (4.33)\\
=&2\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \xi u H_{12}^{\frac{3+[\alpha]}{2}} dx d\tau + \gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \xi^2 \partial_\tau H_{12}^{\frac{3+[\alpha]}{2}} dx d\tau.
\end{aligned}
$$

The second term on the right-hand side of (4.33) can be controlled as

$$
\begin{aligned}
&\gamma \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \xi^2 \partial_\tau H_{12}^{\frac{3+[\alpha]}{2}} dx d\tau \qquad (4.34)\\
\lesssim& \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \left(\partial_\tau^{3+[\alpha]} u_x\right)^2 dx d\tau + \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \left(\partial_\tau H_{12}^{\frac{3+[\alpha]}{2}}\right)^2 dx d\tau.
\end{aligned}
$$

We next examine the second term appearing on the right-hand side of (4.34):

$$
\begin{aligned}
&\int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \left(\partial_\tau H_{12}^{\frac{3+[\alpha]}{2}}\right)^2 dx d\tau\\
\lesssim& \|e^{\frac{S_0}{2}}\|_{L^\infty}^2 \int_0^t \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_\tau^{\frac{5+[\alpha]}{2}} u_x \partial_\tau^{\frac{1+[\alpha]}{2}} u_x\|_0^2 d\tau \qquad (4.35)\\
&+\|e^{\frac{S_0}{2}}\|_{L^\infty}^2 \int_0^t \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_\tau^{\frac{3+[\alpha]}{2}} u_x \partial_\tau^{\frac{3+[\alpha]}{2}} u_x\|_0^2 d\tau\\
&+\|e^{\frac{S_0}{2}}\|_{L^\infty}^2 \int_0^t \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_\tau^{\frac{3+[\alpha]}{2}} u_x \partial_\tau^{\frac{1+[\alpha]}{2}} u_x \partial_\tau \left\{ \xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \right\} \|_0^2 d\tau.
\end{aligned}
$$

Among all terms in (4.35), only the first right-hand term must be estimated:

$$
\int_0^t \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_\tau^{\frac{5+[\alpha]}{2}} u_x \partial_\tau^{\frac{1+[\alpha]}{2}} u_x\|_0^2 d\tau \lesssim \int_0^t \|\sigma^{\frac{[\alpha]+1}{4}} \partial_\tau^{\frac{1+[\alpha]}{2}} u_x\|_{L^\infty}^2 \|x^2 \sigma^{\frac{2\alpha-[\alpha]+1}{4}} \partial_\tau^{\frac{5+[\alpha]}{2}} u_x\|_0^2 d\tau. \qquad (4.36)
$$

Since $2\alpha - [\alpha] - 1 > -1$, by using Hardy's inequality, we obtain

$$
\begin{aligned}
\|x^2 \sigma^{\frac{2\alpha-[\alpha]+1}{4}} \partial_t^{\frac{5+[\alpha]}{2}} u_x\|_0^2 &\lesssim \sum_{l=0}^{\frac{1+[\alpha]}{2}} \int_I x^4 \sigma^{\frac{2\alpha+[\alpha]+3}{2}} \left(\partial_t^{\frac{5+[\alpha]}{2}} \partial_x^{1+l} u\right)^2 dx\\
&\lesssim \sum_{l=0}^{\frac{1+[\alpha]}{2}} \int_I x^4 \sigma^{\alpha+1+l} \sigma^{\frac{[\alpha]-2l+1}{2}} \left(\partial_t^{\frac{5+[\alpha]}{2}} \partial_x^{1+l} u\right)^2 dx \qquad (4.37)
\end{aligned}
$$

$$\lesssim \sum_{l=0}^{\frac{1+[\alpha]}{2}} \int_I x^4 \sigma^{\alpha+1+l} \left(\partial_t^{\frac{5+[\alpha]}{2}} \partial_x^{1+l} u\right)^2 dx$$
$$\lesssim E(t).$$

The remaining terms of D_1 can be estimated similarly, and it follows that D_1 can be controlled by $P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right)$.

For D_5, we restrict our estimation to the case $k = \frac{3+[\alpha]}{2}$. Without loss of generality, assume $[\alpha]$ is an odd integer, hence $3 + [\alpha]$ is even. At this critical exponent, we estimate only the following term:

$$2\gamma C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}} \xi \partial_t^{\frac{3+[\alpha]}{2}} u \left\{ x\sigma^{\alpha+1} e^{S_0} \partial_t^{\frac{1+[\alpha]}{2}} u_x \xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \right\}_x . \tag{4.38}$$

Multiplying (4.38) by $x^3 \partial_t^{4+[\alpha]} u$ and integrating over the domain $(0, t) \times (0, 1)$, we obtain

$$\begin{aligned}
&2\gamma C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}} \int_0^t \int_I x^3 \xi \partial_\tau^{4+[\alpha]} u \partial_\tau^{\frac{3+[\alpha]}{2}} u \left\{ x\sigma^{\alpha+1} e^{S_0} \partial_\tau^{\frac{1+[\alpha]}{2}} u_x \xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \right\}_x dx d\tau \\
=& -2\gamma C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}} \int_0^t \int_I \partial_x \left\{ x^3 \xi \partial_\tau^{4+[\alpha]} u \partial_\tau^{\frac{3+[\alpha]}{2}} u \right\} x\sigma^{\alpha+1} e^{S_0} \xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \partial_\tau^{\frac{1+[\alpha]}{2}} u_x dx d\tau.
\end{aligned} \tag{4.39}$$

From (4.39), we identify the most challenging term to handle as

$$\begin{aligned}
&-2\gamma C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}} \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{4+[\alpha]} u_x \partial_\tau^{\frac{3+[\alpha]}{2}} u \partial_\tau^{\frac{1+[\alpha]}{2}} u_x \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma-1} dx d\tau \\
=& -2\gamma C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}} \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_t^{3+[\alpha]} u_x \partial_t^{\frac{3+[\alpha]}{2}} u \partial_t^{\frac{1+[\alpha]}{2}} u_x \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma-1} dx \\
&+ 2\gamma C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}} \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_t^{3+[\alpha]} u_x \left[\partial_t^{\frac{3+[\alpha]}{2}} u \partial_t^{\frac{1+[\alpha]}{2}} u_x \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \right](0) dx \\
&+ 2\gamma C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}} \int_0^t \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_\tau^{3+[\alpha]} u_x \partial_\tau \left[\partial_\tau^{\frac{3+[\alpha]}{2}} u \partial_\tau^{\frac{1+[\alpha]}{2}} u_x \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \right] dx d\tau.
\end{aligned} \tag{4.40}$$

For any $\varepsilon > 0$, we apply Young's inequality and proceed with the estimation of the first term on the right-hand side of (4.40):

$$\begin{aligned}
&-2\gamma C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}} \int_I x^4 \sigma^{\alpha+1} e^{S_0} \partial_t^{3+[\alpha]} u_x \partial_t^{\frac{3+[\alpha]}{2}} u \partial_t^{\frac{1+[\alpha]}{2}} u_x \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma-1} dx \\
\lesssim& \varepsilon \int_I x^4 \sigma^{\alpha+1} e^{S_0} \left(\partial_t^{3+[\alpha]} u_x \right)^2 dx + C(\varepsilon) \int_I x^4 \sigma^{\alpha+1} e^{S_0} \left[\partial_t^{\frac{3+[\alpha]}{2}} u \partial_t^{\frac{1+[\alpha]}{2}} u_x \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \right]^2 dx.
\end{aligned} \tag{4.41}$$

Then,

$$\begin{aligned}
&\int_I x^4 \sigma^{\alpha+1} e^{S_0} \left[\partial_t^{\frac{3+[\alpha]}{2}} u \partial_t^{\frac{1+[\alpha]}{2}} u_x \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \right]^2 dx \\
=& \|e^{\frac{S_0}{2}}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_t^{\frac{[\alpha]+3}{2}} u \partial_t^{\frac{[\alpha]+1}{2}} u_x \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma-1}\|_0^2 \\
\lesssim& \|e^{\frac{S_0}{2}}\|_{L^\infty}^2 \int_0^t \|\sigma^{\frac{\alpha+1}{4}} \partial_\tau^{\frac{[\alpha]+5}{2}} u\|_{L^\infty}^2 \|x^2 \sigma^{\frac{2\alpha-[\alpha]+1}{4}} \partial_\tau^{\frac{[\alpha]+1}{2}} u_x\|_0^2 d\tau \\
&+ \|e^{\frac{S_0}{2}}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_t^{\frac{[\alpha]+3}{2}} u_x(0) \partial_t^{\frac{[\alpha]+1}{2}} u_x\|_0^2.
\end{aligned} \tag{4.42}$$

By Hardy's inequality, we have

$$
\begin{aligned}
\|x^2\sigma^{\frac{2\alpha-[\alpha]+1}{4}}\partial_t^{\frac{[\alpha]+1}{2}}u_x\|_0^2 \lesssim & \sum_{l=0}^{\frac{5+[\alpha]}{2}} \int_I x^4\sigma^{\frac{2\alpha+[\alpha]+1}{2}}\left(\partial_t^{\frac{1+[\alpha]}{2}}\partial_x^{1+l}u\right)^2 dx \\
\lesssim & \sum_{l=0}^{\frac{5+[\alpha]}{2}} \int_I x^4\sigma^{\alpha+1+l}\sigma^{\frac{[\alpha]-2l+9}{2}}\left(\partial_t^{\frac{1+[\alpha]}{2}}\partial_x^{1+l}u\right)^2 dx \\
\lesssim & \sum_{l=0}^{\frac{5+[\alpha]}{2}} \int_I x^4\sigma^{\alpha+1+l}\left(\partial_t^{\frac{1+[\alpha]}{2}}\partial_x^{1+l}u\right)^2 dx \\
\lesssim & E(t).
\end{aligned}
\tag{4.43}
$$

We next examine the second term appearing on the right-hand side of (4.42):

$$
\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+1}{2}}\partial_t^{\frac{[\alpha]+3}{2}}u_x(0)\partial_t^{\frac{[\alpha]+1}{2}}u_x\|_0^2 \\
\lesssim & \int_0^t \|\sigma^{\frac{[\alpha]-1}{4}}\partial_\tau^{\frac{[\alpha]+3}{2}}u(0)\|_{L^\infty}^2\|x^2\sigma^{\frac{2\alpha-[\alpha]+3}{4}}\partial_\tau^{\frac{[\alpha]+3}{2}}u_x\|_0^2 d\tau + \|x^2\sigma^{\frac{\alpha+1}{2}}\partial_t^{\frac{[\alpha]+3}{2}}u_x(0)\partial_t^{\frac{[\alpha]+1}{2}}u_x(0)\|_0^2.
\end{aligned}
\tag{4.44}
$$

By Hardy's inequality, it follows that

$$
\begin{aligned}
\|x^2\sigma^{\frac{2\alpha-[\alpha]+3}{4}}\partial_t^{\frac{[\alpha]+3}{2}}u_x\|_0^2 \lesssim & \sum_{l=0}^{\frac{3+[\alpha]}{2}} \int_I x^4\sigma^{\frac{2\alpha+[\alpha]+9}{2}}\left(\partial_t^{\frac{3+[\alpha]}{2}}\partial_x^{1+l}u\right)^2 dx \\
\lesssim & \sum_{l=0}^{\frac{3+[\alpha]}{2}} \int_I x^4\sigma^{\alpha+1+l}\sigma^{\frac{[\alpha]-2l+7}{2}}\left(\partial_t^{\frac{3+[\alpha]}{2}}\partial_x^{1+l}u\right)^2 dx \\
\lesssim & \sum_{l=0}^{\frac{3+[\alpha]}{2}} \int_I x^4\sigma^{\alpha+1+l}\left(\partial_t^{\frac{3+[\alpha]}{2}}\partial_x^{1+l}u\right)^2 dx \\
\lesssim & E(t).
\end{aligned}
\tag{4.45}
$$

Proceeding to the third term on the right-hand side of (4.40), we derive

$$
\begin{aligned}
&\left|2\gamma C_{3+[\alpha]}^{\frac{3+[\alpha]}{2}}\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\partial_\tau^{3+[\alpha]}u_x\partial_\tau\left\{\partial_\tau^{\frac{3+[\alpha]}{2}}u\partial_\tau^{\frac{1+[\alpha]}{2}}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right\}dxd\tau\right| \\
\lesssim & \int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\left(\partial_\tau^{3+[\alpha]}u_x\right)^2 dxd\tau \\
&+\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\left\{\partial_\tau\left[\partial_\tau^{\frac{3+[\alpha]}{2}}u\partial_\tau^{\frac{1+[\alpha]}{2}}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right]\right\}^2 dxd\tau.
\end{aligned}
\tag{4.46}
$$

Moreover, it holds that

$$
\begin{aligned}
&\int_0^t\int_I x^4\sigma^{\alpha+1}e^{S_0}\left\{\partial_\tau\left[\partial_\tau^{\frac{3+[\alpha]}{2}}u\partial_\tau^{\frac{1+[\alpha]}{2}}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right]\right\}^2 dxd\tau \\
\lesssim & \int_0^t \|x^2\sigma^{\frac{\alpha+1}{2}}e^{\frac{S_0}{2}}\partial_\tau^{\frac{5+[\alpha]}{2}}u\partial_\tau^{\frac{1+[\alpha]}{2}}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\|_0^2 d\tau
\end{aligned}
\tag{4.47}
$$

$$+\int_0^t \|x^2\sigma^{\frac{\alpha+1}{2}}e^{\frac{S_0}{2}}\partial_\tau^{\frac{3+[\alpha]}{2}}u\partial_\tau^{\frac{3+[\alpha]}{2}}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\|_0^2 d\tau$$
$$+\int_0^t \|x^2\sigma^{\frac{\alpha+1}{2}}e^{\frac{S_0}{2}}\partial_\tau^{\frac{3+[\alpha]}{2}}u\partial_\tau^{\frac{1+[\alpha]}{2}}u_x\partial_\tau\left\{\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right\}\|_0^2 d\tau.$$

We need only handle the first term on the right-hand side of (4.47) by

$$\begin{aligned}&\int_0^t \|x^2\sigma^{\frac{\alpha+1}{2}}e^{\frac{S_0}{2}}\partial_\tau^{\frac{5+[\alpha]}{2}}u\partial_\tau^{\frac{1+[\alpha]}{2}}u_x\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\|_0^2 d\tau\\ \lesssim&\|e^{\frac{S_0}{2}}\|_{L^\infty}^2\int_0^t \|\sigma^{\frac{1+[\alpha]}{4}}\partial_\tau^{\frac{1+[\alpha]}{2}}u_x\|_{L^\infty}^2\|x^2\sigma^{\frac{2\alpha-[\alpha]+1}{2}}\partial_\tau^{\frac{5+[\alpha]}{2}}u\|_0^2 d\tau.\end{aligned}\tag{4.48}$$

Using Hardy's inequality, we obtain

$$\begin{aligned}\|x^2\sigma^{\frac{2\alpha-[\alpha]+1}{2}}\partial_t^{\frac{5+[\alpha]}{2}}u\|_0^2\lesssim&\sum_{l=0}^{\frac{3+[\alpha]}{2}}\int_I x^4\sigma^{\frac{2\alpha+[\alpha]+7}{2}}\left(\partial_t^{\frac{5+[\alpha]}{2}}\partial_x^l u\right)^2 dx\\ \lesssim&\sum_{l=0}^{\frac{3+[\alpha]}{2}}\int_I x^4\sigma^{\alpha+1+l}\sigma^{\frac{[\alpha]-2l+5}{2}}\left(\partial_t^{\frac{5+[\alpha]}{2}}\partial_x^l u\right)^2 dx\\ \lesssim&\sum_{l=0}^{\frac{3+[\alpha]}{2}}\int_I x^4\sigma^{\alpha+1+l}\left(\partial_t^{\frac{5+[\alpha]}{2}}\partial_x^l u\right)^2 dx\\ \lesssim&E(t).\end{aligned}\tag{4.49}$$

Combining the estimates from (4.40) and (4.49), we obtain the bound for D_5. The terms D_2 through D_4 can be handled analogously to D_1 and D_5. Therefore, the remainder in (4.4) can be controlled by $P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right)$. □

5. Higher-order spatial derivative estimates

This section provides the proof of Lemma 5.1, which derives the higher-order estimates (5.1) for spatial derivatives.

Lemma 5.1. *Let $T>0$, and suppose (ξ,u) is a smooth solution to the initial boundary value problem* (2.8)–(2.9) *under condition* (3.1) *on* $(0,1)\times(0,T]$. *Then , the following estimates are valid:*

$$\|(x\sigma^{\frac{\alpha+i-1}{2}}\partial_t^j\partial_x^i u, x^2\sigma^{\frac{\alpha+i+1}{2}}\partial_t^j\partial_x^{i+1}u)\|_0^2\lesssim P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\tag{5.1}$$

Here, $0\le j\le 3+[\alpha]$ *and* $1\le i\le 3+[\alpha]-j$.

Proof. We rewrite (2.8) as the following:

$$\begin{aligned}&x\sigma e^{S_0}u_{xx}+(\alpha+1)x\sigma_x e^{S_0}u_x+4\sigma e^{S_0}u_x\\ =&\frac{x}{\gamma}u_{tt}+x\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\end{aligned}$$

$$
\begin{aligned}
&+(\alpha+1)x\sigma_x e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_x - 2\sigma e^{S_0}[\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma}-1]u_x \\
&-2\sigma e^{S_0}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}-1]u_x - x\sigma\xi^2 e^{S_0}\partial_x[\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_x \\
&-x\sigma\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u_x - (\alpha+1)\sigma_x e^{S_0}\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u \qquad (5.2)\\
&-2\sigma_x e^{S_0}\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma}u + \frac{2}{\gamma}\,\sigma\xi e^{S_0}\partial_x[\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma}]u \\
&-2\sigma\xi^2 e^{S_0}\partial_x[\xi^{-2\gamma-1}(\xi+x\xi_x)^{-\gamma}]u - \sigma\xi^2 e^{S_0}\partial_x[\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u \\
&-\frac{2}{\alpha+1}\sigma\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma}\partial_x(e^{S_0})u - \sigma\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u.
\end{aligned}
$$

By applying the differential operator $\partial_t^j\partial_x^{i-1}$ to both sides of (5.2), we obtain

$$
\begin{aligned}
&\partial_t^j\partial_x^{i-1}(x\sigma e^{S_0}u_{xx}) + (\alpha+1)\partial_t^j\partial_x^{i-1}(x\sigma_x e^{S_0}u_x) + 4\partial_t^j\partial_x^{i-1}(\sigma e^{S_0}u_x) \\
&= \frac{1}{\gamma}\partial_t^{j+2}\partial_x^{i-1}(xu) + \partial_t^j\partial_x^{i-1}\left\{x\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\} \\
&+ (\alpha+1)\partial_t^j\partial_x^{i-1}\left\{x\sigma_x e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_x\right\} - 2\partial_t^j\partial_x^{i-1}\left\{\sigma e^{S_0}[\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma}-1]u_x\right\} \\
&- 2\partial_t^j\partial_x^{i-1}\left\{\sigma e^{S_0}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}-1]u_x\right\} - \partial_t^j\partial_x^{i-1}\left\{x\sigma\xi^2 e^{S_0}\partial_x[\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_x\right\} \\
&- \partial_t^j\partial_x^{i-1}[x\sigma\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u_x] - (\alpha+1)\partial_t^j\partial_x^{i-1}[\sigma_x e^{S_0}\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u] \qquad (5.3)\\
&- 2\partial_t^j\partial_x^{i-1}[\sigma_x e^{S_0}\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma}u] + \frac{2}{\gamma}\,\partial_t^j\partial_x^{i-1}\left\{\sigma\xi e^{S_0}\partial_x[\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma}]u\right\} \\
&- 2\partial_t^j\partial_x^{i-1}\left\{\sigma\xi^2 e^{S_0}\partial_x[\xi^{-2\gamma-1}(\xi+x\xi_x)^{-\gamma}]u\right\} - \partial_t^j\partial_x^{i-1}\left\{\sigma\xi^2 e^{S_0}\partial_x[\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u\right\} \\
&- \frac{2}{\alpha+1}\partial_t^j\partial_x^{i-1}[\sigma\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma}\partial_x(e^{S_0})u] - \partial_t^j\partial_x^{i-1}[\sigma\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u].
\end{aligned}
$$

For mathematical induction purposes, we unify the indices i and j by introducing a reparametrized index μ:

$$
\begin{aligned}
j &= 3 + [\alpha] - 2 - \mu = 1 + [\alpha] - \mu, \\
i &= 3 + [\alpha] - j = 3 + [\alpha] - (1 + [\alpha] - \mu) = 2 + \mu.
\end{aligned}
$$

Therefore, multiplying (5.3) by $x\sigma^{\frac{\alpha+\mu+1}{2}}$ and taking the L^2-norm gives

$$
\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+\mu+3}{2}}e^{S_0}\partial_t^{1+[\alpha]-\mu}\partial_x^{3+\mu}u + (\alpha+\mu+2)x^2\sigma_x\sigma^{\frac{\alpha+\mu+1}{2}}\partial_t^{1+[\alpha]-\mu}\partial_x^{2+\mu}u \\
&\quad + (\mu+5)x\sigma^{\frac{\alpha+\mu+3}{2}}\partial_t^{1+[\alpha]-\mu}\partial_x^{2+\mu}u\|_0^2 \\
&\lesssim \left\|x^2\sigma^{\frac{\alpha+\mu+1}{2}}\partial_t^{3+[\alpha]-\mu}\partial_x^{1+\mu}u\right\|_0^2 + \left\|x\sigma^{\frac{\alpha+\mu+1}{2}}\partial_t^{2+[\alpha]-\mu}\partial_x^{\mu}u\right\|_0^2 \\
&+ \left\|x^2\sigma^{\frac{\alpha+\mu+3}{2}}\partial_x(e^{S_0})\partial_t^{1+[\alpha]-\mu}\partial_x^{2+\mu}u\right\|_0^2 \qquad (5.4)\\
&+ \left\|x\sigma^{\frac{\alpha+\mu+1}{2}}\partial_t^{1+[\alpha]-\mu}\partial_x^{1+\mu}\left\{xe^{S_0}\sigma[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\right\|_0^2 \\
&+ \left\|x\sigma^{\frac{\alpha+\mu+1}{2}}\partial_t^{1+[\alpha]-\mu}\partial_x^{1+\mu}\left\{xe^{S_0}\sigma\xi^2\partial_x[\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_x\right\}\right\|_0^2 \\
&+ \left\|x\sigma^{\frac{\alpha+\mu+1}{2}}\partial_t^{1+[\alpha]-\mu}\partial_x^{1+\mu}\left\{x\sigma\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u_x\right\}\right\|_0^2 + \mathscr{R}^{\mu},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{R}^{\mu} =& \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} \left\{ e^{S_0} \sigma [\xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma} - 1] u_x \right\} \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} \left\{ e^{S_0} \sigma [\xi^{2-2\gamma} (\xi + x\xi_x)^{-\gamma-1} - 1] u_x \right\} \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} \left\{ e^{S_0} \sigma_x \xi^{2-2\gamma} (\xi + x\xi_x)^{-\gamma-1} u \right\} \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} [e^{S_0} \sigma_x \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma} u] \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} \left\{ \xi e^{S_0} \sigma \partial_x [\xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma}] u \right\} \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} \left\{ \xi^2 e^{S_0} \sigma \partial_x [\xi^{-2\gamma-1} (\xi + x\xi_x)^{-\gamma}] u \right\} \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} \left\{ \xi^2 e^{S_0} \sigma \partial_x [\xi^{-2\gamma} (\xi + x\xi_x)^{-\gamma-1}] u \right\} \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} [\sigma \xi^{1-2\gamma} (\xi + x\xi_x)^{-\gamma} \partial_x (e^{S_0}) u] \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} [\sigma \xi^{2-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \partial_x (e^{S_0}) u] \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \partial_t^{1+[\alpha]-\mu} \partial_x^{1+\mu} [\sigma_x \xi^{2-2\gamma} (\xi + x\xi_x)^{-\gamma-1} \partial_x (e^{S_0}) u] \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \sum_{l=2}^{\mu+1} \partial_x^l (x\sigma e^{S_0}) \partial_t^{1+[\alpha]-\mu} \partial_x^{3+\mu-l} u \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \sum_{l=1}^{\mu+1} \partial_x^l (x\sigma_x e^{S_0}) \partial_t^{1+[\alpha]-\mu} \partial_x^{2+\mu-l} u \right\|_0^2 \\
&+ \left\| x\sigma^{\frac{\alpha+\mu+1}{2}} \sum_{l=1}^{\mu+1} \partial_x^l (\sigma_x e^{S_0}) \partial_t^{1+[\alpha]-\mu} \partial_x^{2+\mu-l} u \right\|_0^2 .
\end{aligned}
\tag{5.5}
$$

We shall establish (5.1) for all μ by mathematical induction, which involves two key steps as follows.

Step 1. *To apply mathematical induction, we begin by assuming the validity of the following estimates:*

$$
\|x^2\sigma^{\frac{\alpha+3}{2}} e^{S_0} \partial_t^{1+[\alpha]} \partial_x^3 u\|_0^2 + \|x\sigma^{\frac{\alpha+1}{2}} e^{S_0} \partial_t^{1+[\alpha]} \partial_x^2 u\|_0^2 \lesssim P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau) \right). \tag{5.6}
$$

From (5.4), we compute its left-hand side as

$$
\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+3}{2}} e^{S_0} \partial_t^{1+[\alpha]} \partial_x^3 u + (\alpha+2) x^2 \sigma_x \sigma^{\frac{\alpha+1}{2}} e^{S_0} \partial_t^{1+[\alpha]} \partial_x^2 u + 5x\sigma^{\frac{\alpha+3}{2}} e^{S_0} \partial_t^{1+[\alpha]} \partial_x^2 u\|_0^2 \\
=&\|x^2\sigma^{\frac{\alpha+3}{2}} e^{S_0} \partial_t^{1+[\alpha]} \partial_x^3 u\|_0^2 + (\alpha+2)^2 \|x^2 \sigma_x \sigma^{\frac{\alpha+1}{2}} e^{S_0} \partial_t^{1+[\alpha]} \partial_x^2 u\|_0^2 \\
&+25\|x\sigma^{\frac{\alpha+3}{2}} e^{S_0} \partial_t^{1+[\alpha]} \partial_x^2 u\|_0^2 + \int_I \left[5x^3\sigma^{\alpha+3} + (\alpha+2) x^4 \sigma^{\alpha+2} \sigma_x \right] e^{2S_0} \partial_x \left\{ (\partial_t^{1+[\alpha]} \partial_x^2 u)^2 \right\} dx \\
&+10(\alpha+2) \int_I x^3 \sigma^{\alpha+2} \sigma_x e^{2S_0} (\partial_t^{1+[\alpha]} \partial_x^2 u)^2 dx.
\end{aligned}
\tag{5.7}
$$

Applying integration by parts with the identity $\sigma_x = -2x$, we derive

$$\begin{aligned}
&\int_I \left[5x^3\sigma^{\alpha+3} + (\alpha+2)x^4\sigma^{\alpha+2}\sigma_x\right] e^{2S_0}\partial_x \left\{(\partial_t^{1+[\alpha]}\partial_x^2 u)^2\right\} dx \\
=& -15\int_I x^2\sigma^{\alpha+3}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2 u\right)^2 dx - (\alpha+2)^2\int_I x^4\sigma^{\alpha+1}\sigma_x^2 e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2 u\right)^2 dx \\
&+ 10(2\alpha+5)\int_I x^4\sigma^{\alpha+2}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2 u\right)^2 dx \qquad (5.8)\\
\gtrsim& -15\int_I x^2\sigma^{\alpha+3}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2 u\right)^2 dx - (\alpha+2)^2\int_I x^4\sigma^{\alpha+1}\sigma_x^2\, e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2 u\right)^2 dx \\
&- C\int_I x^4\sigma^{\alpha+2}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2 u\right)^2 dx.
\end{aligned}$$

Assuming temporarily that the right hand side of (5.4) is bounded by $P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right)$ ((5.9) will be subsequently verified), we proceed with

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^3 u + (\alpha+2)x^2\sigma_x\sigma^{\frac{\alpha+1}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2 u + 5x\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2 u\|_0^2 \qquad (5.9)\\
\gtrsim&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^3 u\|_0^2 + 10\|x\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2 u\|_0^2 - C\int_I x^4\sigma^{\alpha+2}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2 u\right)^2 dx.
\end{aligned}$$

At this point, substituting the result from (5.8) into (5.7) and utilizing $\sigma_x = -2x$, we obtain

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^3 u + (\alpha+2)x^2\sigma_x\sigma^{\frac{\alpha+1}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2 u + 5x\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2 u\|_0^2 \\
\lesssim& P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \qquad (5.10)
\end{aligned}$$

Applying the Newton-Leibniz formula gives

$$\int_I x^4\sigma^{\alpha+2}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2 u\right)^2 dx \lesssim P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \qquad (5.11)$$

Combining (5.9)–(5.11), we derive

$$\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^3 u\|_0^2 + 10\|x\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2 u\|_0^2 \lesssim P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \qquad (5.12)$$

Combined with (5.10), this yields

$$\|(\alpha+2)x^2\sigma_x\sigma^{\frac{\alpha+1}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2 u\|_0^2 \lesssim P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \qquad (5.13)$$

From (5.12) and (5.13), we derive

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^3 u\|_0^2 + \|x^2\sigma^{\frac{\alpha+1}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2 u\|_0^2 + \|x\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2 u\|_0^2 \qquad (5.14)\\
\lesssim& P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).
\end{aligned}$$

From (5.14) and $\sigma_x = -2x$, it follows that

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^3u\|_0^2 + \|x^3\sigma^{\frac{\alpha+1}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2u\|_0^2 + \|x\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2u\|_0^2 \\ \lesssim& P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).\end{aligned} \tag{5.15}$$

Combining (5.15) with previous results, we have

$$\begin{aligned}&\|x\sigma^{\frac{\alpha+1}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2u\|_0^2 \\ \lesssim& \int_{I_0} x^2\sigma^{\alpha+3}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2u\right)^2dx + \int_{I_b} x^6\sigma^{\alpha+1}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2u\right)^2dx \\ \lesssim& \int_{I} x^2\sigma^{\alpha+3}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2u\right)^2dx + \int_{I} x^6\sigma^{\alpha+1}e^{2S_0}\left(\partial_t^{1+[\alpha]}\partial_x^2u\right)^2dx \\ \lesssim& P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).\end{aligned} \tag{5.16}$$

The combination of (5.15) and (5.16) leads to

$$\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^3u\|_0^2 + \|x\sigma^{\frac{\alpha+1}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2u\|_0^2 \lesssim P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \tag{5.17}$$

To prove (5.10), it concludes from (5.4) with $\mu = 0$ that

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^3u + (\alpha+2)x^2\sigma_x\sigma^{\frac{\alpha+1}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2u + 5x\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2u\|_0^2 \\ \lesssim& \|x^2\sigma^{\frac{\alpha+1}{2}}\partial_t^{3+[\alpha]}\partial_xu\|_0^2 + \|x\sigma^{\frac{\alpha+1}{2}}\partial_t^{3+[\alpha]}u\|_0^2 + \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\partial_t^{1+[\alpha]}\partial_x^2u\|_0^2 \\ &+\|x\sigma^{\frac{\alpha+1}{2}}\partial_t^{1+[\alpha]}\partial_x\left\{x\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2 \\ &+\|x\sigma^{\frac{\alpha+1}{2}}\partial_t^{1+[\alpha]}\partial_x\left\{x\sigma e^{S_0}\xi^2\partial_x[\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_x\right\}\|_0^2 \\ &+\|x\sigma^{\frac{\alpha+1}{2}}\partial_t^{1+[\alpha]}\partial_x\left\{x\sigma\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x\right\}\|_0^2 + \mathscr{R}^0.\end{aligned} \tag{5.18}$$

We now estimate the third term on the right-hand side of (5.18):

$$\begin{aligned}\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\partial_t^{1+[\alpha]}\partial_x^2u\|_0^2 \lesssim& \|\partial_x(e^{S_0})\|_{L^\infty}^2\int_0^t \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_\tau^{2+[\alpha]}\partial_x^2u\|_0^2d\tau \\ &+ \|\partial_x(e^{S_0})\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{1+[\alpha]}\partial_x^2u(0)\|_0^2 \lesssim P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).\end{aligned} \tag{5.19}$$

Regarding the fourth term on the right-hand side of (5.18), we establish the following estimate:

$$\begin{aligned}&\|x\sigma^{\frac{\alpha+1}{2}}\partial_t^{1+[\alpha]}\partial_x\left\{x\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2 \\ \lesssim& \|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\left\{[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xxx}\right\}\|_0^2 \\ &+ \|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\left\{\partial_x[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2 \\ &+ \|x^2\sigma^{\frac{\alpha+1}{2}}\sigma_xe^{S_0}\partial_t^{1+[\alpha]}\left\{[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2 \\ &+ \|x\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\left\{[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2\end{aligned} \tag{5.20}$$

$$+\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\partial_t^{1+[\alpha]}\left\{[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2.$$

The first right-hand term in (5.20) is estimated via the Newton-Leibniz formula:

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\left\{[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xxx}\right\}\|_0^2\\
\lesssim&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_t^{1+[\alpha]}u_{xxx}\|_0^2\\
+&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\left\{\sum_{m=1}^{1+[\alpha]}C_{1+[\alpha]}^m\partial_t^m[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_t^{1+[\alpha]-m}u_{xxx}\right\}\|_0^2.
\end{aligned}\tag{5.21}$$

For the first right-hand term in (5.21), we apply the Newton-Leibniz formula to obtain

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_t^{1+[\alpha]}u_{xxx}\|_0^2\\
\lesssim&\|e^{S_0}\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha+3}{2}}\int_0^t\partial_\tau\left[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right]d\tau\partial_\tau^{1+[\alpha]}u_{xxx}\|_0^2\\
\lesssim&\|e^{S_0}\|_{L^\infty}^2\int_0^t\left(\|u\|_{L^\infty}^2+\|u_x\|_{L^\infty}^2\right)\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_\tau^{1+[\alpha]}u_{xxx}\|_0^2\,d\tau\\
\lesssim&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}\tag{5.22}$$

Regarding the second term on the right-hand side of (5.21), we specifically analyze the case of $m=1+[\alpha]$ as

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xxx}\|_0^2\\
\lesssim&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{[\alpha]}\left\{[(2-2\gamma)\xi^{1-2\gamma}(\xi+x\xi_x)^{-\gamma-1}+(-\gamma-1)\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}]u\right\}u_{xxx}\|_0^2\\
&+\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{[\alpha]}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}u_x]u_{xxx}\|_0^2.
\end{aligned}\tag{5.23}$$

We apply the Leibniz formula to the second term on the right-hand side of (5.23) and obtain

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{[\alpha]}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}u_x]u_{xxx}\|_0^2\\
=&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\left\{\sum_{n=0}^{[\alpha]}C_{[\alpha]}^n\partial_t^n[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}]\partial_t^{[\alpha]-n}u_x\right\}u_{xxx}\|_0^2.
\end{aligned}$$

For $n=0$, we have

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\partial_t^{[\alpha]}u_xu_{xxx}\|_0^2\\
\lesssim&\|e^{S_0}\|_{L^\infty}^2\int_0^t\|\sigma^2u_{xxx}\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha-1}{2}}\partial_\tau^{[\alpha]+1}u_x\|_0^2d\tau\\
+&\|e^{S_0}\|_{L^\infty}^2\int_0^t\|\sigma^{\frac{[\alpha]}{2}}\partial_\tau^{[\alpha]}u_x(0)\|_{L^\infty}^2d\tau\|x^2\sigma^{\frac{\alpha-[\alpha]+3}{2}}\partial_\tau u_{xxx}\|_0^2\\
+&\|e^{S_0}\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{[\alpha]}u_x(0)u_{xxx}(0)\|_0^2.
\end{aligned}\tag{5.24}$$

Since $\alpha-1>-1$ and $\alpha-[\alpha]+3>-1$, applying Hardy's inequality yields

$$\|x^2\sigma^{\frac{\alpha-1}{2}}\partial_t^{[\alpha]+1}u_x\|_0^2\lesssim\sum_{l=0}^p\int_I x^4\sigma^{\alpha-1+2p}\left(\partial_t^{1+[\alpha]}\partial_x^{l+1}u\right)^2dx$$

$$
\begin{aligned}
&\lesssim \sum_{l=0}^{2} \int_I x^4 \sigma^{\alpha+3} \left(\partial_t^{1+[\alpha]} \partial_x^{l+1} u\right)^2 dx && (5.25)\\
&\lesssim \sum_{l=0}^{2} \int_I x^4 \sigma^{\alpha+1+l} \left(\partial_t^{1+[\alpha]} \partial_x^{l+1} u\right)^2 dx\\
&\lesssim E(t).
\end{aligned}
$$

Since $\alpha - [\alpha] + 3 > -1$, we apply Hardy's inequality to obtain

$$
\|x^2 \sigma^{\frac{\alpha-[\alpha]+3}{2}} \partial_t u_{xxx}\|_0^2 \lesssim \sum_{l=0}^{[\alpha]} \int_I x^4 \sigma^{\alpha+[\alpha]+3} \left(\partial_t \partial_x^{l+3} u\right)^2 dx \tag{5.26}
$$
$$
\lesssim \sum_{l=0}^{[\alpha]} \int_I x^4 \sigma^{\alpha+3+l} \left(\partial_t \partial_x^{l+3} u\right)^2 dx \lesssim E(t).
$$

The second term on the right-hand side of (5.20) is estimated as follows:

$$
\begin{aligned}
&\|x^2 \sigma^{\frac{\alpha+3}{2}} e^{S_0} \partial_t^{1+[\alpha]} \left\{\partial_x [\xi^{2-2\gamma} (\xi + x\xi_x)^{-\gamma-1}] u_{xx}\right\} \|_0^2\\
\lesssim &\|x^2 \sigma^{\frac{\alpha+3}{2}} e^{S_0} \partial_t^{1+[\alpha]} \left\{[(2-2\gamma)\xi^{1-2\gamma}(\xi + x\xi_x)^{-\gamma-1}\xi_x + 2(-\gamma-1)\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2}\xi_x] u_{xx}\right\} \|_0^2\\
&+ \|x^2 \sigma^{\frac{\alpha+3}{2}} e^{S_0} \partial_t^{1+[\alpha]} [\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2}\xi_{xx} u_{xx}]\|_0^2. && (5.27)
\end{aligned}
$$

Applying the Leibniz formula to the second term on the right-hand side of (5.27) yields

$$
\begin{aligned}
&\|x^2 \sigma^{\frac{\alpha+3}{2}} e^{S_0} \partial_t^{1+[\alpha]} [\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2}\xi_{xx} u_{xx}]\|_0^2\\
\lesssim &\|x^2 \sigma^{\frac{\alpha+3}{2}} e^{S_0} \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2}\xi_{xx} \partial_t^{1+[\alpha]} u_{xx}\|_0^2 && (5.28)\\
+&\|x^2 \sigma^{\frac{\alpha+3}{2}} e^{S_0} \left\{\sum_{m=1}^{1+[\alpha]} C_{1+[\alpha]}^m \partial_t^m [\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2}\xi_{xx}] \partial_t^{1+[\alpha]-m} u_{xx}\right\} \|_0^2.
\end{aligned}
$$

The first term on the right-hand side of (5.28) is estimated by

$$
\begin{aligned}
&\|x^2 \sigma^{\frac{\alpha+3}{2}} e^{S_0} \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2}\xi_{xx} \partial_t^{1+[\alpha]} u_{xx}\|_0^2\\
\lesssim &\|e^{S_0}\|_{L^\infty}^2 \int_0^t \|\sigma u_{xx}\|_{L^\infty}^2 \|x \sigma^{\frac{\alpha+1}{2}} \partial_\tau^{1+[\alpha]} u_{xx}\|_0^2 d\tau && (5.29)\\
\lesssim &CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).
\end{aligned}
$$

We now analyze the principal obstacle in (5.29) as

$$
\begin{aligned}
&\|x^2 \sigma^{\frac{\alpha+3}{2}} e^{S_0} \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} \partial_t^{[\alpha]} u_{xx} u_{xx}\|_0^2\\
\lesssim &\|e^{S_0}\|_{L^\infty}^2 \int_0^t \|\sigma u_{xx}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+1}{2}} \partial_\tau^{1+[\alpha]} u_{xx}\|_0^2 d\tau + \|e^{S_0}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+3}{2}} \partial_t^{[\alpha]} u_{xx}(0) u_{xx}\|_0^2 && (5.30)\\
\lesssim &CtP\left(\sup_{0\le\tau\le t} E(\tau)\right) + \|e^{S_0}\|_{L^\infty}^2 \int_0^t \|\sigma^{\frac{[\alpha]+2}{2}} \partial_\tau^{[\alpha]} u_{xx}(0)\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha-[\alpha]+1}{2}} \partial_\tau u_{xx}\|_0^2 d\tau
\end{aligned}
$$

$$+\|e^{S_0}\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{[\alpha]}u_{xx}(0)u_{xx}(0)\|_0^2.$$

Since $\alpha-[\alpha]+1>-1$, applying Hardy's inequality yields

$$\begin{aligned}\|x^2\sigma^{\frac{\alpha-[\alpha]+1}{2}}\partial_t u_{xx}\|_0^2 &\lesssim \sum_{l=0}^{1+[\alpha]}\int_I x^4\sigma^{\alpha+[\alpha]+3}\left(\partial_t\partial_x^{l+2}u\right)^2dx\\ &\lesssim \sum_{l=0}^{1+[\alpha]}\int_I x^4\sigma^{\alpha+2+l}\sigma^{1+[\alpha]-l}\left(\partial_t\partial_x^{l+2}u\right)^2dx\\ &\lesssim \sum_{l=0}^{1+[\alpha]}\int_I x^4\sigma^{\alpha+2+l}\left(\partial_t\partial_x^{l+2}u\right)^2dx\\ &\lesssim E(t).\end{aligned} \tag{5.31}$$

We now compute the sixth term on the right-hand side of (5.18):

$$\begin{aligned}&\|x\sigma^{\frac{\alpha+1}{2}}\partial_t^{1+[\alpha]}\partial_x\left[x\sigma\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u_x\right]\|_0^2\\ \lesssim&\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{1+[\alpha]}\left[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u_{xx}\right]\|_0^2\\ &+\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{1+[\alpha]}\left\{\partial_x[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_x(e^{S_0})u_x\right\}\|_0^2\\ &+\|x^2\sigma^{\frac{\alpha+1}{2}}\sigma_x\partial_t^{1+[\alpha]}\left[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u_x\right]\|_0^2\\ &+\|x\sigma^{\frac{\alpha+3}{2}}\partial_t^{1+[\alpha]}\left[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u_x\right]\|_0^2\\ &+\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{1+[\alpha]}\left[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x^2(e^{S_0})u_x\right]\|_0^2.\end{aligned} \tag{5.32}$$

According to the Leibniz formula, the first term on the right-hand side of (5.32) can be expressed as

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{1+[\alpha]}\left[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})u_{xx}\right]\|_0^2\\ \lesssim&\|x^2\sigma^{\frac{\alpha+3}{2}}\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})\partial_t^{1+[\alpha]}u_{xx}\|_0^2\\ &+\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\left\{\sum_{m=1}^{1+[\alpha]}C_{1+[\alpha]}^m\partial_t^m[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_t^{1+[\alpha]-m}u_{xx}\right\}\|_0^2.\end{aligned} \tag{5.33}$$

We turn now to estimating the first term on the right-hand side of (5.33):

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+3}{2}}\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x(e^{S_0})\partial_t^{1+[\alpha]}u_{xx}\|_0^2\\ \lesssim&\|\partial_x(e^{S_0})\|_{L^\infty}^2\left(\int_0^t\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_\tau^{2+[\alpha]}u_{xx}\|_0^2d\tau+\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{1+[\alpha]}u_{xx}(0)\|_0^2\right)\\ \lesssim&P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned} \tag{5.34}$$

For the second term on the right-hand side of (5.33),we consider only the case $m=1+[\alpha]$. The most difficult term of it is

$$\|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\partial_t^{[\alpha]}u_xu_{xx}\|_0^2$$

$$
\begin{aligned}
\lesssim & \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|\sigma u_{xx}\|_{L^\infty}^2 \int_0^t \|x^2\sigma^{\frac{\alpha+1}{2}}\partial_\tau^{1+[\alpha]}u_x\|_0^2 d\tau \\
& + \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{[\alpha]}u_x(0)u_{xx}\|_0^2 \lesssim CtP\left(\sup_{0\le\tau\le t} E(\tau)\right) \\
& + \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|\sigma^{\frac{[\alpha]}{2}}\partial_t^{[\alpha]}u_x(0)\|_{L^\infty}^2 \int_0^t \|x^2\sigma^{\frac{\alpha-[\alpha]+3}{2}}\partial_\tau u_{xx}\|_0^2 d\tau \\
& + \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{[\alpha]}u_x(0)u_{xx}(0)\|_0^2,
\end{aligned}
\tag{5.35}
$$

where

$$
\begin{aligned}
& \|x^2\sigma^{\frac{\alpha-[\alpha]+3}{2}}\partial_t u_{xx}\|_0^2 \lesssim \sum_{l=0}^{1+[\alpha]} \int_I x^4\sigma^{\alpha+[\alpha]+5}\left(\partial_t\partial_x^{l+2}u\right)^2 dx \\
\lesssim & \sum_{l=0}^{1+[\alpha]} \int_I x^4\sigma^{\alpha+2+l}\left(\partial_t\partial_x^{l+2}u\right)^2 dx \lesssim E(t).
\end{aligned}
\tag{5.36}
$$

We now handle the most challenging term in the second term on the right-hand side of (5.32):

$$
\begin{aligned}
& \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\partial_t^{1+[\alpha]}\left[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\xi_{xx}u_x\right]\|_0^2 \\
\lesssim & \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\xi_{xx}\partial_t^{1+[\alpha]}u_x\|_0^2 \\
& + \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\left\{\sum_{m=1}^{1+[\alpha]} C_{1+[\alpha]}^m \partial_t^m[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\xi_{xx}]\partial_t^{1+[\alpha]-m}u_x\right\}\|_0^2.
\end{aligned}
\tag{5.37}
$$

The first right-hand side term in (5.37) is

$$
\begin{aligned}
& \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\xi_{xx}\partial_t^{1+[\alpha]}u_x\|_0^2 \\
\lesssim & \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|x^2\sigma^{\frac{\alpha+1}{2}}\partial_t^{1+[\alpha]}u_x\|_0^2 \int_0^t \|\sigma u_{xx}\|_{L^\infty}^2 d\tau \\
\lesssim & CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).
\end{aligned}
\tag{5.38}
$$

For the second term on the right-hand side of (5.37), we consider only the case $m = 1 + [\alpha]$. The most difficult term of it is

$$
\begin{aligned}
& \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\partial_t^{[\alpha]}u_{xx}u_x\|_0^2 \\
\lesssim & \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|u_x\|_{L^\infty}^2 \int_0^t \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_\tau^{1+[\alpha]}u_{xx}\|_0^2 d\tau \\
& + \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|x^2\sigma^{\frac{2+\alpha}{2}}\partial_t^{[\alpha]}u_{xx}(0)\|_0^2 \int_0^t \|\sigma^{\frac{1}{2}}\partial_\tau u_x\|_{L^\infty}^2 d\tau \\
& + \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|x^2\sigma^{\frac{\alpha+3}{2}}\partial_t^{[\alpha]}u_{xx}(0)u_x(0)\|_0^2 \\
\lesssim & P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right).
\end{aligned}
\tag{5.39}
$$

Through similar analysis, we handle the remaining terms in (5.18) and obtain

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^3u+(\alpha+2)x^2\sigma_x\sigma^{\frac{\alpha+1}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2u+5x\sigma^{\frac{\alpha+3}{2}}e^{S_0}\partial_t^{1+[\alpha]}\partial_x^2u\|_0^2\\ \lesssim&P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}\tag{5.40}$$

Step 2. *To utilize mathematical induction, we first assume that the following estimates hold for* $0\le\mu\le k$ *with* $1\le k\le[\alpha]$ *and* $1\le a\le\mu+2$*:*

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+\mu+3}{2}}e^{S_0}\partial_t^{1+[\alpha]-\mu}\partial_x^{a+1}u\|_0^2+\|x\sigma^{\frac{\alpha+\mu+1}{2}}e^{S_0}\partial_t^{1+[\alpha]-\mu}\partial_x^{a}u\|_0^2\\ \lesssim&P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}\tag{5.41}$$

Then, it needs to show (5.41) *for* $k+1$ *with* $1\le a\le k+3$.

$$\|x^2\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{a+1}u\|_0^2+\|x\sigma^{\frac{\alpha+k+2}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{a}u\|_0^2\lesssim P(0))+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\tag{5.42}$$

Similar to (5.7), it follows from (5.4) that

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+4}u+(\alpha+k+3)x^2\sigma_x\sigma^{\frac{\alpha+k+2}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+3}u\\ &+(k+6)x\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+3}u\|_0^2\\ &=\|x^2\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+4}u\|_0^2+(\alpha+k+3)^2\|x^2\sigma_x\sigma^{\frac{\alpha+k+2}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+3}u\|_0^2\\ &+(k+6)^2\|x\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+3}u\|_0^2\\ &+\int_I[(k+6)x^3\sigma^{\alpha+k+4}+(\alpha+k+3)\sigma^{\alpha+k+3}\sigma_x]e^{2S_0}\left(\partial_t^{[\alpha]-k}\partial_x^{k+3}u\right)_x^2dx\\ &+2(k+6)(\alpha+k+3)\int_I x^3\sigma^{\alpha+k+3}\sigma_xe^{2S_0}\left(\partial_t^{[\alpha]-k}\partial_x^{k+3}u\right)^2dx.\end{aligned}\tag{5.43}$$

By analogy with the derivations (5.10)–(5.17), the following is obtained:

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+4}u\|_0^2+\|x\sigma^{\frac{\alpha+k+2}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+3}u\|_0^2\\ \lesssim&P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}\tag{5.44}$$

As in the case $\mu=0$, assuming temporarily that the right-hand side of (5.4) is bounded by $P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right)$, we obtain

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+4}u+(\alpha+k+3)x^2\sigma_x\sigma^{\frac{\alpha+k+2}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+3}u\\ &+(k+6)x\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+3}u\|_0^2\lesssim P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}\tag{5.45}$$

We now present a formal proof of (5.45):

$$\|x^2\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+4}u+(\alpha+k+3)x^2\sigma_x\sigma^{\frac{\alpha+k+2}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+3}u$$

$$
\begin{aligned}
&+ (k+6)x\sigma^{\frac{\alpha+k+4}{2}} e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+3}u\|_0^2 \\
&\lesssim \|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{2+[\alpha]-k}\partial_x^{k+2}u\|_0^2 + \|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{1+[\alpha]-k}\partial_x^{k+1}u\|_0^2 \\
&+ \|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}\partial_x^{k+3}u\|_0^2 \\
&+ \|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left\{x\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2 \\
&+ \|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left\{x\sigma\xi^2 e^{S_0}\partial_x[\xi^{-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_x\right\}\|_0^2 \\
&+ \|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left\{x\sigma\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x\right\}\|_0^2 + \mathscr{R}^{k+1},
\end{aligned}
\tag{5.46}
$$

where $\mathscr{R}^{k+1}$ is given by (5.4) with $\mu = k-1$ and the first and second terms on the right-hand side of (5.46) can be derived from the expression for $\mu = k-1$ in (5.41).

For the third term on the right-hand side of (5.46), we proceed to estimate it as follows:

$$
\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}\partial_x^{k+3}u\|_0^2 \\
\lesssim&\|\partial_x(e^{S_0})\|_{L^\infty}^2\int_0^t \|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_\tau^{1+[\alpha]-k}\partial_x^{k+3}u\|_0^2 d\tau \\
+&\|\partial_x(e^{S_0})\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+3}u(0)\|_0^2 \\
\lesssim&P(0) + CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}
\tag{5.47}
$$

Next, we estimate the fourth term on the right-hand side of (5.46) by applying the Leibniz formula:

$$
\begin{aligned}
&\|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left\{x\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2 \\
=&\|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\left\{\sum_{l=0}^{k+2}C_{k+2}^l\partial_x^l x\partial_x^{k+2-l}\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2 \\
\lesssim&\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left\{\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2 \\
+&\|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+1}\left\{\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2.
\end{aligned}
\tag{5.48}
$$

Restricting our attention to the first term on the right-hand side of (5.48), an application of the Leibniz formula yields

$$
\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left\{\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2 \\
=&\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\left\{\sum_{m=0}^{k+2}C_{k+2}^m\partial_x^m e^{S_0}\partial_x^{k+2-m}\sigma[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2.
\end{aligned}
\tag{5.49}
$$

Setting $m = 0$ in (5.49) yields

$$
\begin{aligned}
&x^2\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k}\partial_x^{k+2}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx} \\
=&\sum_{p=0}^{[\alpha]-k}\sum_{n=0}^{k+2}C_{[\alpha]-k}^p C_{k+2}^n x^2\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^p\partial_x^n\left[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right]\partial_t^{[\alpha]-k-p}\partial_x^{k+2-n}u_{xx}
\end{aligned}
\tag{5.50}
$$

$$= \sum_{p=0}^{[\alpha]-k} \sum_{n=0}^{k+2} C^k_{n,p} P^k_{n,p},$$

where $C^k_{n,p}$ is a constant depending on n, p, and k and $P^k_{n,p}$ satisfies

$$P^k_{n,p} = x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \partial_t^p \partial_x^n [1 - \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-1}] \partial_t^{[\alpha]-k-p} \partial_x^{k+2-n} u_{xx}.$$

The principal technical challenge emerges in the treatment of the following terms:

$$\begin{aligned}
P^k_{0,0} &= x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} [1 - \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-1}] \partial_t^{[\alpha]-k} \partial_x^{k+2} u_{xx} \\
P^k_{0,[\alpha]-k} &= x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \partial_t^{\alpha]-k} [1 - \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-1}] \partial_x^{k+2} u_{xx} \\
P^k_{k+2,0} &= x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \partial_x^{k+2} [1 - \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-1}] \partial_t^{[\alpha]-k} u_{xx} \\
P^k_{k+2,[\alpha]-k} &= x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \partial_t^{[\alpha]-k} \partial_x^{k+2} [1 - \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-1}] u_{xx}.
\end{aligned} \tag{5.51}$$

Regarding $\|P^k_{0,0}\|_0^2$, the following holds:

$$\begin{aligned}
\|P^k_{0,0}\|_0^2 =& \|x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} [1 - \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-1}] \partial_t^{[\alpha]-k} \partial_x^{k+2} u_{xx}\|_0^2 \\
\lesssim& \|e^{S_0}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+k+4}{2}} \int_0^t \partial_\tau [1 - \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-1}] d\tau \partial_t^{[\alpha]-k} \partial_x^{k+2} u_{xx}\|_0^2 \\
\lesssim& \|e^{S_0}\|_{L^\infty}^2 \int_0^t \left(\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2 \right) \|x^2 \sigma^{\frac{\alpha+k+4}{2}} \partial_\tau^{[\alpha]-k} \partial_x^{k+4} u\|_0^2 d\tau \\
\lesssim& CtP \left(\sup_{0 \le \tau \le t} E(\tau) \right).
\end{aligned} \tag{5.52}$$

For $\|P^k_{0,[\alpha]-k}\|_0^2$, we apply the Leibniz formula and obtain

$$\begin{aligned}
& \|x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \partial_t^{\alpha]-k-1} [\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} u_x] \partial_x^{k+2} u_{xx}\|_0^2 \\
=& \|x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \left\{ \sum_{a=0}^{[\alpha]-k-1} C^a_{[\alpha]-k-1} \partial_t^a [\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2}] \partial_t^{[\alpha]-k-1-a} u_x \partial_x^{k+4} u \right\} \|_0^2.
\end{aligned} \tag{5.53}$$

Setting $a = 0$ gives

$$\begin{aligned}
& \|x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} \partial_t^{[\alpha]-k-1} u_x \partial_x^{k+4} u\|_0^2 \\
\lesssim& \|e^{S_0}\|_{L^\infty}^2 \int_0^t \|\sigma^{\frac{[\alpha]-k}{2}} \partial_\tau^{[\alpha]-k} u_x\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha-[\alpha]+2k+4}{2}} \partial_x^{k+4} u\|_0^2 d\tau \\
&+ \|e^{S_0}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+k+4}{2}} \partial_t^{[\alpha]-k-1} u_x(0) \partial_x^{k+4} u\|_0^2.
\end{aligned} \tag{5.54}$$

We now handle the first term on the right-hand side of (5.54). Given that $\alpha - \lfloor \alpha \rfloor + 2k + 4 > -1$, it follows that

$$\|x^2 \sigma^{\frac{\alpha-[\alpha]+2k+4}{2}} \partial_x^{k+4} u\|_0^2 \lesssim \sum_{l=0}^{[\alpha]-k} \int_I x^4 \sigma^{\alpha+[\alpha]+4} \left(\partial_x^{k+4+l} u \right)^2 dx$$

$$\lesssim \sum_{l=0}^{[\alpha]-k} \int_I x^4 \sigma^{\alpha+k+4+l} \sigma^{[\alpha]-k-l} \left(\partial_x^{k+4+l} u\right)^2 dx \tag{5.55}$$
$$\lesssim \sum_{l=0}^{[\alpha]-k} \int_I x^4 \sigma^{\alpha+k+4+l} \left(\partial_x^{k+4+l} u\right)^2 dx$$
$$\lesssim E(t).$$

We handle the second term on the right-hand side of (5.54) via Hardy's inequality, obtaining

$$\begin{aligned}
&\|x^2 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \partial_t^{[\alpha]-k-1} u_x(0) \partial_x^{k+4} u\|_0^2 \\
\lesssim &\|e^{S_0}\|_{L^\infty}^2 \int_0^t \|\sigma^{\frac{[\alpha]-k-1}{2}} \partial_\tau^{[\alpha]-k-1} u_x\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha-[\alpha]+2k+5}{2}} \partial_\tau \partial_x^{k+4} u\|_0^2 d\tau \\
+&\|e^{S_0}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+k+4}{2}} \partial_t^{[\alpha]-k-1} u_x(0) \partial_x^{k+4} u(0)\|_0^2.
\end{aligned} \tag{5.56}$$

Since $\alpha - [\alpha] + 2k + 5 > -1$, we apply Hardy's inequality and obtain

$$\begin{aligned}
&\|x^2 \sigma^{\frac{\alpha-[\alpha]+2k+5}{2}} \partial_t \partial_x^{k+4} u\|_0^2 \lesssim \sum_{l=0}^{[\alpha]-k-1} \int_I x^4 \sigma^{\alpha+[\alpha]+3} \left(\partial_t \partial_x^{k+4+l} u\right)^2 dx \\
\lesssim &\sum_{l=0}^{[\alpha]-k-1} \int_I x^4 \sigma^{\alpha+k+4+l} \left(\partial_t \partial_x^{k+4+l} u\right)^2 dx \lesssim E(t).
\end{aligned} \tag{5.57}$$

For $\|P_{k+2,0}^k\|_0^2$, the following holds:

$$\begin{aligned}
&\|x^3 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \xi^{2-2\gamma} (\xi + x\xi_x)^{-\gamma-2} \partial_x^{k+1} \xi_{xx} \partial_t^{[\alpha]-k} u_{xx}\|_0^2 \\
\lesssim &\|e^{S_0}\|_{L^\infty}^2 \int_0^t \|\sigma^{\frac{2+[\alpha]-k}{2}} \partial_\tau^{[\alpha]-k} u_{xx}\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha-[\alpha]+2k+2}{2}} \partial_x^{k+3} u\|_0^2 d\tau.
\end{aligned} \tag{5.58}$$

Since $\alpha - [\alpha] + 2k + 2 > -1$, we apply Hardy's inequality to obtain

$$\begin{aligned}
&\|x^2 \sigma^{\frac{\alpha-[\alpha]+2k+2}{2}} \partial_x^{k+3} u\|_0^2 \lesssim \sum_{l=0}^{[\alpha]-k+1} \int_I x^4 \sigma^{\alpha+[\alpha]+4} \left(\partial_x^{k+3+l} u\right)^2 dx \\
\lesssim &\sum_{l=0}^{[\alpha]-k+1} \int_I x^4 \sigma^{\alpha+k+3+l} \sigma^{[\alpha]-k-1+l} \left(\partial_x^{k+3+l} u\right)^2 dx \lesssim E(t).
\end{aligned} \tag{5.59}$$

For $\|P_{k+2,[\alpha]-k}^k\|_0^2$, the most difficult term is treated by applying the Leibniz formula, which yields

$$\begin{aligned}
&\|x^3 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \partial_t^{[\alpha]-k} [\xi^{2-2\gamma} (\xi + x\xi_x)^{-\gamma-2} \partial_x^{k+1} \xi_{xx}] u_{xx}\|_0^2 \\
\lesssim &\|x^3 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \xi^{2-2\gamma} (\xi + x\xi_x)^{-\gamma-2} \partial_t^{[\alpha]-k} \partial_x^{k+1} \xi_{xx} u_{xx}\|_0^2 \\
+&\|x^3 \sigma^{\frac{\alpha+k+4}{2}} e^{S_0} \left\{ \sum_{n=1}^{[\alpha]-k} C_{[\alpha]-k}^n \partial_t^n [\xi^{2-2\gamma} (\xi + x\xi_x)^{-\gamma-2}] \partial_t^{[\alpha]-k-n} \partial_x^{k+1} \xi_{xx} u_{xx} \right\} \|_0^2.
\end{aligned} \tag{5.60}$$

Next, we handle the first term on the right-hand side of (5.60) by applying the Newton-Leibniz formula, which yields

$$\begin{aligned}&\|x^3\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\partial_t^{[\alpha]-k}\partial_x^{k+1}\xi_{xx}u_{xx}\|_0^2\\ \lesssim&\|e^{S_0}\|_{L^\infty}^2\|\sigma u_{xx}\|_{L^\infty}^2\int_0^t\|x\sigma^{\frac{\alpha+k+2}{2}}\partial_\tau^{[\alpha]-k}\partial_x^{k+3}u\|_0^2d\tau+\|x^3\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\partial_t^{[\alpha]-k-1}\partial_x^{k+3}u(0)u_{xx}\|_0^2\\ \lesssim&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right)+\|e^{S_0}\|_{L^\infty}^2\|x\sigma^{\frac{[\alpha]+k+3}{2}}\partial_t^{[\alpha]-k-1}\partial_x^{k+3}u(0)\|_{L^\infty}^2\int_0^t\|x^2\sigma^{\frac{\alpha-[\alpha]+1}{2}}\partial_\tau\partial_x^2u\|_0^2d\tau\\ &+\|e^{S_0}\|_{L^\infty}^2\|x^3\sigma^{\frac{\alpha+k+4}{2}}\partial_t^{[\alpha]-k-1}\partial_x^{k+3}u(0)u_{xx}(0)\|_0^2\lesssim P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}\tag{5.61}$$

We take the second term on the right-hand side of (5.60), set $n=[\alpha]-k$, and establish the following estimates for its most challenging terms:

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+k+4}{2}}e^{S_0}\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-3}\partial_t^{[\alpha]-k-1}u_x\partial_x^{k+1}\xi_{xx}u_{xx}\|_0^2\\ \lesssim&\|e^{S_0}\|_{L^\infty}^2\|\sigma u_{xx}\|_{L^\infty}^2\|\sigma^{\frac{[\alpha]-k-1}{2}}\partial_t^{[\alpha]-k-1}u_x\|_{L^\infty}^2\int_0^t\|x^2\sigma^{\frac{\alpha-[\alpha]+2k+5}{2}}\partial_x^{k+3}u\|_0^2d\tau\\ \lesssim&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}\tag{5.62}$$

Combining results from (5.48) through (5.62), we deduce that

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left\{\sigma e^{S_0}[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2\\ \lesssim&P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}\tag{5.63}$$

By setting $m=k+2$ in (5.49), it yields that

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+2}(e^{S_0})\partial_t^{[\alpha]-k}\left\{[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_{xx}\right\}\|_0^2\\ =&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+2}(e^{S_0})[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_t^{[\alpha]-k}u_{xx}\|_0^2\\ &+\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+2}(e^{S_0})\left\{\sum_{b=0}^{[\alpha]-k}C_{[\alpha]-k}^b\partial_t^b[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_t^{[\alpha]-k-b}u_{xx}\right\}\|_0^2.\end{aligned}\tag{5.64}$$

Turning to the first term on the right-hand side of (5.64), we establish the following estimate:

$$\begin{aligned}&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+2}(e^{S_0})[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_t^{[\alpha]-k}u_{xx}\|_0^2\\ \lesssim&\|\partial_x^{k+2}(e^{S_0})\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha+k+4}{2}}\int_0^t\partial_\tau\left[1-\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\right]d\tau\partial_\tau^{[\alpha]-k}u_{xx}\|_0^2\\ \lesssim&\|\partial_x^{k+2}(e^{S_0})\|_{L^\infty}^2\int_0^t\left(\|u\|_{L^\infty}^2+\|u_x\|_{L^\infty}^2\right)\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_\tau^{[\alpha]-k}u_{xx}\|_0^2d\tau\\ \lesssim&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).\end{aligned}\tag{5.65}$$

For the second term on the right-hand side of (5.64), we take $b = [\alpha] - k$ and establish the following estimates for its most challenging terms:

$$\begin{aligned}
&\|x^3\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+2}(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\partial_t^{[\alpha]-k-1}u_xu_{xx}\|_0^2\\
\lesssim&\|\partial_x^{k+2}(e^{S_0})\|_{L^\infty}^2\|\sigma u_{xx}\|_{L^\infty}^2\int_0^t\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_\tau^{[\alpha]-k}u_x\|_0^2d\tau\\
&+\|\partial_x^{k+2}(e^{S_0})\|_{L^\infty}^2\|\sigma u_{xx}\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k-1}u_x(0)\|_0^2\\
\lesssim&P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}\tag{5.66}$$

Finally, we turn to estimating the sixth term on the right-hand side of (5.46):

$$\begin{aligned}
&\|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left[x\sigma\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x\right]\|_0^2\\
=&\|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\left\{\sum_{l=0}^{k+2}C_{k+2}^l\partial_x^lx\partial_x^{k+2-l}\left[\sigma\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x\right]\right\}\|_0^2\\
\lesssim&\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left\{\sigma\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x\right\}\|_0^2\\
&+\|x\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+1}\left\{\sigma\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x\right\}\|_0^2.
\end{aligned}\tag{5.67}$$

We only need to estimate the first term on the right-hand side of (5.67):

$$\begin{aligned}
&x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+2}\left\{\sigma\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x\right\}\\
=&x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_t^{[\alpha]-k}\left\{\sum_{m=0}^{k+2}C_{k+2}^m\partial_x^m[\partial_x(e^{S_0})]\partial_x^{k+2-m}[\sigma\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x]\right\}.
\end{aligned}\tag{5.68}$$

In the case $m=0$, it follows that

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}\partial_x^{k+2}[\sigma\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x]\|_0^2\\
\lesssim&\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x^{k+2}(\sigma u_x)]\|_0^2\\
&+\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}\left\{\sum_{n=1}^{k+2}C_{k+2}^n\partial_x^n[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_x^{k+2-n}(\sigma u_x)\right\}\|_0^2.
\end{aligned}\tag{5.69}$$

We now estimate the first term on the right-hand side of (5.69):

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x^{k+2}(\sigma u_x)]\|_0^2\\
=&\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}\left\{\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\sum_{b=0}^{k+2}C_{k+2}^b\partial_x^b\sigma\partial_x^{k+2-b}u_x\right\}\|_0^2.
\end{aligned}\tag{5.70}$$

In the case $b=0$, it follows that

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_x^{k+2}u_x]\|_0^2\\
=&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\left\{\sum_{l=0}^{[\alpha]-k}C_{[\alpha]-k}^l\partial_t^l[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_t^{[\alpha]-k-l}\partial_x^{k+2}u_x\right\}\|_0^2.
\end{aligned}\tag{5.71}$$

When $l = 0$ in (5.71), we obtain

$$
\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_t^{[\alpha]-k}\partial_x^{k+2}u_x\|_0^2\\
\lesssim&\|\partial_x(e^{S_0})\|_{L^\infty}^2\int_0^t\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_\tau^{[\alpha]-k+1}\partial_x^{k+3}u\|_0^2d\tau\\
+&\|\partial_x(e^{S_0})\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_t^{[\alpha]-k}\partial_x^{k+3}u(0)\|_0^2\\
\lesssim&P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}
\tag{5.72}
$$

In the case $l = [\alpha] - k$ of (5.71), we bound the most difficult terms as follows:

$$
\begin{aligned}
&\|x^3\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\partial_t^{[\alpha]-k-1}u_x\partial_x^{k+3}u\|_0^2\\
\lesssim&\|\partial_x(e^{S_0})\|_{L^\infty}^2\int_0^t\|\sigma^{\frac{[\alpha]-k}{2}}\partial_\tau^{[\alpha]-k}u_x\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha-[\alpha]+2k+4}{2}}\partial_x^{k+3}u\|_0^2d\tau\\
+&\|\partial_x(e^{S_0})\|_{L^\infty}^2\|x^3\sigma^{\frac{\alpha+k+4}{2}}\partial_t^{[\alpha]-k-1}u_x(0)\partial_x^{k+3}u\|_0^2\\
\lesssim&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right)+\|\partial_x(e^{S_0})\|_{L^\infty}^2\|x^3\sigma^{\frac{\alpha+k+4}{2}}\partial_t^{[\alpha]-k-1}u_x(0)\partial_x^{k+3}u(0)\|_0^2\\
+&\|\partial_x(e^{S_0})\|_{L^\infty}^2\int_0^t\|\sigma^{\frac{[\alpha]-k-1}{2}}\partial_\tau^{[\alpha]-k-1}u_x(0)\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha-[\alpha]+2k+5}{2}}\partial_\tau\partial_x^{k+3}u\|_0^2d\tau,
\end{aligned}
\tag{5.73}
$$

where

$$
\|x^2\sigma^{\frac{\alpha-[\alpha]+2k+5}{2}}\partial_t\partial_x^{k+3}u\|_0^2\lesssim\sum_{l=0}^{[\alpha]-k}\int_I x^4\sigma^{\alpha+[\alpha]+5}\left(\partial_t\partial_x^{k+3+l}u\right)^2dx
\tag{5.74}
$$

$$
\lesssim\sum_{l=0}^{[\alpha]-k}\int_I x^4\sigma^{\alpha+k+3+l}\left(\partial_t\partial_x^{k+3+l}u\right)^2dx\lesssim E(t).
$$

The second term on the right-hand side of (5.69), where $n = k+2$, can be bounded as follows:

$$
\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}\left\{\partial_x^{k+2}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]u_x\right\}\|_0^2\\
=&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\left\{\sum_{p=0}^{[\alpha]-k}C_{[\alpha]-k}^p\partial_t^p\left\{\partial_x^{k+2}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\right\}\partial_t^{[\alpha]-k-p}u_x\right\}\|_0^2.
\end{aligned}
\tag{5.75}
$$

For (5.75) with $p = 0$, the principal challenging terms admit the estimates

$$
\begin{aligned}
&\|x^3\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\partial_x^{k+1}\xi_{xx}\partial_t^{[\alpha]-k}u_x\|_0^2\\
\lesssim&\|\partial_x(e^{S_0})\|_{L^\infty}^2\int_0^t\|x^2\sigma^{\frac{\alpha-[\alpha]+2k+4}{2}}\partial_x^{k+3}u\|_0^2\|\sigma^{\frac{[\alpha]-k}{2}}\partial_\tau^{[\alpha]-k}u_x\|_{L^\infty}^2d\tau\\
\lesssim&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}
\tag{5.76}
$$

For (5.75), for the case $p = [\alpha] - k$, the most difficult terms are estimated as follows:

$$
x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}\partial_x^{k+1}[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}x\xi_{xx}]u_x
$$

$$
\begin{aligned}
&= \sum_{n=0}^{[\alpha]-k} \sum_{p=0}^{k+1} C_{[\alpha]-k}^{n} C_{k+2}^{p} x^2 \sigma^{\frac{\alpha+k+4}{2}} \partial_x(e^{S_0}) \partial_t^n \partial_x^p \left\{ x\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} \right\} \partial_t^{[\alpha]-k-n} \partial_x^{k+1-p} \xi_{xx} u_x \\
&= \sum_{n=0}^{[\alpha]-k} \sum_{p=0}^{k+1} C_{n,p}^{k} G_{n,p}^{k},
\end{aligned} \tag{5.77}
$$

where $C_{n,p}^k$ is a constant depending on n, p, and k and $G_{n,p}^k$ satisfies

$$
G_{n,p}^k = x^2 \sigma^{\frac{\alpha+k+4}{2}} \partial_x(e^{S_0}) \partial_t^n \partial_x^p \left\{ x\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} \right\} \partial_t^{[\alpha]-k-n} \partial_x^{k+1-p} \xi_{xx} u_x.
$$

The principal technical challenge emerges in the treatment of the following terms:

$$
\begin{aligned}
&G_{0,0}^k = x^3 \sigma^{\frac{\alpha+k+4}{2}} \partial_x(e^{S_0}) \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} \partial_t^{[\alpha]-k} \partial_x^{k+1} \xi_{xx} u_x \\
&G_{[\alpha]-k,0}^k = x^3 \sigma^{\frac{\alpha+k+4}{2}} \partial_x(e^{S_0}) \partial_t^{[\alpha]-k} \left\{ \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} \right\} \partial_x^{k+1} \xi_{xx} u_x \\
&G_{0,k+1}^k = x^2 \sigma^{\frac{\alpha+k+4}{2}} \partial_x(e^{S_0}) \partial_x^{k+1} \left\{ x\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} \right\} \partial_t^{[\alpha]-k} \xi_{xx} u_x \\
&G_{[\alpha]-k,k+1}^k = x^2 \sigma^{\frac{\alpha+k+4}{2}} \partial_x(e^{S_0}) \partial_t^{[\alpha]-k} \partial_x^{k+1} \left\{ x\xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} \right\} \xi_{xx} u_x.
\end{aligned} \tag{5.78}
$$

For $\|G_{0,0}^k\|_0^2$, the estimate is as follows:

$$
\begin{aligned}
\|G_{0,0}^k\|_0^2 =& \|x^3 \sigma^{\frac{\alpha+k+4}{2}} \partial_x(e^{S_0}) \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-2} \partial_t^{[\alpha]-k} \partial_x^{k+1} \xi_{xx} u_x\|_0^2 \\
\lesssim& \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|u_x\|_{L^\infty}^2 \int_0^t \|x^2 \sigma^{\frac{\alpha+k+4}{2}} \partial_\tau^{[\alpha]-k} \partial_x^{k+3} u\|_0^2 d\tau \\
\lesssim& CtP\left(\sup_{0\le\tau\le t} E(\tau) \right).
\end{aligned} \tag{5.79}
$$

For $\|G_{[\alpha]-k,0}^k\|_0^2$, the estimates for its most challenging terms are as follows:

$$
\begin{aligned}
& \|x^4 \sigma^{\frac{\alpha+k+4}{2}} \partial_x(e^{S_0}) \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-3} \partial_t^{[\alpha]-k-1} u_x \partial_x^{k+1} \xi_{xx} u_x\|_0^2 \\
\lesssim& \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|u_x\|_{L^\infty}^2 \int_0^t \|\sigma^{\frac{[\alpha]-k}{2}} \partial_\tau^{[\alpha]-k} u_x\|_{L^\infty}^2 d\tau \int_0^t \|x^2 \sigma^{\frac{\alpha-[\alpha]+2k+4}{2}} \partial_x^{k+3} u\|_0^2 d\tau \\
&+ \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|u_x\|_{L^\infty}^2 \|x^2 \sigma^{\frac{\alpha+k+4}{2}} \partial_t^{[\alpha]-k-1} u_x(0) \partial_x^{k+3} \xi\|_0^2 \\
\lesssim& P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau) \right).
\end{aligned} \tag{5.80}
$$

For $\|G_{0,k+1}^k\|_0^2$, the estimates for its most challenging terms are as follows:

$$
\begin{aligned}
& \|x^4 \sigma^{\frac{\alpha+k+4}{2}} \partial_x(e^{S_0}) \xi^{2-2\gamma}(\xi + x\xi_x)^{-\gamma-3} \partial_x^k \xi_{xx} \partial_t^{[\alpha]-k} \xi_{xx} u_x\|_0^2 \\
\lesssim& \|\partial_x(e^{S_0})\|_{L^\infty}^2 \|u_x\|_{L^\infty}^2 \int_0^t \|\sigma^{\frac{[\alpha]-k+2}{2}} \partial_\tau^{[\alpha]-k} u_{xx}\|_{L^\infty}^2 d\tau \int_0^t \|x^2 \sigma^{\frac{\alpha-[\alpha]+2k+2}{2}} \partial_x^{k+2} u\|_0^2 d\tau \\
\lesssim& CtP\left(\sup_{0\le\tau\le t} E(\tau) \right).
\end{aligned} \tag{5.81}
$$

For $\|G^k_{[\alpha]-k,k+1}\|_0^2$, the estimates for its most challenging terms are as follows:

$$\begin{aligned}
&\|x^4\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\partial_t^{[\alpha]-k}\left\{x^2\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-3}\partial_x^k\xi_{xx}\right\}\xi_{xx}u_x\|_0^2\\
\lesssim&\|x^6\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-3}\partial_t^{[\alpha]-k}\partial_x^k\xi_{xx}\xi_{xx}u_x\|_0^2\\
&+\|x^6\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\left\{\sum_{n=1}^{[\alpha]-k}C^n_{[\alpha]-k}\partial_t^n[x^2\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-3}]\partial_t^{[\alpha]-k-n}\partial_x^k\xi_{xx}\right\}\xi_{xx}u_x\|_0^2.
\end{aligned}\tag{5.82}$$

The first term on the right-hand side of (5.82) requires the following estimate:

$$\begin{aligned}
&\|x^6\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-3}\partial_t^{[\alpha]-k}\partial_x^k\xi_{xx}\xi_{xx}u_x\|_0^2\\
\lesssim&\|\partial_x(e^{S_0})\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\int_0^t\|\sigma u_{xx}\|_{L^\infty}^2d\tau\int_0^t\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_\tau^{[\alpha]-k}\partial_x^{k+2}u\|_0^2d\tau\\
\lesssim&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}\tag{5.83}$$

For the second term on the right-hand side of (5.82), we set $n=[\alpha]-k$ and obtain these bounds for the critical terms

$$\begin{aligned}
&\|x^7\sigma^{\frac{\alpha+k+4}{2}}\partial_x(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-4}\partial_t^{[\alpha]-k-1}\partial_x^k\xi_{xx}\xi_{xx}u_x\|_0^2\\
\lesssim&\|\partial_x(e^{S_0})\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\int_0^t\|\sigma u_{xx}\|_{L^\infty}^2d\tau\int_0^t\|x^2\sigma^{\frac{\alpha+k+2}{2}}\partial_\tau^{[\alpha]-k-1}\partial_x^{k+2}u\|_0^2d\tau\\
\lesssim&CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}\tag{5.84}$$

Setting $m=k+2$ in (5.68) yields

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+3}(e^{S_0})\partial_t^{[\alpha]-k}\left\{\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}u_x\right\}\|_0^2\\
\lesssim&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+3}(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_t^{[\alpha]-k}u_x\|_0^2\\
&+\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+3}(e^{S_0})\left\{\sum_{q=1}^{[\alpha]-k}C^q_{[\alpha]-k}\partial_t^q[\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}]\partial_t^{[\alpha]-k-q}u_x\right\}\|_0^2.
\end{aligned}\tag{5.85}$$

We now bound the first term on the right-hand side of (5.85):

$$\begin{aligned}
&\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+3}(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-1}\partial_t^{[\alpha]-k}u_x\|_0^2\\
\lesssim&\|\partial_x^{k+3}(e^{S_0})\|_{L^\infty}^2\int_0^t\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_\tau^{[\alpha]-k+1}u_x\|_0^2d\tau\\
&+\|\partial_x^{k+3}(e^{S_0})\|_{L^\infty}^2\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_t^{[\alpha]-k}u_x(0)\|_0^2\\
\lesssim&P(0)+CtP\left(\sup_{0\le\tau\le t}E(\tau)\right).
\end{aligned}\tag{5.86}$$

For the second term on the right-hand side of (5.85), we take $q=[\alpha]-k$ and establish the following bounds for their principal challenging terms:

$$\|x^3\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+3}(e^{S_0})\xi^{2-2\gamma}(\xi+x\xi_x)^{-\gamma-2}\partial_t^{[\alpha]-k-1}u_xu_x\|_0^2$$

$$\lesssim \|\partial_x^{k+3}(e^{S_0})\|_{L^\infty}^2 \|u_x\|_{L^\infty}^2 \int_0^t \|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_\tau^{[\alpha]-k}u_x\|_0^2 d\tau + \|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+3}(e^{S_0})\partial_t^{[\alpha]-k-1}u_x(0)u_x\|_0^2. \tag{5.87}$$

The second term on the right-hand side of (5.87) requires the following estimate:

$$\begin{aligned} &\|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_x^{k+3}(e^{S_0})\partial_t^{[\alpha]-k-1}u_x(0)u_x\|_0^2 \\ \lesssim &\|\partial_x^{k+3}(e^{S_0})\|_{L^\infty}^2 \int_0^t \|\sigma^{\frac{1}{2}}\partial_\tau u_x\|_{L^\infty}^2 \|x^2\sigma^{\frac{\alpha+k+3}{2}}\partial_\tau^{[\alpha]-k-1}u_x(0)\|_0^2 d\tau \\ +&\|\partial_x^{k+3}(e^{S_0})\|_{L^\infty}^2 \|x^2\sigma^{\frac{\alpha+k+4}{2}}\partial_t^{[\alpha]-k-1}u_x(0)u_x(0)\|_0^2 \\ \lesssim &P(0) + CtP\left(\sup_{0\le\tau\le t} E(\tau)\right). \end{aligned} \tag{5.88}$$

We can also estimate the other terms on the right-hand side of (5.46) and obtain (5.45) from (5.47)–(5.88). □

Author contributions

Jiarong Dong: Writing-original draft; La-Su Mai: Supervision, formal analysis, methodology.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We would like to express our sincere thanks to the referee for the valuable and helpful suggestions and comments, which made a significant change of the paper. The research of L. Mai was partially supported by the National Natural Science Foundation of China (No. 11601246, No. 11971014 and No. 12461045), Young science and technology talents cultivation project of Inner Mongolia University (No.21221505), Research and Educational Reform Project for Graduate Education of Inner Mongolia Autonomous Region in 2023 (No. JGCG2023007) and Outstanding Youth fund of Inner Mongolia Natural Science Foundation (No. 2023JQ13).

Conflict of interest

The authors declare no conflict of interest.

References

1. D. Coutand, H. Lindblad, S. Shkoller, A priori estimates for the free-boundary 3D compressible Euler equations in physical vacuum, *Comm. Math. Phys.*, **296** (2010), 559–587. https://doi.org/10.1007/s00220-010-1028-5

2. F. M. Huang, M. Mei, Y. Wang, M. H. Yu, Asymptotic convergence to stationary waves for unipolar hydrodynamic model of semiconductors, *SIAM J. Math. Anal.*, **43** (2011), 411–429. https://doi.org/10.1137/100793025

3. F. M. Huang, M. Mei, Y. Wang, M. H. Yu, Asymptotic convergence to planar stationary waves for multi-dimensional unipolar hydrodynamic model of semiconductors, *J. Differ. Equ.*, **251** (2011), 1305–1331. https://doi.org/10.1016/j.jde.2011.04.007

4. F. M. Huang, H. M. Yu, D. F. Yuan, Large time behavior of entropy solutions to one-dimensional unipolar hydrodynamic model for semiconductor devices, *Z. Angew. Math. Phys.*, **69** (2018), 69. https://doi.org/10.1007/s00033-018-0968-z

5. F. M. Huang, R. Pan, H. M. Yu, Large time behavior of Euler-Poisson system for semiconductor, *Sci. China Ser. A.*, **51** (2008), 965–972. https://doi.org/10.1007/s11425-008-0049-4

6. A. Majda, Compressible Fluid Flow and System of Conservation Laws in Several Space Variables, In: *Applied Mathematical Sciences*, New York: Springer, 1984. https://doi.org/10.1007/978-1-4612-1116-7

7. H. M. Yu, Large time behavior of Euler-Poisson equations for isothermal fluids with spherical symmetry, *J. Math. Anal. Appl.*, **363** (2010), 302–309. https://doi.org/10.1016/j.jmaa.2009.08.047

8. D. Coutand, S. Shkoller, Well-posedness in smooth function spaces for the moving-boundary three-dimensional compressible Euler equations in physical vacuum, *Arch. Rational Mech. Anal.*, **206** (2012), 515–616. https://doi.org/10.1007/s00205-012-0536-1

9. X. M. Gu, Z. Lei, Well-posedness of 1D compressible Euler-Poisson equations with physical vacuum, *J. Differ. Equ.*, **252** (2012), 2160–2188. https://doi.org/10.1016/j.jde.2011.10.019

10. X. M. Gu, Z. Lei, Local well-posedness of the three dimensional compressible Euler-Poisson equations with physical vacuum, *J. Math. Pures Appl.*, **105** (2016), 662–723. https://doi.org/10.1016/j.matpur.2015.11.010

11. J. Jang, Nonlinear instability in gravitational Euler-Poisson system for $\gamma = 6/5$, *Arch. Ration. Mech. Anal.*, **188** (2008), 265–307. https://doi.org/10.1007/s00205-007-0086-0

12. J. Jang, N. Masmoudi, Well-posedness for compressible Euler equations with physical vacuum singularity, *Comm. Pure Appl. Math.*, **62** (2009), 1327–1385. https://doi.org/10.1002/cpa.20285

13. J. Jang, Nonlinear instability theory of Lane-Emden stars, *Comm. Pure Appl. Math.*, **67** (2014), 1418–1465. https://doi.org/10.1002/cpa.21499

14. J. Jang, Masmoudi, N. Well-posedness of compressible Euler equations in a physical vacuum, *Comm. Pure Appl. Math.*, **68** (2015), 61–111. https://doi.org/10.1002/cpa.21517

15. T. Luo, Z. Xin, H. H. Zeng, Well-posedness for the motion of physical vacuum of the three dimensional compressible Euler equations with or without self-gravitation, *Arch. Ration. Mech. Anal.*, **213** (2014), 763–831. https://doi.org/10.1007/s00205-014-0742-0

16. T. Luo, J. Rauch, C. Xie, Z. Xin, Stability of transonic shock solutions for one-dimensional Euler-Poisson equations, *Arch. Ration. Mech. Anal.*, **202** (2011), 787–827. https://doi.org/10.1007/s00205-011-0433-z

17. C. Rickard, M. Hadžić, J. Jang, Global existence of the nonisentropic compressible Euler equations with vacuum boundary surrounding a variable entropy state, *Nonlinearity* , **34** (2021), 1–52. https://doi.org/10.1088/1361-6544/abb03b

18. S. Liu, T. Luo, Physical Vacuum Problems for the Full Compressible Euler Equations: Hadamard-style Local Well-posedness, arXiv: 2411.17153, 2024. https://doi.org/10.48550/arXiv.2411.17153

19. T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Ration. Mech. Anal.*, **58** (1975), 181–205. https://doi.org/10.1007/BF00280740

20. H. L. Li, P. Markowich, M. Mei, Asymptotic behavior of solutions of the hydrodynamic model of semiconductors, *Proc. Roy. Soc. Edinburgh Sect. A.*, **132** (2002), 359–378. https://doi.org/10.1017/S0308210500001670