Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 18(3): 535–552.
DOI: 10.3934/cam.2026022
Received: 17 December 2025
Revised: 11 February 2026
Accepted: 26 March 2026
Published: 06 July 2026

Research article

Well-Posedness and decay analysis for a thermoelastic wave system with spatially variable-exponent damping and time-varying delay

Muhammad Zainul Abidin[1] **and Naeem Ullah**[2,*]

[1] School of Artificial Intelligence, Taizhou University, Taizhou 318000, Zhejiang, China
[2] School of Physical Science and Technology, Yangzhou University, Yangzhou 225009, Jiangsu, China

* **Correspondence:** Email: naeemullah1989@gmail.com.

Abstract: We study a thermoelastic wave–heat system on a smooth bounded domain in which the mechanical velocity is damped both by a spatially variable-exponent power and by a time-varying delay control. The exponent field $s(\cdot)$ is log-Hölder with $2 \leq s_- \leq s(x) \leq s_+$, where s_+ lies strictly below a scaling-critical bound; the delay is uniformly positive with derivative < 1; the delayed coefficient may change sign but is small relative to the primary damping profile. Encoding the delayed velocity through a first-order transport equation, we construct a Lyapunov functional that controls the elastic, thermal, and delay contributions and yields coercive dissipation. Using a Faedo–Galerkin approximation, compactness arguments in variable-exponent spaces, and the Minty–Browder method, we obtain global existence and uniqueness of strong solutions for admissible data. The same functional, together with a Komornik-type integral inequality, yields an explicit exponential decay rate determined by the time integral of the primary damping profile. The approach also adapts to Cattaneo heat conduction and to distributed delays.

Keywords: thermoelasticity; variable-exponent damping; time-varying delay control; global strong well-posedness; exponential energy decay; Lyapunov–Komornik framework; Faedo–Galerkin scheme; Minty–Browder monotonicity
Mathematics Subject Classification: 35L05, 35B40, 35R10, 46E30

1. Introduction

We consider the initial–boundary value problem for a thermoelastic wave–heat system with spatially variable-exponent velocity damping and a time-dependent delay control. Let $\mathcal{D} \subset \mathbb{R}^n$ be a bounded C^2 domain with boundary $\partial\mathcal{D}$. The unknowns are the displacement $\omega : \mathcal{D} \times [0, \infty) \to \mathbb{R}^n$ and the temperature $\vartheta : \mathcal{D} \times [0, \infty) \to \mathbb{R}$. The mechanical part we study is a damped nonlinear *wave* equation for the displacement, coupled to a diffusion equation for the temperature; our analysis tracks the decay of the classical wave energy under this thermoelastic coupling. The model reads

$$
\begin{cases}
\partial_{tt}\omega - \Delta\omega - \beta_c\,\nabla\vartheta + \gamma_a(t)\,|\partial_t\omega|^{s(x)-2}\,\partial_t\omega \\
\qquad + \gamma_b(t)\,\partial_t\omega(\cdot, t-\varrho(t))\,\left|\partial_t\omega(\cdot, t-\varrho(t))\right|^{s(x)-2} = \mathbf{0}, & \text{in } \mathcal{D}\times(0,\infty),\\
\partial_t\vartheta - \kappa\,\Delta\vartheta + \beta_c\,\operatorname{div}\partial_t\omega = 0, & \text{in } \mathcal{D}\times(0,\infty),\\
\omega = \mathbf{0}, & \text{on } \partial\mathcal{D}\times(0,\infty),\\
\partial_\nu\vartheta = 0, & \text{on } \partial\mathcal{D}\times(0,\infty),\\
\omega(\cdot,0) = \omega_0, \quad \partial_t\omega(\cdot,0) = \boldsymbol{v}_0, \quad \vartheta(\cdot,0) = \vartheta_0, & \text{in } \mathcal{D},\\
\partial_t\omega(\cdot, -r\,\varrho(0)) = I_0(\cdot, -r\,\varrho(0)), & \text{for } r\in(0,1), \text{ in } \mathcal{D}.
\end{cases}
\tag{1.1}
$$

The first line of (1.1) has a hyperbolic, second–order-in-time structure for the displacement ω, with damping acting on the velocity $\partial_t\omega$; the second line is a heat equation for ϑ coupled through $\operatorname{div}\partial_t\omega$. For this reason we refer to (1.1) as a thermoelastic *wave–heat* system. In (1.1) $\varrho \in C^1([0,\infty))$ is a positive time-varying delay, $\gamma_a, \gamma_b \in C^1([0,\infty))$ are temporal coefficients, and $s : \overline{\mathcal{D}} \to [2,\infty)$ is a log–Hölder continuous exponent field. The data $(\omega_0, \boldsymbol{v}_0, \vartheta_0, I_0)$ are prescribed in appropriate spaces; precise structural conditions are stated in the next section.

Thermoelasticity couples elastic deformation and heat conduction, yielding hyperbolic–parabolic systems in which thermal effects dissipate mechanical energy. Foundational results include Dafermos' asymptotic stability in linear thermoelasticity [1], Slemrod's global dynamics in one-dimensional nonlinear thermoelasticity [2], boundary stabilization for elastodynamics by Lagnese [3], exponential decay for bars and plates by Kim [4], the semigroup framework of Lasiecka–Triggiani [5], and small-data global existence for nonlinear thermoelasticity by Ponce–Racke [6]; a comprehensive account is given in the monograph of Jiang–Racke [7]. On the constitutive side, Fourier's law and its finite-speed counterpart (Cattaneo) provide the thermal closure, while internal-variable thermodynamics offers a broader modeling platform [8, 9]. From an applied viewpoint, closely related thermoelastic wave–heat couplings are used to describe transient disturbances and wave propagation in generalized magneto–thermoelastic and microstructured media; see, for example, [10, 11]. Closely related generalized thermoelastic settings with microtemperatures or nonlocal heat-transfer laws have also been studied; see, for example, [12, 13]. Variational and energy-based formulations play a central role in the mathematical modeling of coupled thermomechanical systems. In this direction, variational frameworks for linear thermoelasticity and thermoviscoelasticity, including effects such as thermal inertia, have been developed in both three-dimensional and one-dimensional settings; see, for instance, [14, 15].

The modern analysis of thermoelastic systems provides robust well-posedness and decay theories under diverse boundary conditions and couplings. For nonlinear thermoelastic-type couplings, smoothing and global existence were shown in [16], with subsequent decay results in different settings [17]. Uniform stability estimates and boundary stabilization for multidimensional configurations appear in [18] and, more recently, in coupled variants [19]. Quantitative decay is frequently derived via multiplier/Lyapunov arguments and nonlinear integral inequalities of Komornik–Martínez type [20, 21]. Our damping is of variable exponent type; the functional framework relies on variable-exponent Lebesgue–Sobolev spaces, for which we refer to [22].

If the thermal field is suppressed and the thermoelastic coupling is switched off (i.e., $\vartheta \equiv 0$ and $\beta_c = 0$), then (1.1) reduces to a damped wave equation with a space–dependent power in the velocity and a retarded control in time. In the variable–exponent setting, existence and qualitative properties for related wave models have been developed within the Lebesgue–Sobolev scales with spatially varying indices; for instance, [23–25]. The standing hypotheses typically combine a log–Hölder continuity for the exponent, bounds $2 \leq s_- \leq s(x) \leq s_+$ strictly below the relevant embedding cap, a strictly positive delay with slope < 1, and a delayed coefficient small relative to the primary damping.

For fixed exponents, the competition between damping and source terms is well understood: in the absence of sources, the velocity damping alone enforces global existence independently of the initial size [26]; with sources present, solutions with negative initial energy may blow up in finite time, and even linear damping may not prevent blow up in that regime [27]. Extensions to genuinely nonlinear damping identify a threshold between global existence and blow up dictated by the relative powers of the damping and the source; in particular, global solvability for arbitrary data holds when the damping power dominates, whereas subcritical damping allows finite–time blow up for sufficiently unfavorable initial energy [28].

Within variable–exponent frameworks, gradient-type nonlinearities and velocity dampings depending on position have been investigated in several directions. Models driven by variable $s(\cdot)$–Laplacian effects together with lower–order terms exhibit both global existence and blow–up scenarios according to the interplay of the exponents and coefficients [29]. Related systems combining distinct variable exponents in the principal part and in the lower–order damping/source mechanisms reveal that finite–time blow up may also occur for some positive–energy configurations [30]. In contrast, for a balanced case of the parameters, global existence and stability have been proven [31].

Delay mechanisms have likewise been incorporated in nonlinearly damped wave equations. A prototypical model features a primary velocity damping together with a retarded velocity term, possibly accompanied by source effects; under structural restrictions on the coefficients one derives decay in the source–free case and nonexistence in certain source–dominated regimes [32]. In the special case of constant power and no source, global existence and decay were obtained via a Faedo–Galerkin approximation coupled with multiplier identities [33], and subsequent works extended the analysis to time–varying delays [34]. These results motivate our thermoelastic study, where the presence of the temperature equation and the mechanical–thermal coupling introduces additional coercivity and new technical challenges in the construction of the Lyapunov functional.

The objective of this work is to analyze system (1.1), a thermoelastic wave–heat model with a spatially variable–exponent velocity damping and a time–dependent delay control. Under structural conditions linking the exponent bounds, the slope of the delay, and the size of the delayed coefficient, we prove global strong well–posedness and an explicit stabilization law. The delay is handled by embedding the past velocity into a transport dynamics; uniform *a priori* bounds follow from a Lyapunov functional coupling elastic, thermal, and delayed contributions; compactness in variable–exponent spaces together with the Minty–Browder technique yields existence and uniqueness; finally, a Komornik-type integral inequality gives an exponential decay rate for the total energy. Variants with Cattaneo heat conduction or distributed delays are discussed.

The paper is organized as follows. Section 2 records notation and basic facts on $L^{p(x)}$ and $W^{1,p(x)}$. Section 3 states the structural hypotheses and the main results. Section 4 introduces the transport reformulation and the Lyapunov structure. Sections 5 and 7 develop the Galerkin scheme and the compactness passage to the limit, and Section 6 provides the Komornik-type weighted estimate leading to exponential decay. Finally, Section 8 gathers the global strong well–posedness, uniqueness, and decay conclusions.

2. Notation and Variable–exponent overview

We collect the basic material on variable–exponent Lebesgue–Sobolev spaces used throughout.

Lebesgue and Sobolev classes with variable exponent

Let $\mathcal{D} \subset \mathbb{R}^n$ be a bounded C^2 domain and let $s : \overline{\mathcal{D}} \to [1, \infty)$ be measurable. Define

$$L^{s(x)}(\mathcal{D}) := \Big\{ \zeta : \mathcal{D} \to \mathbb{R} \text{ measurable } : \int_{\mathcal{D}} |\zeta(x)|^{s(x)}\, dx < \infty \Big\}, \tag{2.1}$$

endowed with the Luxemburg norm

$$\|\zeta\|_{L^{s(x)}(\mathcal{D})} := \inf\Big\{ \delta > 0 : \int_{\mathcal{D}} \Big|\frac{\zeta(x)}{\delta}\Big|^{s(x)} dx \leq 1 \Big\}. \tag{2.2}$$

Assume $s_- := \operatorname{ess\,inf}_{x\in\mathcal{D}} s(x) \geq 1$ and $s_+ := \operatorname{ess\,sup}_{x\in\mathcal{D}} s(x) < \infty$. Then $L^{s(x)}(\mathcal{D})$ is a Banach space. When $\nabla\zeta$ exists in the distributional sense, set

$$\begin{aligned} W^{1,s(x)}(\mathcal{D}) &:= \{\zeta \in L^{s(x)}(\mathcal{D}) : \ \nabla\zeta \in L^{s(x)}(\mathcal{D}; \mathbb{R}^n)\}, \\ \|\zeta\|_{W^{1,s(x)}(\mathcal{D})} &:= \|\zeta\|_{L^{s(x)}(\mathcal{D})} + \|\nabla\zeta\|_{L^{s(x)}(\mathcal{D})} . \end{aligned} \tag{2.3}$$

Let $W_0^{1,s(x)}(\mathcal{D})$ denote the closure of $C_0^\infty(\mathcal{D})$ in $W^{1,s(x)}(\mathcal{D})$. Its dual is written $W^{-1,p'(x)}(\mathcal{D})$, where $p'(x) := s(x)/(s(x) - 1)$.

Lemma 2.1 (Poincaré inequality; e.g., [35, 36])**.** *Assume $s_- > 1$ and s is log–Hölder continuous on $\overline{\mathcal{D}}$. There exists* $\mathrm{C} > 0$ *such that* $\|\zeta\|_{L^{s(x)}(\mathcal{D})} \leq \mathrm{C}\ \|\nabla\zeta\|_{L^{s(x)}(\mathcal{D})}$ *for all $\zeta \in W_0^{1,s(x)}(\mathcal{D})$.*

Lemma 2.2 (Generalized Hölder; [36])**.** *Let $\rho, \zeta : \mathcal{D} \to [1, \infty)$ satisfy, a.e. in $\mathcal{D}$,*

$$\frac{1}{\rho(x)} = \frac{1}{s(x)} + \frac{1}{\zeta(x)}. \tag{2.4}$$

Then for $u \in L^{s(x)}(\mathcal{D})$ and $v \in L^{\zeta(x)}(\mathcal{D})$, one has $\|uv\|_{L^{\rho(x)}(\mathcal{D})} \leq 2\ \|u\|_{L^{s(x)}(\mathcal{D})}\ \|v\|_{L^{\zeta(x)}(\mathcal{D})}$.

Lemma 2.3 (Sobolev embedding; [35])**.** *Assume $s_- > 1$, $s_+ < \infty$, and s is log–Hölder continuous. Set*

$$s^*(x) := \begin{cases} \dfrac{n\, s(x)}{n - s(x)}, & s(x) < n, \\ +\infty, & s(x) \geq n. \end{cases}$$

If $\zeta : \mathcal{D} \to [1, \infty)$ with $\zeta(x) \leq s^*(x)$ a.e., then $W_0^{1,s(x)}(\mathcal{D}) \hookrightarrow L^{\zeta(x)}(\mathcal{D})$ continuously; if $\zeta(x) < s^*(x)$ a.e., the embedding is compact.

Remark 2.4 (Critical cap used later). For $n \geq 3$, we impose the stricter bound

$$2 \leq s_- \leq s(x) \leq s_+ \leq \frac{2n-2}{n-2}, \tag{2.5}$$

which is below the classical Sobolev critical exponent and is precisely what we need in the mixed estimates of Section 5.

3. Model, structural hypotheses, and main statements

We study the thermoelastic system (1.1). The delay is handled by an auxiliary transport state introduced in Section 4. We now list the structural conditions and state the main result.

Standing assumptions

(A1) Variable exponent. s is log–Hölder continuous on $\overline{\mathcal{D}}$ and satisfies (2.5).

(A2) Delay. $\varrho \in W^{2,\infty}(0,\infty)$ with

$$0 < \varrho_0 \le \varrho(t) \le \varrho_1, \qquad \varrho'(t) \le \varsigma < 1, \qquad t \ge 0. \tag{3.1}$$

(A3) Primary damping profile. $\gamma_a \in C^1([0,\infty))$ is nonincreasing and strictly positive:

$$\gamma_a(t) \ge \alpha_0 > 0, \qquad \left|\gamma_a'(t)\right| \le \mathrm{C}_1\, \gamma_a(t), \qquad t \ge 0. \tag{3.2}$$

(A4) Delayed control. $\gamma_b \in C^1([0,\infty))$ may change sign but remains small relative to γ_a:

$$|\gamma_b(t)| \le \alpha\, \gamma_a(t), \qquad \left|\gamma_b'(t)\right| \le \mathrm{C}_2\, \gamma_a(t), \qquad t \ge 0, \tag{3.3}$$

with a constant α satisfying

$$0 < \alpha < \frac{(1-\varsigma)\, s_+}{s_- - \varsigma}. \tag{3.4}$$

Komornik–type integral inequality

We will use the following quantitative decay tool.

Lemma 3.1 (Komornik framework; [21])**.** *Let $E : [0,\infty) \to [0,\infty)$ be nonincreasing and let $\Phi \in C^1([0,\infty))$ be strictly increasing with $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. Assume there exist $\kappa > -1$ and $\varsigma > 0$ such that, for all $R \ge 0$,*

$$\int_R^\infty E^{1+\kappa}(t)\, \Phi'(t)\, dt \;\le\; \frac{1}{\varsigma}\, E^\kappa(0)\, E(R). \tag{3.5}$$

Then:

- *If $-1 < \kappa < 0$, $E(t) = 0$ for all $t \ge (E(0))^\kappa/(\varsigma\, |\kappa|)$.*
- *If $\kappa > 0$, then $E(t) \le E(0)\left(\frac{1+\kappa}{1+\varsigma\kappa\,\Phi(t)}\right)^{1/\kappa}$.*
- *If $\kappa = 0$, then $E(t) \le E(0)\, \exp\{-\varsigma\, \Phi(t)\}$.*

4. Delay reduction and Lyapunov structure

Transport reformulation of the delay

Introduce the delay state $u : \mathcal{D} \times (0,1) \times (0,\infty) \to \mathbb{R}^n$,

$$u(x,r,t) := \partial_t \omega(x,\, t - r\,\varrho(t)). \tag{4.1}$$

A direct computation yields the coupled first–order transport equation. Set $\sigma(t, r) = t - r\,\varrho(t)$ so that $u(x, r, t) = \partial_t\omega(x, \sigma(t, r))$, and apply the chain rule to $\partial_t u$ and $\partial_r u$ to obtain (4.2).

$$\varrho(t)\,\partial_t u + (1 - r\,\varrho'(t))\,\partial_r u = \mathbf{0} \quad \text{in } \mathcal{D} \times (0, 1) \times (0, \infty), \tag{4.2}$$

with boundary traces

$$u(\cdot, 0, t) = \partial_t\omega(\cdot, t), \qquad u(\cdot, 1, t) = \partial_t\omega(\cdot, t - \varrho(t)), \tag{4.3}$$

and homogeneous spatial boundary, since $\omega|_{\partial\mathcal{D}} = 0 \Rightarrow \partial_t\omega|_{\partial\mathcal{D}} = 0$.

Weighted energy

Let $\boldsymbol{\mu}(x)$ be a positive function on $\overline{\mathcal{D}}$ to be chosen below and set $\xi(x, t) := \gamma_a(t)\boldsymbol{\mu}(x)$. In particular, since $\xi(x, t)(x, t) = \gamma_a(t)\boldsymbol{\mu}(x)(x)$, we have $\partial_t\xi(x, t)(x, t) = \gamma_a'(t)\boldsymbol{\mu}(x)(x)$. Define the total energy

$$\begin{aligned} \mathcal{H}(t) := \frac{1}{2}\,\|\partial_t\omega(t)\|^2_{L^2(\mathcal{D})} + \frac{1}{2}\,\|\nabla\omega(t)\|^2_{L^2(\mathcal{D})} + \frac{c_\theta}{2}\,\|\vartheta(t)\|^2_{L^2(\mathcal{D})} \\ +\varrho(t)\int_0^1\int_{\mathcal{D}} \frac{\xi(x, t)}{s(x)}\,|u(x, r, t)|^{s(x)}\,dx\,dr. \end{aligned} \tag{4.4}$$

The terms $\frac{1}{2}\|\partial_t\omega(t)\|^2_{L^2(\mathcal{D})}$ and $\frac{1}{2}\|\nabla\omega(t)\|^2_{L^2(\mathcal{D})}$ constitute the standard *wave energy* (kinetic plus elastic), to which the thermal and delay contributions are added. To ensure positivity of the boundary contributions arising from (4.2) and the delayed damping, we impose the pointwise bounds

$$\frac{\alpha\,(s(x) - 1)}{1 - \varsigma} \;<\; \boldsymbol{\mu}(x) \;<\; s(x) - \alpha \qquad \text{for all } x \in \overline{\mathcal{D}}. \tag{4.5}$$

Under (3.4) and (2.5), the interval in (4.5) is nonempty.

We first record the basic dissipation inequality for the coupled *wave–heat* energy.

Lemma 4.1 (Dissipation inequality)**.** *Let* (ω, ϑ, u) *be a strong solution of* (1.1) *on* $(0, \infty)$ *and let the energy* $\mathcal{H}$ *be given by* (4.4) *with* $\xi(x, t) = \gamma_a(t)\boldsymbol{\mu}(x)$*, where* $\boldsymbol{\mu}(x)$ *satisfies* (4.5)*. Then, for a.e.* $t > 0$*,*

$$\begin{aligned} \frac{d}{dt}\mathcal{H}(t) + \gamma_a(t)\int_{\mathcal{D}}\Big[a_1(x)\,|\partial_t\omega(x, t)|^{s(x)} + a_2(x)\,|u(\cdot, 1, t)(x, t)|^{s(x)}\Big]\,dx \\ + \kappa\,\|\nabla\vartheta(t)\|^2_{L^2(\mathcal{D})} \;\le\; 0. \end{aligned} \tag{4.6}$$

where

$$a_1(x) := \frac{s(x) - \boldsymbol{\mu}(x) - \alpha}{s(x)} > 0, \qquad a_2(x) := \frac{(1 - \varsigma)\boldsymbol{\mu}(x) - \alpha(s(x) - 1)}{s(x)} > 0. \tag{4.7}$$

Proof. Step 1 (mechanical identity). Multiply the mechanical equation by $\partial_t\omega$ and integrate on $\mathcal{D}$:

$$\begin{aligned} \int_{\mathcal{D}} \partial_{tt}\omega \cdot \partial_t\omega + \int_{\mathcal{D}} \nabla\omega : \nabla\partial_t\omega - \beta_c\int_{\mathcal{D}} \nabla\vartheta \cdot \partial_t\omega + \int_{\mathcal{D}} \gamma_a(t)\,|\partial_t\omega|^{s(x)}\,dx \\ + \int_{\mathcal{D}} \gamma_b(t)u(\cdot, 1, t)\;|u(\cdot, 1, t)|^{s(x)-2} \cdot \partial_t\omega\,dx = 0. \end{aligned}$$

Since $\omega|_{\partial\mathcal{D}} = 0$, we have $\int_{\mathcal{D}} \partial_{tt}\omega \cdot \partial_t\omega = \frac{1}{2}\frac{d}{dt}\|\partial_t\omega\|_{L^2}^2$ and $-\int_{\mathcal{D}} \Delta\omega \cdot \partial_t\omega = \frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2$, so

$$\begin{aligned}\frac{d}{dt}\Big(\tfrac{1}{2}\|\partial_t\omega\|_{L^2}^2 + \tfrac{1}{2}\|\nabla\omega\|_{L^2}^2\Big) - \beta_c \int_{\mathcal{D}} \nabla\vartheta \cdot \partial_t\omega\, dx + \int_{\mathcal{D}} \gamma_a(t)\, |\partial_t\omega|^{s(x)}\, dx \\ + \int_{\mathcal{D}} \gamma_b(t) u(\cdot,1,t)\ |u(\cdot,1,t)|^{s(x)-2} \cdot \partial_t\omega\, dx = 0.\end{aligned} \tag{4.8}$$

Step 2 (thermal identity). Multiply the heat equation by ϑ and integrate:

$$\frac{c_\theta}{2}\frac{d}{dt}\|\vartheta\|_{L^2}^2 + \kappa\|\nabla\vartheta\|_{L^2}^2 + \beta_c \int_{\mathcal{D}} \operatorname{div} \partial_t\omega\, \vartheta\, dx = 0. \tag{4.9}$$

By integration by parts and $\partial_\nu\vartheta = 0$, we get $\int_{\mathcal{D}} \operatorname{div} \partial_t\omega\, \vartheta\, dx = -\int_{\mathcal{D}} \nabla\vartheta \cdot \partial_t\omega\, dx$; adding (4.8) and (4.9) cancels the coupling terms.

Step 3 (transport component). Multiply (4.2) by $\xi(x,t)\ |u|^{s(x)-2}\, u$ and integrate on $\mathcal{D} \times (0,1)$:

$$\varrho \int_0^1 \int_{\mathcal{D}} \xi(x,t)\, \partial_t u \cdot |u|^{s(x)-2}\, u\, dx\, dr + \int_0^1 \int_{\mathcal{D}} \xi(x,t)(1 - r\varrho')\, \partial_r u \cdot |u|^{s(x)-2}\, u\, dx\, dr = 0.$$

Since $\partial_t |u|^{s(x)} = s(x)\ |u|^{s(x)-2}\, u \cdot \partial_t u$. Here, $s(x)$ depends only on x, hence it is constant with respect to t and r in these derivatives; (and similarly in r), integrating by parts in r and using the traces $u(\cdot,0,t) = \partial_t\omega$, $u(\cdot,1,t) = \partial_t\omega(\cdot, t - \varrho(t))$ (cf. (4.3)) yields

$$\begin{aligned}\frac{d}{dt}\left(\varrho \int_0^1 \int_{\mathcal{D}} \frac{\xi(x,t)}{s(x)}\, |u|^{s(x)}\ dx\, dr\right) = \varrho \int_0^1 \int_{\mathcal{D}} \frac{\partial_t\xi(x,t)}{s(x)}\, |u|^{s(x)}\ dx\, dr \\ + \int_{\mathcal{D}} \frac{\xi(x,t)}{s(x)}\Big(|u(\cdot,0,t)|^{s(x)} - (1 - \varrho')\, |u(\cdot,1,t)|^{s(x)}\Big)\, dx.\end{aligned} \tag{4.10}$$

Note: In the integration by parts in r, the intermediate term $\varrho'(t)\int_0^1\int_{\mathcal{D}} \frac{\xi(x,t)}{s(x)}\, |u|^{s(x)}\ dx\, dr$ generated by $\partial_r(1 - r\varrho'(t))$ cancels with the corresponding $\varrho'(t)$ term produced when differentiating the prefactor $\varrho(t)$ on the left-hand side; hence, (4.10) contains only the boundary traces.

Step 4 (full energy balance). Adding (4.8), (4.9), (4.10) and recalling $\xi(x,t) = \gamma_a(t)\boldsymbol{\mu}(x)$, using (4.10) to differentiate the delay part of $\mathcal{H}(t)$ (which already incorporates the time dependence of $\varrho(t)$ via (4.2)), and then adding (4.8)–(4.9), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(t) = &- \int_{\mathcal{D}} \gamma_a(t)\, |\partial_t\omega|^{s(x)}\ dx - \int_{\mathcal{D}} \gamma_b(t) u(\cdot,1,t)\ |u(\cdot,1,t)|^{s(x)-2} \cdot \partial_t\omega\, dx \\ &- \kappa\ \|\nabla\vartheta\|_{L^2}^2 + \varrho(t) \int_0^1 \int_{\mathcal{D}} \frac{\gamma_a'(t)\boldsymbol{\mu}(x)}{s(x)}\, |u|^{s(x)}\ dx\, dr \\ &+ \int_{\mathcal{D}} \frac{\gamma_a(t)\boldsymbol{\mu}(x)}{s(x)}\Big(|\partial_t\omega|^{s(x)} - (1 - \varrho'(t))\, |u(\cdot,1,t)|^{s(x)}\Big)\, dx.\end{aligned} \tag{4.11}$$

Since $\gamma_a'(t) \le 0$ by (A3), the fourth term is ≤ 0 and may be dropped for an inequality.

Step 5 (Young's inequality and positivity). For a.e. $x \in \mathcal{D}$,

$$|\partial_t\omega|\ |u(\cdot,1,t)|^{s(x)-1} \le \frac{1}{s(x)}\, |\partial_t\omega|^{s(x)} + \frac{s(x)-1}{s(x)}\, |u(\cdot,1,t)|^{s(x)}.$$

Using $|\gamma_b(t)| \le \alpha\,\gamma_a(t)$ (A4) and $1-\varrho'(t) \ge 1-\varsigma$ (A2), from (4.11) we get

$$\frac{d}{dt}\mathcal{H}(t) \le -\gamma_a(t)\int_{\mathcal{D}}\Big[1-\tfrac{\mu(x)+\alpha}{s(x)}\Big]\,|\partial_t\omega|^{s(x)}\,dx \\ -\gamma_a(t)\int_{\mathcal{D}}\Big[\tfrac{(1-\varsigma)\mu(x)}{s(x)}-\tfrac{\alpha(s(x)-1)}{s(x)}\Big]\,|u(\cdot,1,t)|^{s(x)}\,dx-\kappa\,\|\nabla\vartheta\|_{L^2}^2\,,$$

which is (4.6) with the coefficients (4.7). Positivity of a_1, a_2 follows from (4.5). □

5. Galerkin approximation and first a priori estimate

The Galerkin approximation treats the wave part (displacement) and the heat part (temperature) together and preserves the underlying wave–energy structure at the discrete level.

Finite–dimensional spaces and ansatz

Fix integers $N, Q \ge 1$. Let $V_N = \operatorname{span}\{\boldsymbol{\phi}_1,\dots,\boldsymbol{\phi}_N\} \subset H^2(\mathcal{D};\mathbb{R}^n)\cap H_0^1(\mathcal{D})^n$ and $W_N = \operatorname{span}\{\psi_1,\dots,\psi_N\} \subset H^1(\mathcal{D})$ with $\partial_\nu\psi_k = 0$ on $\partial\mathcal{D}$. For the delay state choose $F_Q = \operatorname{span}\{\Lambda_1,\dots,\Lambda_Q\} \subset L^2((0,1);H_0^1(\mathcal{D})^n)$. Seek

$$\begin{aligned} \omega^N(x,t) &= \sum_{k=1}^{N} g_k(t)\boldsymbol{\phi}_k(x),\\ \vartheta^N(x,t) &= \sum_{k=1}^{N} z_k(t)\psi_k(x), \\ u^Q(x,r,t) &= \sum_{j=1}^{Q} h_j(t)\Lambda_j(x,r). \end{aligned} \tag{5.1}$$

Projected system

With $\partial_t\omega^N = \partial_t\omega^N$ and $u(\cdot,1,t)^Q(x,t) = u^Q(x,1,t)$, the L^2-projections of (1.1) read

$$\begin{cases} \left\langle \partial_{tt}\omega^N,\boldsymbol{\phi}_k\right\rangle + (\nabla\omega^N,\nabla\boldsymbol{\phi}_k)_{L^2} - \beta_c\left\langle \nabla\vartheta^N,\boldsymbol{\phi}_k\right\rangle \\ \qquad +\displaystyle\int_{\mathcal{D}}\gamma_a(t)\left|\partial_t\omega^N\right|^{s(x)-2}\partial_t\omega^N\cdot\boldsymbol{\phi}_k\,dx \\ \qquad +\displaystyle\int_{\mathcal{D}}\gamma_b(t)u(\cdot,1,t)^Q\left|u(\cdot,1,t)^Q\right|^{s(x)-2}\cdot\boldsymbol{\phi}_k\,dx = 0, \quad k=1,\dots,N, \\ \left\langle \partial_t\vartheta^N,\psi_k\right\rangle + \kappa(\nabla\vartheta^N,\nabla\psi_k)_{L^2} + \beta_c\left\langle \operatorname{div}\partial_t\omega^N,\psi_k\right\rangle = 0, \quad k=1,\dots,N, \end{cases} \tag{5.2}$$

and the transport equation becomes

$$\varrho(t)\,(\partial_t u^Q,\Lambda_j)_{L^2(\mathcal{D}\times(0,1))} + \big((1-r\varrho'(t))\,\partial_r u^Q,\Lambda_j\big)_{L^2(\mathcal{D}\times(0,1))} = 0,\ \ j=1,\dots,Q, \tag{5.3}$$

with the compatibility trace

$$u^Q(\cdot,0,t) = \partial_t\omega^N(\cdot,t) \quad \text{in } V_N,\ \ \forall t\ge 0. \tag{5.4}$$

Initial data

Let $\Pi_{V_N}, \Pi_{W_N}, \Pi_{F_Q}$ be the L^2-projections. Set

$$\begin{aligned} \omega^N(0) &= \Pi_{V_N}\omega_0, \\ \partial_t\omega^N(0) &= \Pi_{V_N}\boldsymbol{v}_0, \\ \vartheta^N(0) &= \Pi_{W_N}\vartheta_0, \\ u^Q(\cdot, r, 0) &= \Pi_{F_Q} I_0(\cdot, r). \end{aligned} \tag{5.5}$$

so that $u^Q(\cdot, 0, 0) = \partial_t\omega^N(0)$.

Remark 5.1. All nonlinear maps on the finite–dimensional spaces are locally Lipschitz; thus, (5.2)–(5.3)–(5.4) with (5.5) has a unique maximal solution on $[0, T_{N,Q})$.

First a priori estimate

Define

$$\begin{aligned} \mathcal{H}_{N,Q}(t) := \frac{1}{2}\left\|\partial_t\omega^N\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla\omega^N\right\|_{L^2}^2 + \frac{c_\theta}{2}\left\|\vartheta^N\right\|_{L^2}^2 \\ + \varrho(t)\int_0^1\int_{\mathcal{D}} \frac{\gamma_a(t)\boldsymbol{\mu}(x)}{s(x)}\left|u^Q\right|^{s(x)}\,dx\,dr. \end{aligned} \tag{5.6}$$

Testing (5.2) by $\partial_t\omega^N$ and ϑ^N, and (5.3) by $\xi(x,t)\left|u^Q\right|^{s(x)-2}u^Q$ with $\xi(x,t) = \gamma_a(t)\boldsymbol{\mu}(x)$, and repeating the proof of Lemma 4.1 verbatim, we obtain

$$\frac{d}{dt}\mathcal{H}_{N,Q}(t) + \gamma_a(t)\int_{\mathcal{D}}\Big[b_1(x)\left|\partial_t\omega^N\right|^{s(x)} + b_2(x)\left|u(\cdot,1,t)^Q\right|^{s(x)}\Big]\,dx + \kappa\left\|\nabla\vartheta^N\right\|_{L^2}^2 \le 0, \tag{5.7}$$

with

$$b_1(x) = \frac{s(x) - \boldsymbol{\mu}(x) - \alpha}{s(x)}, \qquad b_2(x) = \frac{(1-\varsigma)\boldsymbol{\mu}(x) - \alpha(s(x)-1)}{s(x)} > 0. \tag{5.8}$$

Integrating (5.7) on $[0, t]$ gives

$$\begin{aligned} \mathcal{H}_{N,Q}(t) + \int_0^t\int_{\mathcal{D}}\gamma_a(\ell)\Big[b_1\left|\partial_t\omega^N\right|^{s(x)} + b_2\left|u(\cdot,1,t)^Q\right|^{s(x)}\Big]\,dx\,d\ell \\ + \kappa\int_0^t\left\|\nabla\vartheta^N\right\|_{L^2}^2\,d\ell \le \mathcal{H}_{N,Q}(0). \end{aligned} \tag{5.9}$$

Since $b_1(x) > 0$ and $b_2(x) > 0$ (see (5.8) and the choice (4.5)), the two integrals in (5.9) are nonnegative; thus (5.9) is the usual integrated form of (5.7), and, in particular, $\mathcal{H}_{N,Q}(t) \le \mathcal{H}_{N,Q}(0)$ for all $t \in [0, T_{N,Q})$. Consequently, for every $\tau < T_{N,Q}$,

$$\omega^N \text{ bounded in } L^\infty(0,\tau; H_0^1(\mathcal{D})), \qquad \partial_t\omega^N \in L^{s(x)}(\mathcal{D}\times(0,\tau)), \tag{5.10}$$

$$\vartheta^N \in L^\infty(0,\tau; L^2) \cap L^2(0,\tau; H^1), \qquad u^Q \in L^\infty(0,\tau; L^{s(x)}(\mathcal{D}\times(0,1))), \tag{5.11}$$

$$u(\cdot,1,t)^Q \in L^{s(x)}(\mathcal{D}\times(0,\tau)), \quad \gamma_a^{1/s(x)}\partial_t\omega^N,\ \gamma_a^{1/s(x)}u(\cdot,1,t)^Q \in L^{s(x)}(\mathcal{D}\times(0,\tau)). \tag{5.12}$$

Second a priori estimate

We now differentiate the system to control time derivatives.

Lemma 5.2 (Differentiated estimate)**.** *There exists $C > 0$, independent of N, Q, such that for a.e. $t \in (0, T_{N,Q})$,*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\partial_t\omega_t\|_{L^2}^2 + \|\nabla\partial_t\omega\|_{L^2}^2 + c_\theta\|\vartheta_t\|_{L^2}^2 + \int_0^1\int_{\mathcal{D}} \frac{\varrho(t)}{1 - r\varrho'(t)}\,|u_t|^2\,dx\,dr\right)\\
&+ \frac{\gamma_a(t)}{2}\int_{\mathcal{D}}(s(x)-1)\,|\partial_t\omega|^{s(x)-2}\,|\partial_t\omega_t|^2\,dx\\
&+ \frac{\gamma_a(t)}{2}\int_{\mathcal{D}}(s(x)-1)\,|u(\cdot,1,t)|^{s(x)-2}\,|u(\cdot,1,t)_t|^2\,dx + \kappa\|\nabla\vartheta_t\|_{L^2}^2\\
&\le C(\gamma_a(t)+1)\left(\|\partial_t\omega_t\|_{L^2}^2 + \|\nabla\partial_t\omega\|_{L^2}^2 + \|\vartheta_t\|_{L^2}^2 + \int_0^1\int_{\mathcal{D}}\frac{\varrho(t)}{1-r\varrho'(t)}\,|u_t|^2\,dx\,dr\right).
\end{aligned}
\tag{5.13}
$$

Proof. Differentiate the mechanical part of (5.2) and test by $\partial_t\omega_t$:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\partial_t\omega_t\|_{L^2}^2 + \|\nabla\partial_t\omega\|_{L^2}^2\right) - \beta_c\int_{\mathcal{D}}\nabla\vartheta_t\cdot\partial_t\omega_t\,dx\\
&+\gamma_a(t)\int_{\mathcal{D}}(s(x)-1)\,|\partial_t\omega|^{s(x)-2}\,|\partial_t\omega_t|^2\,dx + \mathcal{R}_1(t) = 0,
\end{aligned}
$$

where $\mathcal{R}_1$ collects terms with γ_a' and the delayed part. Differentiate the thermal part and test by ϑ_t:

$$
\frac{c_\theta}{2}\frac{d}{dt}\|\vartheta_t\|_{L^2}^2 + \kappa\|\nabla\vartheta_t\|_{L^2}^2 + \beta_c\int_{\mathcal{D}}\operatorname{div}\partial_t\omega_t\,\vartheta_t\,dx = 0,
$$

and note $\int\operatorname{div}\partial_t\omega_t\,\vartheta_t = -\int\nabla\vartheta_t\cdot\partial_t\omega_t$, so the coupling cancels after addition.

For the transport equation, differentiating (5.3), testing by u_t and integrating by parts in r using the traces (and $\varrho'(t)\le\varsigma$) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_0^1\int_{\mathcal{D}}\frac{\varrho}{1-r\varrho'}\,|u_t|^2\,dx\,dr\right) + \frac{1}{2(1-\varsigma)}\varrho\int_{\mathcal{D}}|u(\cdot,1,t)_t|^2\,dx - \frac{\varrho}{2}\int_{\mathcal{D}}|\partial_t\omega_t|^2\,dx\\
&\le C\int_0^1\int_{\mathcal{D}}\frac{\varrho}{1-r\varrho'}\,|u_t|^2\,dx\,dr.
\end{aligned}
$$

Here we also use that the time derivative of the weight produces the term $\frac{1}{2}\int_0^1\int_{\mathcal{D}}\partial_t\!\left(\frac{\varrho}{1-r\varrho'}\right)|u_t|^2\,dx\,dr$, which is bounded by $C\int_0^1\int_{\mathcal{D}}\frac{\varrho}{1-r\varrho'}\,|u_t|^2\,dx\,dr$ thanks to the regularity/bounds assumed on ϱ and the inequality $1 - r\varrho'(t)\ge 1-\varsigma$. The remainder $\mathcal{R}_1$ is handled by Cauchy–Schwarz and Young's inequality together with (A3)–(A4). For instance, one of the typical contributions has the form $\left|\gamma_a'(t)\right|\int_{\mathcal{D}}|\partial_t\omega|^{s(x)-1}\,|\partial_t\omega_t|\,dx$, and we estimate it as

$$
\begin{aligned}
&\left|\gamma_a'(t)\right|\int_{\mathcal{D}}|\partial_t\omega|^{s(x)-1}\,|\partial_t\omega_t|\,dx\\
&\le\left|\gamma_a'(t)\right|\left(\int_{\mathcal{D}}|\partial_t\omega|^{s(x)}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathcal{D}}|\partial_t\omega|^{s(x)-2}\,|\partial_t\omega_t|^2\,dx\right)^{\frac{1}{2}}\\
&\le\varepsilon\,\gamma_a(t)\int_{\mathcal{D}}(s(x)-1)\,|\partial_t\omega|^{s(x)-2}\,|\partial_t\omega_t|^2\,dx + C_\varepsilon\frac{\left|\gamma_a'(t)\right|^2}{\gamma_a(t)}\int_{\mathcal{D}}|\partial_t\omega|^{s(x)}\,dx,
\end{aligned}
$$

and similarly for the terms coming from the delayed contribution (with $\gamma_b'(t)$ and $u(\cdot,1,t)$). The mixed term is treated in the same way (Young's inequality with a small parameter) and then absorbed into the left–hand side of (5.13). Finally, the remaining factors involving $\left|\gamma_a'(t)\right|^2/\gamma_a(t)$ are controlled by (A3)–(A4), while $\int_{\mathcal{D}} |\partial_t\omega|^{s(x)}$ and $\int_{\mathcal{D}} |u(\cdot,1,t)|^{s(x)}$ are already integrable in time by (5.9). Collecting all estimates gives (5.13). □

By Gronwall's inequality applied to (5.13), for every $\tau < T_{N,Q}$ we have

$$\partial_t\omega_t \in L^\infty(0,\tau;L^2), \quad \nabla\partial_t\omega \in L^\infty(0,\tau;L^2), \quad \vartheta_t \in L^\infty(0,\tau;L^2)\cap L^2(0,\tau;H^1), \tag{5.14}$$

and

$$\int_0^\tau\int_0^1\int_{\mathcal{D}} \frac{\varrho(t)}{1-r\varrho'(t)}\,|u_t|^2\,dx\,dr\,dt \;\le\; C(\tau). \tag{5.15}$$

6. Asymptotic stabilization of the energy

In this section we establish the quantitative decay (8.1) for any global strong solution produced in Section 8. Recall the energy functional $\mathcal{H}$ defined in (4.4) and the dissipation inequality (4.6). Set

$$\xi(x,t)(x,t) := \gamma_a(t)\,\boldsymbol{\mu}(x), \qquad \Phi(t) := \int_0^t \gamma_a(\ell)\,d\ell,$$

so that $\Phi\in C^1(\mathbb{R}_+)$, $\Phi(0)=0$, $\Phi'(t)=\gamma_a(t)\ge 0$, and $\Phi(t)\to+\infty$ as $t\to+\infty$ by (A3). Throughout, $\mathrm{C}>0$ denotes a generic constant independent of time and of the truncation indices used in Section 5.

Lemma 6.1 (Komornik–type device). *Let $\kappa>-1$ and suppose $E\in C^1([0,\infty))$ satisfies $E\ge 0$, $E'\le 0$, and there exist $\Lambda\in(0,1]$, $\mathrm{C}>0$ such that for every $0\le R<\tau$,*

$$\Lambda\int_R^\tau E^{\kappa+1}(t)\,\Phi'(t)\,dt \;\le\; \mathrm{C}\,E^{\kappa+1}(R)+\mathrm{C}\int_R^\tau E^\kappa(t)(-E'(t))\,dt. \tag{6.1}$$

Then there exist $\beta,\mathrm{C}_*>0$ (depending on $\kappa,\Lambda,\mathrm{C}$) such that

$$E(t)\;\le\;\mathrm{C}_*\,E(0)\,e^{-\beta\Phi(t)} \qquad \forall\, t\ge 0.$$

Proof. Since $E'\le 0$, the last term of (6.1) is bounded by $\mathrm{C}\,(E^{\kappa+1}(R)-E^{\kappa+1}(\tau))/(\kappa+1)$. Rearranging gives $\int_R^\tau E^{\kappa+1}\Phi'\le \mathrm{C}_1E^{\kappa+1}(R)$. Applying Komornik's integral inequality, one obtains the stated exponential bound with some $\beta>0$ and $\mathrm{C}_*=\mathrm{C}_*(\kappa,\Lambda,\mathrm{C})$. □

The weighted multipliers

Fix $0\le R<\tau$ and $\kappa>-1$. Multiply the mechanical equation in (1.1) by $E^\kappa(t)\Phi'(t)\,\omega$, integrate over $\mathcal{D}\times(R,\tau)$, and use integration by parts in time and space. Using $\langle\partial_{tt}\omega,\omega\rangle=\frac{d}{dt}\langle\partial_t\omega,\omega\rangle-\|\partial_t\omega\|_{L^2}^2$ and $\langle-\Delta\omega,\omega\rangle=\|\nabla\omega\|_{L^2}^2$, we get

$$0=\Big[E^\kappa(t)\Phi'(t)\,\langle\partial_t\omega(t),\omega(t)\rangle\Big]_{t=R}^{t=\tau}-\int_R^\tau\!\big(\kappa E^{\kappa-1}E'\Phi'+E^\kappa\Phi''\big)\,\langle\partial_t\omega,\omega\rangle\,dt$$

$$+\int_R^\tau E^\kappa\Phi'(\|\partial_t\omega\|^2+\|\nabla\omega\|^2)dt+\int_R^\tau E^\kappa\Phi'\int_{\mathcal{D}}\gamma_a(t)\,|\partial_t\omega|^{s(x)}\,dx\,dt$$
$$+\int_R^\tau E^\kappa\Phi'\int_{\mathcal{D}}\gamma_b(t)u(\cdot,1,t)\,|u(\cdot,1,t)|^{s(x)-2}\cdot\omega\,dx\,dt-\beta_c\int_R^\tau E^\kappa\Phi'\,\langle\nabla\vartheta,\omega\rangle\,dt. \tag{6.2}$$

Next, multiply the transport equation (4.2) by $E^\kappa\Phi'(t)\,\xi(x,t)(x,t)\,e^{-r\varrho(t)}\,|u|^{s(x)-2}\,u$ and integrate over $\mathcal{D}\times(0,1)\times(R,\tau)$. Using $\partial_t(\frac{\varrho(t)}{1-r\varrho'(t)})=\frac{\varrho(t)\varrho'(t)}{(1-r\varrho'(t))^2}\geq 0$, integrating by parts in r, and applying the trace conditions (4.3) leads to

$$\begin{aligned}0=&\int_R^\tau E^\kappa\Phi'\int_{\mathcal{D}}\xi(x,t)\int_0^1 e^{-r\varrho(t)}\,|u|^{s(x)}\,dr\,dx\,dt\\&-\int_R^\tau E^\kappa\Phi'\int_{\mathcal{D}}\xi(x,t)(|u(\cdot,1,t)|^{s(x)}-e^{-\varrho(t)}\,|\partial_t\omega|^{s(x)})\,dx\,dt\\&+\int_R^\tau(\kappa E^{\kappa-1}E'\Phi'+E^\kappa\Phi'')\int_{\mathcal{D}}\xi(x,t)\int_0^1 e^{-r\varrho(t)}\,|u|^{s(x)-2}\,u\,u\,dr\,dx\,dt.\end{aligned} \tag{6.3}$$

Model-dependent bounds

We now estimate every term on the right of (6.2)–(6.3) by the energy and the damping contributions. First, by Cauchy–Schwarz, Young and Poincaré,

$$\left|E^\kappa\Phi'\,\langle\partial_t\omega,\omega\rangle\right|\;\leq\;\mathrm{C}\,E^{\kappa+1}(t),\qquad\left|E^\kappa\Phi'\,\langle\nabla\vartheta,\omega\rangle\right|\;\leq\;\tfrac{\kappa}{4}E^\kappa\Phi'\,\|\nabla\vartheta\|^2+\mathrm{C}\,E^{\kappa+1}(t).$$

Next, using (A1)–(A4), (4.5), and the positivity of a_1,a_2 (see (4.6)), the nonlinear terms satisfy

$$\begin{aligned}\int_{\mathcal{D}}\gamma_a(t)\,|\partial_t\omega|^{s(x)}\,dx\;&\geq\;\int_{\mathcal{D}}\gamma_a(t)\,a_1(x)\,|\partial_t\omega|^{s(x)}\,dx,\\\int_{\mathcal{D}}\gamma_b(t)u(\cdot,1,t)\,|u(\cdot,1,t)|^{s(x)-2}\cdot\omega\,dx\;&\geq\;\int_{\mathcal{D}}\gamma_a(t)\,a_2(x)\,|u(\cdot,1,t)|^{s(x)}\,dx-\mathrm{C}\,E(t).\end{aligned}$$

Since $0<\varrho(t)\leq\varrho_1$, for a.e. (x,t)

$$\int_0^1 e^{-r\varrho(t)}\,dr=\frac{1-e^{-\varrho(t)}}{\varrho(t)}\geq\frac{1-e^{-\varrho_1}}{\varrho_1}=:\Lambda\in(0,1).$$

Hence

$$\int_0^1 e^{-r\varrho(t)}\,|u(x,r,t)|^{s(x)}\,dr\;\geq\;\Lambda\;|u(\cdot,1,t)(x,t)|^{s(x)}\,.$$

Collecting these bounds and using the definition of $\mathcal{H}$, we infer from (6.2)–(6.3) the key estimate

$$\Lambda\int_R^\tau\Phi'(t)\,E^\kappa(t)\left[\int_{\mathcal{D}}(a_1\,|\partial_t\omega|^{s(x)}+a_2\,|u(\cdot,1,t)|^{s(x)})dx+\kappa\,\|\nabla\vartheta(t)\|_{L^2}^2\right]dt \tag{6.4}$$
$$\leq\;\mathrm{C}\,E^{\kappa+1}(R)+\mathrm{C}\int_R^\tau E^\kappa(t)(-E'(t))\,dt. \tag{6.5}$$

By the dissipation inequality (4.6), the square brackets control $E(t)$ from below (up to a harmless constant), so (6.5) implies precisely (6.1) with $E=\mathcal{H}$.

Exponential decay

Applying Lemma 6.1 with $E = \mathcal{H}$, $\Phi(t) = \int_0^t \gamma_a(\ell)\,d\ell$, and $\kappa > -1$ arbitrary, we conclude that there exist $\mathrm{C}, \beta > 0$, depending only on the structural parameters in (A1)–(A4), such that

$$\mathcal{H}(t) \;\le\; \mathrm{C}\,\mathcal{H}(0)\,e^{-\beta\int_0^t \gamma_a(\ell)\,d\ell} \qquad \forall\, t \ge 0,$$

which is exactly (8.1).

7. Compactness and passage to the limit

From (5.9), (5.10)–(5.12), (5.14)–(5.15), and Aubin–Lions, we extract (non–relabeled) subsequences such that, for every fixed $\tau > 0$,

$$\omega^N \overset{*}{\rightharpoonup} \omega \text{ in } L^\infty(0,\tau;H_0^1(\mathcal{D})^n), \qquad \partial_t\omega^N \rightharpoonup \partial_t\omega \text{ in } L^{s(x)}(\mathcal{D}\times(0,\tau)), \tag{7.1}$$

$$\vartheta^N \overset{*}{\rightharpoonup} \vartheta \text{ in } L^\infty(0,\tau;L^2), \quad \vartheta^N \rightharpoonup \vartheta \text{ in } L^2(0,\tau;H^1), \tag{7.2}$$

$$\begin{aligned} &u^Q \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,\tau;L^{s(x)}(\mathcal{D}\times(0,1))),\\ &\quad u(\cdot,1,t)^Q \rightharpoonup u(\cdot,1,t) \text{ in } L^{s(x)}(\mathcal{D}\times(0,\tau)). \end{aligned} \tag{7.3}$$

Moreover,

$$\partial_t\omega^N \to \partial_t\omega \quad \text{in } L^2(\mathcal{D}\times(0,\tau)). \tag{7.4}$$

By monotonicity and Minty–Browder (variable–exponent setting; cf. [35, 36]),

$$\begin{aligned} &\left|\partial_t\omega^N\right|^{s(x)-2}\partial_t\omega^N \rightharpoonup |\partial_t\omega|^{s(x)-2}\,\partial_t\omega,\\ &\left|u(\cdot,1,t)^Q\right|^{s(x)-2} u(\cdot,1,t)^Q \rightharpoonup |u(\cdot,1,t)|^{s(x)-2}\,u(\cdot,1,t) \quad \text{in } L^{p'(x)}. \end{aligned} \tag{7.5}$$

Passing to the limit in the Galerkin formulations with dense test spaces, we obtain, for a.e. $t > 0$,

$$\begin{aligned} &\langle \partial_{tt}\omega(t),\boldsymbol{\varphi}\rangle + (\nabla\omega(t),\nabla\boldsymbol{\varphi})_{L^2} - \beta_c\,\langle\nabla\vartheta(t),\boldsymbol{\varphi}\rangle + \int_{\mathcal{D}}\gamma_a(t)\,|\partial_t\omega|^{s(x)-2}\,\partial_t\omega\cdot\boldsymbol{\varphi}\,dx\\ &\quad + \int_{\mathcal{D}}\gamma_b(t)u(\cdot,1,t)\,|u(\cdot,1,t)|^{s(x)-2}\cdot\boldsymbol{\varphi}\,dx = 0, \quad \forall\,\boldsymbol{\varphi}\in H_0^1(\mathcal{D})^n, \end{aligned} \tag{7.6}$$

$$\langle\partial_t\vartheta(t),\psi\rangle + \kappa(\nabla\vartheta(t),\nabla\psi)_{L^2} + \beta_c\,\langle \operatorname{div}\partial_t\omega(t),\psi\rangle = 0, \quad \forall\,\psi\in H^1(\mathcal{D}), \tag{7.7}$$

$$\begin{aligned} &\varrho(t)\,(\partial_t u(t),\Phi)_{L^2(\mathcal{D}\times(0,1))} + ((1-r\varrho'(t))\,\partial_r u(t),\Phi)_{L^2(\mathcal{D}\times(0,1))} = 0,\\ &\forall\,\Phi\in L^2((0,1);H_0^1(\mathcal{D})^n), \end{aligned} \tag{7.8}$$

$$u(\cdot,0,t) = \partial_t\omega(\cdot,t), \qquad u(\cdot,1,t)(\cdot,t) = u(\cdot,1,t). \tag{7.9}$$

Initial conditions are inherited from the weak–$*$ continuity and (5.5). Hence (ω,ϑ,u) is a global solution of (1.1).

8. Global strong Well–Posedness and decay

Theorem 8.1 (Global strong solution, uniqueness, and decay)**.** *Under* (A1)–(A4)*, for initial data*

$$\omega_0 \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}), \quad \boldsymbol{v}_0 \in H_0^1(\mathcal{D}), \quad \vartheta_0 \in H^1(\mathcal{D}), \quad I_0 \in H^1((0,1); H_0^1(\mathcal{D})),$$
$$I_0(\cdot, 0) = \boldsymbol{v}_0,$$

there exists a unique global strong solution (ω, ϑ, u) *to* (1.1) *such that*

$$\omega \in L^\infty(0,\infty; H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})), \qquad \partial_t\omega = \omega_t \in L^\infty(0,\infty; H_0^1(\mathcal{D})) \cap L^{s(x)}(\mathcal{D} \times (0,\infty)),$$

$$\vartheta \in L^\infty(0,\infty; L^2(\mathcal{D})) \cap L^2(0,\infty; H^1(\mathcal{D})), \qquad \omega_{tt} \in L^\infty(0,\infty; L^2(\mathcal{D})),$$

and the transport state satisfies

$$u \in L^\infty((0,1) \times (0,\infty); H_0^1(\mathcal{D})) \cap L^\infty(0,\infty; L^{s(x)}(\mathcal{D} \times (0,1))),$$
$$u(\cdot, 1, t)(\cdot, t) = u(\cdot, 1, t) \in L^{s(x)}(\mathcal{D}) \quad \textit{a.e. } t > 0.$$

Moreover, the energy $\mathcal{H}$ *is nonincreasing and obeys the decay estimate*

$$\mathcal{H}(t) \le \mathrm{C}\,\mathcal{H}(0)\, \exp\Big(-\beta \int_0^t \gamma_a(\ell)\, d\ell\Big), \qquad t \ge 0, \tag{8.1}$$

for some $\mathrm{C}, \beta > 0$ *depending only on the structural parameters of the model.*

Proof. Existence. The Faedo–Galerkin scheme of Section 5 (with projections (5.2)–(5.4) and initial data (5.5)) produces approximate solutions on any finite interval. The uniform bounds (5.9), (5.10)–(5.12) and (5.14)–(5.15), together with the Aubin–Lions compactness lemma and the Minty–Browder identification in variable-exponent spaces, yield the convergences (7.1)–(7.5) and the limit relations (7.6)–(7.9). Hence a global strong solution in the stated class exists.

Uniqueness. Let $(\omega^{(i)}, \vartheta^{(i)}, u^{(i)})$ be two solutions with the same data, and set differences $w = \omega^{(1)} - \omega^{(2)}$, $\theta = \vartheta^{(1)} - \vartheta^{(2)}$, $g = u^{(1)} - u^{(2)}$. Subtract the weak formulations (7.6)–(7.8), test respectively by w_t, θ, and $\xi(x,t)\,|g|^{s(x)-2}g$, and add the three identities. Using the monotonicity of $\eta \mapsto |\eta|^{s(x)-2}\eta$ and the transport boundary coupling $u(\cdot, 0, t) = \partial_t\omega(\cdot, t)$, we obtain for a.e. $t > 0$

$$\begin{aligned}&\frac{d}{dt}\Big(\tfrac{1}{2}\|w_t\|_{L^2}^2 + \tfrac{1}{2}\|\nabla w\|_{L^2}^2 + \tfrac{c_\theta}{2}\|\theta\|_{L^2}^2\Big)\\ &+\gamma_a(t)\int_{\mathcal{D}}\big(|\partial_t\omega^{(1)} - \partial_t\omega^{(2)}|^{s(x)} + |u(\cdot,1,t)^{(1)} - u(\cdot,1,t)^{(2)}|^{s(x)}\big)\,dx + \kappa\|\nabla\theta\|_{L^2}^2 \le 0.\end{aligned}$$

Gronwall's lemma yields $w \equiv 0$, $\theta \equiv 0$, $g \equiv 0$; hence uniqueness.

Decay. By Lemma 4.1, the energy satisfies the differential inequality (4.6). Section 6 constructs the Komornik-type weighted estimate (6.5) and applies Lemma 6.1 with $\Phi(t) = \int_0^t \gamma_a(\ell)\, d\ell$, which gives (8.1). $\square$

9. Concluding remarks

This study establishes a rigorous framework for a thermoelastic wave system equipped with spatially variable–exponent damping and a time–dependent delay. By recasting the delay through a transport variable and working in the $L^{s(x)(\cdot)}$–$W^{1,s(x)(\cdot)}$ setting, we derived a weighted dissipation inequality with strictly positive coefficients, constructed global approximate solutions via a Faedo–Galerkin scheme, and passed to the limit using compactness together with a Minty–Browder identification of the nonlinearities. The resulting solution is global and unique, and a Lyapunov multiplier combined with a Komornik–type integral argument yields the quantitative stabilization law $\mathcal{H}(t) \leq C\,\mathcal{H}(0)\exp\left(-\beta\int_0^t \gamma_a(\ell)\,d\ell\right)$, showing that the decay profile is governed by the effective damping γ_a under the admissible delay constraint $0 \leq \varrho'(t) \leq \sigma < 1$. The methodology is robust and readily adapts to variants such as Cattaneo-type heat conduction, distributed or state-dependent delays, and source perturbations (under standard smallness or sign conditions), as well as to weaker data or less regular domains. These results provide a tractable template for the analysis and control of thermoelastic systems with variable growth and delayed feedback.

Author Contributions

Muhammad Zainul Abidin: conceptualization, investigation, methodology, writing—original draft; Naeem Ullah: investigation, methodology, validation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. C. M. Dafermos, On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity, *Arch. Ration. Mech. Anal.*, **29** (1968), 241–271. https://doi.org/10.1007/BF00276727
2. M. Slemrod, Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one-dimensional non-linear thermoelasticity, *Arch. Ration. Mech. Anal.*, **76** (1981), 97–133. https://doi.org/10.1007/BF00251248
3. J. E. Lagnese, Boundary stabilization of linear elastodynamic systems, *SIAM J. Control Optim.*, **21** (1983), 968–984. https://doi.org/10.1137/0321059
4. J. U. Kim, On the energy decay of a linear thermoelastic bar and plate, *SIAM J. Math. Anal.*, **23** (1992), 889–899. https://doi.org/10.1137/0523047

5. I. Lasiecka, R. Triggiani, Analyticity of thermo-elastic semigroups with free boundary conditions, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **27** (1998), 457–482.
6. G. Ponce, R. Racke, Global existence of small solutions to the initial value problem for nonlinear thermoelasticity, *J. Differential Equations*, **87** (1990), 70–83. https://doi.org/10.1016/0022-0396(90)90016-I
7. S. Jiang, R. Racke, *Evolution Equations in Thermoelasticity*, Chapman & Hall/CRC, Boca Raton, FL, 2000. https://doi.org/10.1201/9781482285789
8. C. Cattaneo, Sulla conduzione del calore, *Atti Semin. Mat. Fis. Univ. Modena*, **3** (1948), 83–101.
9. B. D. Coleman, M. E. Gurtin, Thermodynamics with internal state variables, *J. Chem. Phys.*, **47** (1967), 597–613. https://doi.org/10.1063/1.1711937
10. M. I. A. Othman, Kh. Lotfy, R. M. Farouk, Transient disturbance in a half-space under generalized magneto-thermoelasticity with internal heat source, *Acta Phys. Pol. A*, **116** (2009), 185–192. https://doi.org/10.12693/APhysPolA.116.185
11. G. M. Ismail, K. A. Gepreel, Kh. Lotfy, A. M. S. Mahdy, A. El-Bary, A. M. Saeed, Influence of variable thermal conductivity on thermal-plasma-elastic waves of excited microelongated semiconductor, *Alexandria Eng. J.*, **61** (2022), 12271–12282. https://doi.org/10.1016/j.aej.2022.06.024
12. M. I. Marin, R. P. Agarwal, I. A. Abbas, Effect of intrinsic rotations, microstructural expansion and contractions in initial boundary value problem of thermoelastic bodies, *Bound. Value Probl.*, **2014** (2014), 129. https://doi.org/10.1186/1687-2770-2014-129
13. A. E. Abouelregal, M. Marin, A. Öchsner, The influence of a non-local Moore–Gibson–Thompson heat transfer model on an underlying thermoelastic material under the model of memory-dependent derivatives, *Contin. Mech. Thermodyn.*, **35** (2023), 545–562. https://doi.org/10.1007/s00161-023-01195-y
14. I. Giorgio, L. Placidi, A variational formulation for three-dimensional linear thermoelasticity with "thermal inertia", *Meccanica*, **59** (2024), 1745–1756. https://doi.org/10.1007/s11012-024-01796-0
15. I. Giorgio, A variational formulation for one-dimensional linear thermoviscoelasticity, *Math. Mech. Complex Syst.*, **9** (2021), 397–412. https://doi.org/10.2140/memocs.2021.9.397
16. J. E. Muñoz Rivera, R. Racke, Smoothing properties, decay, and global existence of solutions to nonlinear coupled systems of thermoelastic type, *SIAM J. Math. Anal.*, **26** (1995), 1547–1563. https://doi.org/10.1137/S0036142993255058
17. B. Lazzari, R. Nibbi, On the energy decay of a linear hyperbolic thermoelastic system with dissipative boundary, *J. Thermal Stresses*, **30** (2007), 1159–1172. https://doi.org/10.1080/01495730701416101
18. M. I. Mustafa, Uniform stability for thermoelastic systems with boundary time-varying delay, *J. Math. Anal. Appl.*, **383** (2011), 490–498. https://doi.org/10.1016/j.jmaa.2011.05.066
19. S. Mansouri, L. Tebou, Stabilization of coupled thermoelastic Kirchhoff plate and wave equations, *Electron. J. Differential Equations*, **2020** (2020), 1–16. https://doi.org/10.58997/ejde.2020.121
20. V. Komornik, *Exact Controllability and Stabilization: The Multiplier Method*, Wiley–Masson, Paris, 1994.

21. P. Martínez, A new method to obtain decay rate estimates for dissipative systems, *ESAIM Control Optim. Calc. Var.*, **4** (1999), 419–444. https://doi.org/10.1051/cocv:1999116

22. L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, In: *Lecture Notes in Mathematics,* Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8

23. A. Choucha, S. Boulaaras, B. Djafari-Rouhani, R. Guefaifia, A. Alharbi, Global existence and general decay for a nonlinear wave equation with acoustic and fractional boundary conditions coupling by source and delay terms, *Results Appl. Math.*, **23** (2024), 100476. https://doi.org/10.1016/j.rinam.2024.100476

24. M. Z. Abidin, J. C. Chen, Global well-posedness of generalized magnetohydrodynamics equations in variable exponent Fourier–Besov–Morrey spaces, *Acta Math. Sin. (Engl. Ser.)*, **38** (2022), 2187–2198. https://doi.org/10.1007/s10114-022-0581-0

25. A. Fahad, S. M. Boulaaras, H. U. Rehman, I. Iqbal, D. Chou, Probing nonlinear wave dynamics: insights from the (2 + 1)-dimensional Konopelchenko–Dubrovsky system, *Results Phys.*, **57** (2024), 107370. https://doi.org/10.1016/j.rinp.2024.107370

26. A. Haraux, E. Zuazua, Decay estimates for some semilinear damped hyperbolic problems, *Arch. Ration. Mech. Anal.*, **100** (1988), 191–206. https://doi.org/10.1007/BF00282203

27. H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.*, **5** (1974), 138–146. https://doi.org/10.1137/0505015

28. V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differential Equations*, **109** (1994), 295–308. https://doi.org/10.1006/jdeq.1994.1051

29. S. A. Messaoudi, A. A. Talahmeh, A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities, *Appl. Anal.*, **96** (2017), 1509–1515. https://doi.org/10.1080/00036811.2016.1276170

30. S. A. Messaoudi, A. A. Talahmeh, Blowup in solutions of a quasilinear wave equation with variable-exponent nonlinearities, *Math. Methods Appl. Sci.*, **40** (2017), 6976–6986. https://doi.org/10.1002/mma.4505

31. S. Ghegal, I. Hamchi, S. A. Messaoudi, Global existence and stability of a nonlinear wave equation with variable-exponent nonlinearities, *Appl. Anal.*, **99** (2020), 1333–1343. https://doi.org/10.1080/00036811.2018.1530760

32. M. Kafini, S. Messaoudi, On the decay and global nonexistence of solutions to a damped wave equation with variable-exponent nonlinearity and delay, *Ann. Polon. Math.*, **122** (2019), 49–70.

33. A. Benaissa, A. Benguessoum, S. A. Messaoudi, Energy decay of solutions for a wave equation with a constant weak delay and a weak internal feedback, *Electron. J. Qual. Theory Differ. Equ.*, **2014** (2014), 1–13. https://doi.org/10.14232/ejqtde.2014.1.11

34. A. Benguessoum, Global existence and energy decay of solutions for a wave equation with a time-varying delay term, *Mathematica*, **63** (2021), 32–46. https://doi.org/10.24193/MATHCLUJ.2021.1.04

35. S. Antontsev, Wave equation with $p(x,t)$-Laplacian and damping term: existence and blow-up, *Differ. Equ. Appl.*, **3** (2011), 503–525. https://doi.org/10.7153/dea-03-32

36. S. Antontsev, Wave equation with $p(x,t)$-Laplacian and damping term: blow-up of solutions, *C. R. Mécanique*, **339** (2011), 751–755. https://doi.org/10.1016/j.crme.2011.09.001