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Research article

Nonlinear boundary control with gain scheduled control for the porous elastic system

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Abstract: In this paper, nonlinear boundary controls associated with two control gain for a one-dimensional continuous porous elastic system are designed, with the primary goal of uniformly stabilizing the model. The nonlinear boundary controllers were designed based on the spillover effect generated by high-frequency problems that generate so-called spurious oscillations. The existence and uniqueness of the solutions are proven using the Galerkin approximation method with a special basis constructed using the Gram-Schmidt process. Uniform stabilization is achieved using the method introduced by Lasiecka-Tataru.

Keywords: Galerkin method; porous elastic materials; uniform decay rates
Mathematics Subject Classification: 65N30, 76S05, 70K15.

1. Introduction

The motivation for this paper comes from a linear theory of elastic materials with voids developed by Cowin and Nunziato in [1] (see also [2–4]). According to the authors, this theory significantly differs from the classical linear elasticity because the volume fraction corresponding to the void volume is considered an independent kinematic variable. Several applications of this theory can be considered, including the response to homogeneous deformations, pure bending of a beam, and small-amplitude acoustic waves. In each of these applications, the variation in the void volume induced by the deformation is determined.

Regarding the objective of the paper, in recent decade, an increasing interest has been developed to determine the asymptotic behavior of the solutions for porous elastic systems (see [5, 6] and the references therein). In order to describe such results (see [1]), let us consider the evolution equations for the one-dimensional porous elastic system

$$\begin{cases} \rho u_{tt} = T_x, \\ J\phi_{tt} = H_x + G, \end{cases} \tag{1.1}$$

where T is the stress, H is the equilibrated stress, and G is the equilibrated body force. Here, u represents the displacement of a solid elastic material and ϕ is the volume fraction.

The constitutive equations are as follows

$$\begin{cases} T = \mu u_x + b\phi, \\ H = \delta\phi_x, \\ G = -bu_x - \xi\phi - \tau\phi_t \end{cases} \tag{1.2}$$

where ϕ_t represents the time rate of change of the volume fraction.

By substituting (1.2) into (1.1), we arrive at the following Cauchy problem:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \text{ in } (0, L) \times (0, \infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t = 0, \text{ in } (0, L) \times (0, \infty), \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ \phi(x, 0) = \phi_0(x), \ \phi_t(x, 0) = \phi_1(x) \ x \in (0, L). \end{cases} \tag{1.3}$$

To the system above, we add Dirichlet boundary conditions

$$u(0, t) = u(L, t) = \phi(0, t) = \phi(L, t) = 0, \ \forall t > 0 \tag{1.4}$$

or Dirichlet-Neumann boundary conditions

$$u(0, t) = u(L, t) = \phi_x(0, t) = \phi_x(L, t) = 0, \ \forall t > 0. \tag{1.5}$$

Here, ρ is the bulk density, $J = \rho k$, k is the equilibrated inertia, and $\mu, \ J, \ \delta, \ \tau,$ and b are constants that depend upon the volume fraction in the reference state ν_R, where ν_R is the matrix volume fraction. The above coefficients satisfy the following

$$\rho > 0, \ \mu > 0, \ J > 0, \ \delta > 0, \ \xi > 0, \ \tau > 0, \ b \neq 0 \text{ and } \mu\xi - b^2 > 0. \tag{1.6}$$

Remark 1.1. Here, we highlight a implicit point from [7]. For this purpose, we refer to the internal mechanical energy density of (1.3) (see [8, 9]) given by

$$\mathbb{E} := \mu u_x^2 + 2bu_x\phi + \xi\phi^2 + \delta\phi_x^2 = \left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)^2 + \delta\phi_x^2, \tag{1.7}$$

which is positive for the condition

$$\mu > 0, \ \xi > 0, \ \delta > 0, \ \mu\xi - b^2 > 0,$$

thus justifying the hypotheses adopted in (1.6).

1.1. State of the art

Among hundreds of results in the literature on the stabilization of partially dissipative systems of the porous elastic type, the respective exponential decay of the energy of the solutions occurs if, and only if, the wave propagation velocities, $\frac{\rho}{\mu}$ and $\frac{J}{\delta}$, are equal, that is,

$$\frac{\rho}{\mu} = \frac{J}{\delta}. \tag{1.8}$$

This has been the focus of the researchers since the pioneering article by R. Quintanilla [9], where the author proved that the system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \text{ in } (0,L)\times(0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t = 0, \text{ in } (0,L)\times(0,\infty), \\ u(0,t) = u(L,t) = \phi_x(0,t) = \phi_x(L,t) = 0, \ \forall t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ \phi(x,0) = \phi_0(x), \ \phi_t(x,0) = \phi_1(x) \ x \in (0,L). \end{cases} \tag{1.9}$$

has a slow decay whenever

$$\frac{\rho}{\mu} \neq \frac{J}{\delta}. \tag{1.10}$$

In general, researchers (see [10, 11] and the references therein) proved the following:
Porous elastic systems partially dissipative are exponentially stable if, and only if condition (1.8) is satisfied. In other words, there exist positive constants α and β such that

$$\mathcal{E}(t) \leq \alpha\mathcal{E}(0)e^{-\beta t}, \tag{1.11}$$

where $\mathcal{E}(t)$ is the model energy (1.3) given in (2.1).

In the general case, where the propagation speeds satisfy condition (1.10), the decay is of the polynomial type.

In the directions indicated above, J.E.M. Rivera and R. Quintanilla [12] proved that if (1.10) holds, then the energy of (1.9) is controlled by a rate decay of type $\frac{1}{t}$.

In [10], Santos et. al. studied the following porous elastic system:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x + \gamma u_t = 0, \text{ in } (0,L)\times(0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t = 0, \text{ in } (0,L)\times(0,\infty), \\ u(0,t) = u(L,t) = \phi_x(0,t) = \phi_x(L,t) = 0, \ \forall t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ \phi(x,0) = \phi_0(x), \ \phi_t(x,0) = \phi_1(x) \ x \in (0,L), \end{cases} \tag{1.12}$$

and considered the cases $\gamma = 0$ and $\tau \neq 0$ or $\gamma \neq 0$ and $\tau = 0$. Then, using the Prüss's Theorem (see [13]), they proved that (1.12) is exponentially stable if and only if (1.8) holds. When (1.10) is valid, the system does not exhibit an exponential decay, but decays polynomially as $\frac{1}{t^{\frac{1}{4}}}$ with an optimal rate in both cases. When $\gamma \neq 0$ and $\tau \neq 0$, the model decays exponentially (see (1.11), independent of the relationships between the velocity coefficients (1.8) and (1.10).

In the paper [7], A. Magaña and R. Quintanilla considered the following system

$$\begin{cases}\rho u_{tt} - \mu u_{xx} - b\phi_x - \gamma u_{xxt} = 0, \text{ in } (0,L)\times(0,\infty),\\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t = 0, \text{ in } (0,L)\times(0,\infty),\\ u(0,t) = u(L,t) = \phi_x(0,t) = \phi_x(L,t) = 0, \ \forall t > 0,\\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ \phi(x,0) = \phi_0(x), \ \phi_t(x,0) = \phi_1(x) \ x \in (0,L),\end{cases} \tag{1.13}$$

They proved that (1.13) is exponentially stable (see (1.11)) using semigroup arguments due to Liu and Zheng [14].

Many other important papers have emerged which studied the problem with several dissipative mechanisms (e.g., frictional damping, thermal effect, and memory effect, among others (for examples, see [7,8,11,15–18] and the references therein)). In all articles, the results showed exponential stability if the model is fully damped (see [7]) or depends on the relationship (1.8) between the velocity coefficients in the case of single damping (see [10]).

The long-term behavior of the porous-elastic system, thereby focusing on the interaction between nonlinear damping and source terms, was studied in [19]. The authors proved that the total energy of the system decays either exponentially or algebraically, depending on the dissipation behavior of the system near the origin. Additionally, they proved the existence of a global attractor.

One of the questions regarding stability of the porous elastic system concerns the minimum dissipation needed to obtain an exponential decay. It is interesting to know if the energy is controlled by an exponential or polynomial function. In [20], Santos and Almeida Júnior studied the following porous elastic system with localized damping:

$$\begin{cases}\rho u_{tt} - \mu u_{xx} - b\phi_x + \gamma(x)(u_t + v_t) = 0, \text{ in } (0,L)\times(0,\infty),\\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \gamma(x)(u_t + v_t) = 0, \text{ in } (0,L)\times(0,\infty),\\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ \phi(x,0) = \phi_0(x), \ \phi_t(x,0) = \phi_1(x), \ x \in (0,L),\end{cases} \tag{1.14}$$

with boundary conditions (1.4) or (1.5). The main contribution in [20] was to provide a necessary and sufficient condition for the strong stability and the exponential decay of the porous elastic system with the rank-one localized damping, where the boundary of the damping region must contain at least one of the end points of the spatial domain.

In short, two independent damping, one in each equation, cause exponential stability of the model. In the same way, a single damping acting on a model equation causes exponential stability depending on a relationship between the speed coefficients. On the other hand, the same damping acting on both equations causes exponential stability of the model (see [20]).

Here, we highlight the main differences between nonlinear boundary damping, which will be discussed in this paper (see (1.15)), nonlinear localized damping, discussed in [8], and damping of memory type, discussed in [15–17].

The main differences between nonlinear boundary damping and nonlinear localized damping lie in the physical location of energy dissipation and the impact on the system's vibration modes. While boundary damping acts at a point on the boundary, localized damping acts in a specific region of the domain.

On the other hand, memory type damping assumes that energy dissipation depends not only on the current velocity, but on the entire history of the boundary motion, usually modeled by integro-differential equations.

Here are the detailed differences:

- Nonlinear damping at a boundary point strongly influences the overall vibration modes, as the energy constraint is at the edges. It is widely used to stabilize systems subject to high-amplitude oscillations at the edges.
- Localized nonlinear damping tends to affect specific vibration modes that have a large displacement amplitude in the region where the damper was placed.
- Memory-type damping provides a smoother and more continuous energy decay over time, without relying on a high instantaneous speed to dissipate energy, thus making it superior for complex structural damping.

1.2. The problem

To stabilize the system given by governing Eqs. (1.3) with $\tau = 0$, we propose the following boundary control laws:

$$\begin{cases} \mu u_x(L,t) + b\phi(L,t) = -\alpha_1 g_1(u_t(L,t)) - \beta_1 u_x(L,t), \quad \forall t \geq 0, \\ \delta\phi_x(L,t) = -\alpha_2 g_2(\phi_t(L,t)) - \beta_2\phi_x(L,t), \quad \forall t \geq 0 \end{cases} \tag{1.15}$$

where $\alpha_i,\ i = 1,2$, are positive constants, $g_i(\cdot),\ i = 1,2$, are the nonlinear boundary control, and $\beta_i > 0,\ i = 1,2$, are the control gains.

Remark 1.2. The justification for using scheduled control gains in this paper comes from Inequalities (3.14) and (3.15), where terms of the form $\frac{L\beta_1}{2}|u_x(L,t)|^2$ and $\frac{L\beta_2}{2}|\phi_x(L,t)|^2$ arise, which need to be controlled. In this paper, we chose to control them using scheduled control gains of the type $-\beta_1 u_x(L,t)$ and $-\beta_2\phi_x(L,t)$, where the coefficients of the control gains $\beta_1,\ \beta_2$ are used as a counter of repetitions of the terms $\frac{L\beta_1}{2}|u_x(L,t)|^2$ and $\frac{L\beta_1}{2}|\phi_x(L,t)|^2$. Thus, the combination of Inequalities (3.13), (3.14), and (3.15) leads to the elimination of these terms, thus making the general decay of the model possible.

The open question is whether two nonlinear controls associated with two control gains acting at a point on the boundary cause uniform decay rates, which will be addressed in this paper.

Taking what has been said above into account, the purpose of this paper is to investigate uniform decay rates of the porous elastic system with nonlinear boundary controls associated with two control gain given by the following:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0, \quad \text{in} \quad (0,L)\times(0,\infty), \\ u(0,t) = \phi(0,t) = 0, \quad \forall t > 0, \\ \mu u_x(L,t) + b\phi(L,t) = -\alpha_1 g_1(u_t(L,t)) - \beta_1 u_x(L,t),\ \forall t > 0 \\ \delta\phi_x(L,t) = -\alpha_2 g_2(\phi_t(L,t)) - \beta_2\phi_x(L,t), \forall t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x),\ \forall x \in (0,L) \end{cases} \tag{1.16}$$

where $\alpha_i,\ i = 1,2$, are positive constants, and $g_i,\ i = 1,2$, are the boundary controls.

Remark 1.3. According to Cowin-Nunziato [1] (page 146 (A.23)), $\tau \geq 0$. Therefore, it is perfectly possible to consider $\tau = 0$. Physically, when we consider $\tau = 0$, it means that the volume fraction of the matrix in a porous elastic material does not change during deformation, which implies that the energy dissipation of the model tends to significantly decrease or become zero, thereby assuming that the main dissipation of the model comes from the structural rearrangement of the porosities (viscoplasticity/damage). This means that the material behaves as a simple linear elastic solid with zero energy dissipation. In this context, the presence of non-linear dissipation terms at the boundary, combined with the scheduled control gains (as given in (1.15)), is justifiable.

The main objectives of this paper are as follows:

- We provide a rigorous proof of the existence and uniqueness of the solutions of System (1.16). To do this, we use the Faedo-Galerkin method with a special basis constructed using the Gram-Schmidt process.
- We provide the uniform decay rates for the System (1.16). To do this, we use the method introduced by Lasiecka-Tataru ([20]) (additionally, see [21–24]).

The outline of paper is given as follows: in Section 2, we provide the existence and uniqueness of regular global solutions; in Section 3, we prove the uniform decay rates; in Section 4, we present some examples of applications of the main result; in Section 5, we discuss the application of controls combined with a scheduled gain control.

2. Existence of global solutions

In what follows, the well-posedness of Problem (1.16) shall be proved by exploiting the Faedo-Galerkin method. To this end, we first introduce the following spaces:

$$H^1_*(0,L) = \{\varphi \in H^1(0,L) : \ \varphi(0) = 0\} \quad \text{and} \quad \mathcal{V} = H^2(0,L) \cap H^1_*(0,L).$$

It follows immediately from [17] that both $H^1_*(0,L)$ and $\mathcal{V}$ are closed subspaces of $H^1(0,L)$ and $H^2(0,L)$, respectively. In order to apply the Faedo-Galerkin method, we need a sequence $\{\varphi_i\}_{i=1}^\infty$ that is complete in $\mathcal{V}$ and orthogonal in $L^2(0,L)$. To construct such a sequence, we must proceed as follows. First, since $\mathcal{V}$, as a closed subspace of $H^2(0,L)$, which is a separable Hilbert space, there exists a sequence of $\{\psi_i\}_{i=1}^\infty \in \mathcal{V}$, which forms a Schauder basis for $\mathcal{V}$, i.e., every $\psi \in \mathcal{V}$ can be uniquely represented as

$$\psi = \sum_{i=1}^{\infty} a_i \psi_i \in \mathcal{V}.$$

Second, using the Gram-Schmidt process, we orthogonalize $\{\psi_i\}_{i=1}^\infty$ in $L^2(0,L)$ as follows:

$$\varphi_1 = \psi_1, \quad \varphi_n = \psi_n - \sum_{i=1}^{n-1} \frac{\langle \psi_i, \varphi_i \rangle}{\langle \varphi_i, \varphi_i \rangle} \varphi_i, \quad n \geq 2$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and norm of $L^2(0,L)$, respectively. As a result, for any $n \geq 1$, $\mathrm{Span}\{\psi_1, \psi_2, \ldots, \psi_n\} = \mathrm{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. Therefore, $\{\varphi_i\}_{i=1}^\infty$ is complete in $\mathcal{V}$.

The following assumptions are made on the nonlinear functions $g_i, \ i = 1, 2$.

Assumptions 2.1:

(I) $g_i(s)$ are continuous, monotone, increasing functions on $\mathbb{R}$;

(II) $g_i(s)s > 0, \ \text{for } s \neq 0$;

(III) $M_i s^2 \leq g_i(s)s \leq K_i s^2, \ \text{for } |s| > N, \ \text{for some } N > 0, \ M_i, \ K_i, \ 0 < M_i \leq K_i, \ i = 1, 2.$

Remark 2.1. Limiting nonlinear damping in the form

$$M_i s^2 \le g_i(s)s \le K_i s^2$$

physically means that the system's energy dissipation is neither constant nor linear, but varies in a controlled manner between two quadratic limits. The above expression ensures that the dissipation force does not become excessively weak (which would cause prolonged vibrations) nor excessively strong (which would stall the system). In summary, $M_i s^2$ guarantees a minimum level of dissipation to avoid instability, and $K_i s^2$ prevents excessive damping forces that could damage the structure or render the system inefficient.

The energy of system (1.16) is as follows:

$$\begin{aligned} \mathcal{E}(t) &= \frac{\rho}{2}\int_0^L |u_t|^2\,dx + \frac{J}{2}\int_0^L |\phi_t|^2\,dx + \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right)\int_0^L |u_x|^2\,dx \\ &+ \frac{1}{2}\int_0^L \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + \frac{\delta}{2}\int_0^L |\phi_x|^2\,dx. \end{aligned} \tag{2.1}$$

The following theorem gives us the existence of global solutions for System (1.16).

Theorem 2.2. *Suppose the initial values satisfy* $u_0,\ u_1,\ \phi_0,$ *and* $\phi_1 \in \mathcal{V}$, *and the feedback nonlinearities* $g_i(\cdot),\ i = 1, 2,$ *satisfy the* **Assumptions 2.1** *and compatibility conditions*

$$\begin{cases} \mu u_{0x}(L) + b\phi_0(L) = -\alpha_1 g_1(u_1(L)) - \beta_1 u_{0x}(L), \\ \delta\phi_{0x}(L) = -\alpha_2 g_2(\phi_1(L)) - \beta_2\phi_{0x}(L). \end{cases} \tag{2.2}$$

Then, System (1.16) admits a unique global solution which continuously depends on the initial value. Moreover, for all $T > 0$

$$\begin{cases} u,\ \phi\ \in L^\infty([0,T); H^2(0,L)), \\ u,\ \phi \in L^\infty([0,T);\ H^1_*(0,L)), \\ u_t,\ \phi_t \in L^\infty([0,T);\ H^1_*(0,L)), \\ u_{tt},\ \phi_{tt} \in L^\infty([0,T);\ L^2(0,L)). \end{cases} \tag{2.3}$$

Proof. Since $u_0, u_1, \phi_0, \phi_1 \in \mathcal{V}$, we assume without a loss of generality that $u_0, u_1, \phi_0, \phi_1 \in \mathrm{Span}\{\varphi_1, \varphi_2\}$. Let $\mathbb{V}_n := \mathrm{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ for $n \in \mathbb{N}$. Then, for a fixed $n \in \mathbb{N}$, with $n \ge 2$, set

$$\begin{cases} Z_n(t) = (a_{1n}(t), a_{2n}(t), \ldots, a_{nn}(t))^\top,\ \ Y_n(t) = (b_{1n}(t), b_{2n}(t), \ldots, b_{nn}(t))^\top \\ \Phi_n(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))^\top, \\ u^n(x,t) = \Phi_n(x)^\top Z_n(t),\ \ \phi^n(x,t) = \Phi_n(x)^\top Y_n(t), \end{cases} \tag{2.4}$$

where $a_{ij}(t)$ and $b_{ij}(t)$ are the values of $u^n(t)$ and $\phi^n(t)$, respectively, in the nodal points i at time t.

From (2.4), we see that the variational structure of the Faedo-Galerkin approximation problem is as follows:

$$\begin{cases} \rho \int_0^L u_{tt}^n \omega \, dx + \mu \int_0^L u_x^n \omega_x \, dx + b \int_0^L \phi^n \omega_x \, dx \\ = -\alpha_1 g_1(u_t^n(L,t))\omega(L) - \beta_1 u_x^n(L,t)\omega(L), \ \forall \omega \in \mathbb{V}_n, \\ J \int_0^L \phi_{tt}^n \psi \, dx + \delta \int_0^L \phi_x^n \psi_x \, dx + b \int_0^L u_x^n \psi \, dx + \xi \int_0^L \phi^n \psi \, dx \\ = -\alpha_2 g_2(\phi_t^n(L,t))\psi(L) - \beta_2 \phi_x^n(L,t)\psi(L), \ \forall \psi \in \mathbb{V}_n, \\ u^n(x,0) = u_0(x), \ u_t^n(x,0) = u_1(x), \ \phi^n(x,0) = \phi_0(x), \ \phi_t^n(x,0) = \phi_1(x). \end{cases} \tag{2.5}$$

In order to derive the autonomous system, from definition of $\Phi_n(x) \in \mathbb{V}_n$ stated above, let

$$\omega(x) = \sum_{i=1}^n \gamma_i \varphi_i(x) = \gamma^\top \Phi_n(x) \in \mathbb{V}_n$$

for all $x \in [0, L]$, where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^\top \in \mathbb{R}^n$.

Then, it is straightforward to deduce that

$$\begin{aligned} &\int_0^L u_{tt}^n(x,t)\omega(x) \, dx = \gamma^\top \int_0^L \Phi_n(x)\Phi_n^\top(x)\ddot{Z}_n(t) \, dx = \gamma^\top A_n \ddot{Z}_n(t), \\ &\int_0^L \phi_{tt}^n(x,t)\omega(x) \, dx = \gamma^\top \int_0^L \Phi_n(x)\Phi_n^\top(x)\ddot{Y}_n(t) \, dx = \gamma^\top A_n \ddot{Y}_n(t), \\ &\int_0^L u_x^n(x,t)\omega_x(x) \, dx = \gamma^\top U_n Z_n(t), \\ &\int_0^L \phi^n(x,t)\omega_x(x) \, dx = \gamma^\top V_n Y_n(t), \\ &\int_0^L \phi_x^n(x,t)\omega_x(x) \, dx = \gamma^\top U_n Y_n(t), \\ &\int_0^L u_x^n(x,t)\omega(x) \, dx = \gamma^\top K_n Z_n(t), \\ &\int_0^L \phi^n(x,t)\omega(x) \, dx = \gamma^\top P_n Y_n(t) \end{aligned} \tag{2.6}$$

where $A_n = \int_0^L \Phi_n(x)\Phi_n^\top(x) \, dx \in \mathbb{R}^{n\times n}$, and the coefficient matrices $U_n, \ V_n, \ K_n,$ and $P_n \in \mathbb{R}^{n\times n}$ are given by

$$\begin{aligned} U_n &= \int_0^L \frac{d\Phi_n(x)}{dx}\left(\frac{d\Phi_n(x)}{dx}\right)^\top dx, \\ V_n &= \int_0^L \Phi_n(x)\left(\frac{d\Phi_n(x)}{dx}\right)^\top dx, \\ K_n &= \int_0^L \frac{d\Phi_n(x)}{dx}(\Phi_N(x))^\top \, dx, \\ P_n &= \int_0^L \Phi_n(x)(\Phi_N(x))^\top \, dx. \end{aligned}$$

Since the basis $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is orthogonal in $L^2(0, L)$, the mass matrixes A_n are reduced to the identity matrix $\mathcal{I}_n$ of size n. Substitute (2.6) into (2.5), with the above observation, to obtain the following:

$$\begin{cases} \gamma^\top \left(\rho \ddot{Z}_n(t) + \mu U_n Z_n(t) + b V_n Y_n(t)\right) \\ = -\gamma^\top \left(\alpha_1 g_1((\Phi_n(L))^\top \dot{Z}_n(t)) + \beta_1 E_n Z_n(t)\right) \Phi_n(L), \\ \gamma^\top \left(J \ddot{Y}_n(t) + \delta U_n Y_n(t) + b K_n Z_n(t) + \xi P_n Y_n(t)\right) \\ = -\gamma^\top \left(\alpha_2 g_2((\Phi_n(L))^\top \dot{Y}_n(t)) + \beta_2 E_n Y_n(t)\right) \Phi_n(L) \end{cases} \tag{2.7}$$

where $E_n = \left(\frac{d\Phi_n}{dx}(L)\right)^\top \in \mathbb{R}^n$. Furthermore, the initial conditions can be written as

$$\begin{cases} u^n(x,0) = (\Phi_n(x))^\top Z_n(0), & u^n_t(x,0) = (\Phi_n(x))^\top \dot{Z}_n(0), \\ \phi^n(x,0) = (\Phi_n(x))^\top Y_n(0), & \phi^n_t(x,0) = (\Phi_n(x))^\top \dot{Y}_n(0) \end{cases}$$

and

$$\begin{aligned} u_0(x) &= d_1\varphi_1 + d_2\varphi_2 = (\Phi_n(x))^\top f_n, \\ u_1(x) &= d_3\varphi_1 + d_4\varphi_2 = (\Phi_n(x))^\top g_n, \\ \phi_0(x) &= d_5\varphi_1 + d_6\varphi_2 = (\Phi_n(x))^\top h_n, \\ \phi_1(x) &= d_7\varphi_1 + d_8\varphi_2 = (\Phi_n(x))^\top m_n \end{aligned}$$

where $f_n = (d_1, d_2, 0, \ldots, 0)^\top$, $g_n = (d_3, d_4, 0, \ldots, 0)^\top$, $h_n = (d_5, d_6, 0, \ldots, 0)^\top$, and $m_n = (d_7, d_8, 0, \ldots, 0)^\top \in \mathbb{R}^n$. Due to the arbitrariness of $\omega \in \mathbb{V}_n$, which is equivalent to the arbitrariness of γ, System (2.5) becomes the following:

$$\begin{cases} \rho \ddot{Z}_n(t) + \mu U_n Z_n(t) + b V_n Y_n(t) \\ = -\left(\alpha_1 g_1((\Phi_n(L))^\top \dot{Z}_n(t)) + \beta_1 E_n Z_n(t)\right) \Phi_n(L), \\ J \ddot{Y}_n(t) + \delta U_n Y_n(t) + b K_n Z_n(t) + \xi P_n Y_n(t) \\ = -\left(\alpha_2 g_2((\Phi_n(L))^\top \dot{Y}_n(t)) + \beta_2 E_n Y_n(t)\right) \Phi_n(L), \\ Z_n(0) = f_n,\ \dot{Z}_n(0) = g_n,\ Y_n(0) = h_n,\ \dot{Y}_n(0) = m_n. \end{cases} \tag{2.8}$$

System (2.8) can be reduced to the following first-order autonomous system in $\mathbb{R}^{4n}$:

$$\begin{cases} \dot{\mathcal{X}}_n = \mathcal{P}_n(\mathcal{X}_n), \\ \mathcal{X}_n(0) = \mathcal{X}_0. \end{cases} \tag{2.9}$$

Here,

$$\mathcal{X}_n(t) = \begin{bmatrix} Z_n \\ \dot{Z}_n \\ Y_n \\ \dot{Y}_n \end{bmatrix}, \quad \mathcal{X}_0 = \begin{bmatrix} f_n \\ g_n \\ h_n \\ m_n \end{bmatrix}, \quad \mathcal{P}_n \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} q \\ F_n(p,q) \\ s \\ G_n(r,s) \end{bmatrix}$$

where the map $F_n,\ G_n :\ \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is given by

$$F_n(p,q) = -\left(\alpha_1 g_1((\Phi_n(L))^\top q) + \beta_1 E_n p\right) \Phi_n(L)$$

and

$$G_n(r,s) = -\left(\alpha_2 g_2((\Phi_n(L))^\top s) + \beta_2 E_n r\right)\Phi_n(L).$$

Using the fact that $g_i,\ i=1,2$, are continuous, Peano's or Caratheodory's theorem guarantees that there exists a local solution to the System (2.9). The extension of the solution of System (2.9) and the original System (2.5) to the whole interval $[0,\infty)$ is a consequence of the following three priori estimates.

We begin the first estimate with the following remark.

Remark 2.3. To get the first estimate, we will use an unusual inequality of the form $ab \geq \frac{a^2}{2} - \frac{b^2}{2}$ that is valid for real numbers a and b that satisfy $a+b \leq -\sqrt{2}a$ or $a+b \geq \sqrt{2}a$.

Taking $\omega = u_t^n$ and $\psi = \phi_t^n$ in the first and second equations of (2.5), respectively, and summing the results obtained, we arrive at the following:

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_n(t) &= \frac{\rho}{2}\frac{d}{dt}\int_0^L |u_t^n|^2\,dx + \frac{J}{2}\frac{d}{dt}\int_0^L |\phi_t^n|^2\,dx + \frac{1}{2}\frac{d}{dt}\left(\mu - \frac{b^2}{\xi}\right)\int_0^L |u_x^n|^2\,dx \\
&+ \frac{1}{2}\frac{d}{dt}\int_0^L \left|\frac{b}{\sqrt{\xi}}u_x^n + \sqrt{\xi}\phi^n\right|^2 dx + \frac{\delta}{2}\frac{d}{dt}\int_0^L |\phi_x^n|^2\,dx \\
&= -\alpha_1 u_t^n(L,t) g_1(u_t^n(L,t)) - \beta_1 u_x^n(L,t) u_t^n(L,t) \\
&- \alpha_2 \phi_t^n(L,t) g_2(\phi_t^n(L,t)) - \beta_2 \phi_x^n(L,t)\phi_t^n(L,t).
\end{aligned} \tag{2.10}$$

From Remark 2.3, we have the following

$$\begin{aligned}
u_x^n(L,t)u_t^n(L,t) &\geq \frac{1}{2}[u_x^n(L,t)]^2 - \frac{1}{2}[u_t^n(L,t)]^2, \\
\phi_x^n(L,t)\phi_t^n(L,t) &\geq \frac{1}{2}[\phi_x^n(L,t)]^2 - \frac{1}{2}[\phi_t^n(L,t)]^2.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
-\beta_1 u_x^n(L,t)u_t^n(L,t) &\leq -\frac{\beta_1}{2}[u_x^n(L,t)]^2 + \frac{\beta_1}{2}[u_t^n(L,t)]^2, \\
-\beta_2 \phi_x^n(L,t)\phi_t^n(L,t) &\leq -\frac{\beta_2}{2}[\phi_x^n(L,t)]^2 + \frac{\beta_2}{2}[\phi_t^n(L,t)]^2.
\end{aligned} \tag{2.11}$$

By combining (2.10) with (2.11) and using the Assumptions 2.1 (III), we get the following

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}^n(t) &= -\alpha_1 u_t^n(L,t) g_1(u_t^n(L,t)) - \beta_1 u_x^n(L,t)u_t^n(L,t) \\
&- \alpha_2 \phi_t^n(L,t) g_2(\phi_t^n(L,t)) - \beta_2 \phi_x^n(L,t)\phi_t^n(L,t) \\
&\leq -(\alpha_1 M_1 - \frac{\beta_1}{2})|u_t^n(L,t)|^2 - (\alpha_2 M_2 - \frac{\beta_2}{2})|\phi_t^n(L,t)|^2 \\
&- \frac{\beta_1}{2}[u_x^n(L,t)]^2 - \frac{\beta_2}{2}[\phi_x^n(L,t)]^2 \leq 0
\end{aligned} \tag{2.12}$$

which means that $\mathcal{E}_n(t) \leq \mathcal{E}_n(0)$ for any $t>0$ and for $\alpha_1 > \frac{\beta_1}{2M_1},\ \alpha_2 > \frac{\beta_2}{2M_2}$. Notice that for any $n>2$, we have that $u_0,\ u_1,\ \phi_0,\ \phi_1 \in \mathbb{V}_n$, and $\mathcal{E}^n(0) = \mathcal{E}(0)$ from which it follows that for any $t>0$, we conclude that

$$\sup_{n>2} \mathcal{E}_n(t) \leq \mathcal{E}(0). \tag{2.13}$$

In order to start the second estimate, we must estimate $||u_{tt}^n(0)||$ and $||\phi_{tt}^n(0)||$. Invoking the variational structure regarding (1.16) and the compatibility condition, by setting $t = 0$ in (2.5), we have the following for $\omega,\ \psi \in H_*^1(0, L)$:

$$\begin{cases} \rho \int_0^L u_{tt}^n(x,0)\omega\, dx - \mu \int_0^L u_{0xx}^n(x)\omega\, dx - b \int_0^L \phi_{0x}(x)\omega\, dx = 0 \\ J \int_0^L \phi_{tt}^n(x,0)\psi\, dx - \delta \int_0^L \phi_{0xx}^n(x)\psi\, dx + b \int_0^L u_{0x}^n(x)\psi + \xi \int_0^L \phi_0(x)\psi\, dx = 0. \end{cases} \tag{2.14}$$

Taking $\omega = u_{tt}^n(x,0),\ \psi = \psi_{tt}^n(x,0)$ in (2.14), together with the initial condition, gives the following:

$$\begin{cases} \rho||u_{tt}^n(x,0)||^2 \le \mu||u_{0xx}(x)||||u_{tt}^n(x,0)|| + b||\phi_{0x}(x)||||u_{tt}^n(x,0)||, \\ J||\phi_{tt}^n(x,0)||^2 \le a_{12}||u_{0xx}(x)||||\phi_{tt}^n(x,0)|| + a_{22}||w_{0xx}(x)||||\phi_{tt}^n(x,0)||. \end{cases}$$

Thus, it follows that there exists a positive constant C such that

$$\begin{cases} ||u_{tt}^n(x,0)|| \le C, \\ ||\phi_{tt}^n(x,0)|| \le C. \end{cases} \tag{2.15}$$

Differentiating the first and second equations of (2.5) and setting $\omega = u_{tt}^n$ and $\psi = \phi_{tt}^n$, leads to the following:

$$\begin{cases} \frac{\rho}{2}\frac{d}{dt}\int_0^L |u_{tt}^n|^2\, dx + \frac{\mu}{2}\frac{d}{dt}\int_0^L |u_{xt}^n|^2\, dx + b\int_0^L \phi_t^n u_{xtt}^n\, dx \\ = -\alpha_1 \frac{d}{dt} g_1(u_t^n(L,t))|u_{tt}^n(L,t)|^2 - \beta_1 u_{xt}^n(L,t)u_{tt}^n(L,t), \\ \frac{J}{2}\frac{d}{dt}\int_0^L |\phi_{tt}^n|^2\, dx + \frac{\delta}{2}\frac{d}{dt}\int_0^L |\phi_{xt}^n|^2\, dx + b\int_0^L u_{xt}^n\phi_{tt}^n\, dx + \frac{\xi}{2}\frac{d}{dt}\int_0^L |\phi_t|^2\, dx \\ = -\alpha_2 \frac{d}{dt} g_2(\phi_t^n(L,t))|\phi_{tt}^n(L,t)|^2 - \beta_2 \phi_{xt}^n(L,t)\phi_{tt}^n(L,t). \end{cases} \tag{2.16}$$

Again, from Remark 2.3, we have the following

$$\begin{aligned} u_{xt}^n(L,t)u_{tt}^n(L,t) &\ge \frac{1}{2}[u_{xt}^n(L,t)]^2 - \frac{1}{2}[u_{tt}^n(L,t)]^2, \\ \phi_{xt}^n(L,t)\phi_{tt}^n(L,t) &\ge \frac{1}{2}[\phi_{xt}^n(L,t)]^2 - \frac{1}{2}[\phi_{tt}^n(L,t)]^2. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} -\beta_1 u_{xt}^n(L,t)u_{tt}^n(L,t) &\le -\frac{\beta_1}{2}[u_{xt}^n(L,t)]^2 + \frac{\beta_1}{2}[u_{tt}^n(L,t)]^2, \\ -\beta_2 \phi_{xt}^n(L,t)\phi_{tt}^n(L,t) &\le -\frac{\beta_2}{2}[\phi_{xt}^n(L,t)]^2 + \frac{\beta_2}{2}[\phi_{tt}^n(L,t)]^2. \end{aligned} \tag{2.17}$$

Adding the two above equations and using the Assumptions 2.1 and (2.17), we get the following:

$$\begin{aligned} &\frac{\rho}{2}\frac{d}{dt}\int_0^L |u_{tt}^n|^2\, dx + \frac{J}{2}\frac{d}{dt}\int_0^L |\phi_{tt}^n|^2\, dx + \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right)\frac{d}{dt}\int_0^L |u_{xt}^n|^2\, dx \\ &+ \frac{1}{2}\frac{d}{dt}\int_0^L \left|\frac{b}{\sqrt{\xi}}u_{xt}^n + \sqrt{\xi}\phi_t^n\right|^2 dx + \frac{\delta}{2}\frac{d}{dt}\int_0^L |\phi_{xt}|^2\, dx \\ &\le -(\alpha_1 M_1 - \frac{\beta_1}{2})|u_{tt}^n(L,t)|^2 - (\alpha_2 M_2 - \frac{\beta_2}{2})|\phi_{tt}^n(L,t)|^2 \\ &- \frac{\beta_1}{2}[u_{xt}^n(L,t)]^2 - \frac{\beta_2}{2}[\phi_{xt}^n(L,t)]^2 \le 0 \end{aligned} \tag{2.18}$$

for $\alpha_1 > \frac{\beta_1}{2M_1}$ and $\alpha_2 > \frac{\beta_2}{2M_2}$. Then, it follows then that

$$\begin{aligned} &\rho\int_0^L |u_{tt}^n|^2\,dx + J\int_0^L |\phi_{tt}^n|^2\,dx + \left(\mu - \frac{b^2}{\xi}\right)\int_0^L |u_{xt}^n|^2\,dx \\ &+ \int_0^L \left|\frac{b}{\sqrt{\xi}}u_{xt}^n + \sqrt{\xi}\phi_t^n\right|^2\,dx + \delta\int_0^L |\phi_{xt}^n|^2\,dx \\ &\leq \rho\int_0^L |u_{tt}^n(x,0)|^2\,dx + J\int_0^L |\phi_{tt}^n(x,0)|^2\,dx + \left(\mu - \frac{b^2}{\xi}\right)\int_0^L |u_{1x}|^2\,dx \\ &+ \int_0^L \left|\frac{b}{\sqrt{\xi}}u_{1x} + \sqrt{\xi}\phi_1\right|^2\,dx + \delta\int_0^L |\phi_{1x}|^2\,dx. \end{aligned} \tag{2.19}$$

Estimates (2.15) and taking in mind that the initial data are in $\mathcal{V}$, we conclude that there exists a positive constant C such that

$$\begin{aligned} &\rho\int_0^L |u_{tt}^n|^2\,dx + J\int_0^L |\phi_{tt}^n|^2\,dx + \left(\mu - \frac{b^2}{\xi}\right)\int_0^L |u_{xt}^n|^2\,dx \\ &+ \int_0^L \left|\frac{b}{\sqrt{\xi}}u_{xt}^n + \sqrt{\xi}\phi_t^n\right|^2\,dx + \delta\int_0^L |\phi_{xt}|^2\,dx \leq C, \quad \forall n > 2 \end{aligned} \tag{2.20}$$

for almost all $t \in [0,T)$. To conclude, let $u^n(\cdot,t),\ \phi^n(\cdot,t)$ be the corresponding maximal solution Problem (2.5), defined on $0 < T_{\max}^n \leq \infty$, which is equivalent to $(Z_n(t), Y_n(t))$, $t \in [0, T_{\max}^n)$ being the maximal solution to Problem (2.5). Using Estimates (2.13) and (2.20), we conclude that $T_{\max}^n = \infty$.

In order to start the third estimate, from (2.5) with $\omega = -u_{xx}^n$ and $\psi = -\phi_{xx}^n$, we get the following:

$$\begin{cases} -\rho\int_0^L u_{tt}^n u_{xx}^n\,dx + \mu\int_0^L |u_{xx}^n|^2\,dx + b\int_0^L \phi_x^n u_{xx}^n\,dx = 0, \\ -J\int_0^L \phi_{tt}^n\phi_{xx}^n\,dx + \delta\int_0^L |\phi_{xx}^n|^2\,dx - b\int_0^L u_x^n\phi_{xx}^n\,dx - \xi\int_0^L \phi^n\phi_{xx}^n\,dx = 0. \end{cases}$$

Applying the Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} \mu\|u_{xx}^n\|^2 + \delta\|\phi_{xx}^n\|^2 \leq{}& \rho\|u_{tt}^n\|\|u_{xx}^n\| + b\|\phi_x^n\|\|u_{xx}^n\| + J\|\phi_{tt}^n\|\|\phi_{xx}^n\| \\ &+ b\|u_x^n\|\|\phi_{xx}^n\| + \xi\|\phi^n\|\|\phi_{xx}^n\|. \end{aligned}$$

Taking Estimates (2.13), (2.20) into account and conveniently applying Hölder's inequality, we conclude that there exists a positive constant C such that

$$\begin{cases} \|u_{xx}^n\|^2 \leq C, \\ \|\phi_{xx}^n\|^2 \leq C. \end{cases} \tag{2.21}$$

From (2.13), (2.20), and (2.21), we have the following:

$$\begin{cases} \{u^n\},\ \{\phi^n\} \text{ are bounded in } L^\infty([0,T);\ H_*^1(0,L)), \\ \{u_t^n\},\ \{\phi_t^n\} \text{ are bounded in } L^\infty([0,T);\ L^2(0,L)), \\ \{u_t^n\},\ \{\phi_t^n\} \quad \text{are bounded in} \quad L^\infty([0,T);\ H_*^1(0,L)), \\ \{u_{xx}^n\},\ \{\phi_{xx}^n\} \quad \text{are bounded in} \quad L^\infty([0,T);\ L^2(0,L)), \\ \{u_{tt}^n\},\ \{\phi_{tt}^n\} \quad \text{are bounded in} \quad L^\infty([0,T);\ L^2(0,L)). \end{cases} \tag{2.22}$$

Consequently, from (2.22) and by the Banach-Alaoglu-Bourbaki Theorem (see [25]), subsequences can be extracted from $\{u^n\}$ and $\{\phi^n\}$, respectively, still denoted by $\{u^n\}$ and $\{\phi^n\}$, and functions $u,\ \phi \in L^\infty([0,T);\ H^1_*(0,L))$, $h_1,\ h_2 \in L^\infty([0,T);\ H^1_*(0,L))$, and $f_1,\ f_2 \in L^\infty([0,T);\ L^2(0,L))$ can be found such that

$$\begin{cases} u^n \to u, \quad \phi^n \to \phi \text{ in } L^\infty([0,T);\ H^1_*(0,L)) \text{ weak}^*,\\ u^n_{xx} \to \theta, \quad \phi^n_{xx} \to \Psi \text{ in } L^\infty([0,T);\ L^2(0,L)) \text{ weak}^*,\\ u^n_t \to h_1, \quad \phi^n_t \to h_2 \text{ in } L^\infty([0,T);\ H^1_*(0,L)) \text{ weak}^*,\\ u^n_{tt} \to f_1, \quad \phi^n_{tt} \to f_2 \text{ in } L^\infty([0,T);\ L^2(0,L)) \text{ weak}^*. \end{cases} \tag{2.23}$$

Thanks to Lemma 4 of [26], we conclude that $\theta = u_{xx},\ \Psi = \phi_{xx},\ h_1 = u_t,\ h_2 = \phi_t$ and $f_1 = u_{tt},\ f_2 = \phi_{tt}$. Since $u,\ \phi \in L^\infty(0,T;H^1_*(0,L))$ and $u_{xx},\ \phi_{xx} \in L^\infty(0,T;L^2(0,L))$ we infer that $u,\ \phi \in L^\infty(0,T;H^2(0,L))$.

From (2.20), we see that $\{u^n_{xt}\}_{n=1}^\infty,\ \{\phi^n_{xt}(\cdot,t)\}_{n=1}^\infty$ are bounded in $L^2(0,L)$. Since $H^1(0,L) \hookrightarrow L^2(0,L) \hookrightarrow H^{-1}(0,L)$ with compact embedding, by the Rellich-Kondrachoff theorem, it follows that $\{u^n_t(L,t)\}$ and $\{\phi^n_t(L,t)\}$ are compact in $L^2(0,L)$.

Now, using the fact that $u^n_t(L,t) = \int_0^L u^n_{xt}(x,t)\,dx$, $\phi^n_t(L,t) = \int_0^L \phi^n_{xt}(x,t)\,dx$, and combining with the above, guarantees that there exist subsequences of $\{u^n_t(L,t)\}$ and $\{\phi^n_t(L,t)\}$, respectively, still denoted by itself, such that

$$u^n_t(L,t) \to u_t(L,t),\ \phi^n_t(L,t) \to \phi_t(L,t) \text{ for almost all } t \in [0,T).$$

Applying the Lebesgue dominated convergence theorem and the continuity of $g_i(\cdot),\ i = 1,2$, we have the following:

$$\begin{cases} u^n_t(L,t) \to u_t(L,t),\ \phi^n_t(L,t) \to \phi_t(L,t),\\ g_1(u^n_t(L,t)) \to g_1(u_t(L,t)),\ g_2(\phi^n_t(L,t)) \to g_2(\phi_t(L,t)) \text{ in } L^\infty(0,T). \end{cases} \tag{2.24}$$

Using the same argument as above, we conclude that

$$u^n_x(L,t) \to u_x(L,t), \quad \phi^n_x(L,t) \to \phi_x(L,t), \text{ for almost all } t \in [0,T). \tag{2.25}$$

Passing to the limit as $n \to \infty$ for (2.5) and taking (2.23), (2.24), and (2.25) into account, we get the following

$$\begin{cases} \rho \int_0^L u_{tt}\omega\,dx + \mu \int_0^L u_x\omega_x\,dx + b\int_0^L \phi\omega_x\,dx\\ = -\alpha_1 g_1(u_t(L,t))\omega(L) - \beta_1 u_x(L,t)\omega(L),\ \forall \omega \in H^1_*(0,L),\\ J\int_0^L \phi_{tt}\psi\,dx + \delta\int_0^L \phi_x\psi_x\,dx + b\int_0^L u_x\psi\,dx + \xi\int_0^L \phi\psi\,dx\\ = -\alpha_2 g_2(\phi_t(L,t))\psi(L) - \beta_2\phi_x(L,t)\psi(L),\ \forall\psi \in H^1_*(0,L). \end{cases} \tag{2.26}$$

In the particular case when $\omega,\ \psi \in H^1_0(0,L)$, we have

$$\begin{cases} \rho\int_0^L u_{tt}\omega\,dx + \mu\int_0^L u_x\omega_x\,dx + b\int_0^L \phi\omega_x\,dx = 0,\\ J\int_0^L \phi_{tt}\psi\,dx + \delta\int_0^L \phi_x\psi_x\,dx + b\int_0^L u_x\psi\,dx + \xi\int_0^L \phi\psi\,dx = 0, \end{cases} \tag{2.27}$$

from where it follows that

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \quad \text{in} \quad L^2(0,L), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0, \quad \text{in} \quad L^2(0,L). \end{cases} \tag{2.28}$$

Integration by parts of (2.27), gives us the following:

$$\begin{cases} \int_0^L (\rho u_{tt} - \mu u_{xx} - b\phi_x)\omega \, dx \\ +(\mu u_x(L,t) + b\phi(L,t))\omega(L) = -\alpha_1 g_1(u_t(L,t)\omega(L) - \beta_1 u_x(L,t)\omega(L), \ \forall \omega \in H_*^1(0,L), \\ \int_0^L (J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi)\psi \, dx \\ +(\delta\phi_x(L,t))\psi(L) = -\alpha_2 g_2(\phi_t(L,t)\psi(L) - \beta_2\phi_x(L,t)\psi(L), \ \forall \psi \in H_*^1(0,L) \end{cases} \tag{2.29}$$

for almost all $t \in [0,t)$. This, together with (2.28), gives the following:

$$\begin{cases} \mu u_x(L,t) + b\phi(L,t) = -\alpha_1 g_1(u_t(L,t)) - \beta_1 u_x(L,t), \\ \delta\phi_x(L,t) = -\alpha_2 g_2(\phi_t(L,t)) - \beta_2\phi_x(L,t). \end{cases} \tag{2.30}$$

This concludes the existence of global solutions to System (1.16) in $[0,T)$, for all $T > 0$. □

To conclude this section, the following remark is necessary.

Remark 2.4. The compatibility conditions in Theorem 2.1, whici are necessary to guarantee the existence and regularity of solutions, cause significant restrictions on the physical or mathematical model. These conditions arise from the need for the initial and boundary data to be consistent with each other at the beginning of the evolution process, thereby avoiding discontinuities or contradictions at the domain boundary.

On the other hand, if the model does not respect these conditions, then singularities may arise at the edges (boundary) of the space-time domain. This affects the modeling, thus forcing the adjustment of the initial data or the use of weak solutions, which do not require this rigorous conformity but lose regularity.

Since the method we are using in this paper requires regular solutions to obtain uniform stability, the use of the compatibility conditions in Theorem 2.1 is justifiable.

In summary, compatibility conditions require that the physical model be “well-behaved” at the initial instant, thus limiting how “arbitrary” the initial and boundary data in the model can be.

3. Uniform stability

The main purpose of this section is to determine the uniform stability to System (1.16). To do this, we will use the method introduced in Lasiecka and Tataru [20].

First, let us consider a function $h : \mathbb{R} \to \mathbb{R}$ defined by the following

$$h(s) = h_1(s) + h_2(s), \ s \in \mathbb{R}, \tag{3.1}$$

where $h_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are both continuous, concaves, strictly increasing, and satisfy

$$h_i(g_i(s)s) \geq s^2 + [g_i(s)]^2, \ |s| \leq 1, \tag{3.2}$$

with $h_i(0) = 0,\ i = 1, 2$. Note that h has the same properties of its composing functions. With this in mind, we also define the auxiliar function r by the following:

$$r(s) = h\left(\frac{s}{T}\right),\ s \in \mathbb{R} \tag{3.3}$$

where T is a constant to be determined later.

Since r is a monotone increasing function, the function $\delta I + r$ is invertible for any constant $\delta \geq 0$, where I is a identity function. In this way, for nonnegative constants δ and β, we define the following:

$$p(s) = (\delta I + r)^{-1}(\beta s),\ s \in \mathbb{R}. \tag{3.4}$$

Note that p is a continuous, positive, and strictly increasing function. Additionally, $p(0) = 0$. Finally, we define the following:

$$q(s) = s - (I + p)^{-1}(s),\ s \in \mathbb{R}. \tag{3.5}$$

In this setting, the following technical lemma due to Lasiecka and Tataru (see [18]) will be admitted.

Lemma 3.1. *Let the functions p, q be defined as above. For any sequence (s_n) of positive numbers that satisfies*

$$s_{m+1} + p(s_{m+1}) \leq s_m,$$

we have $s_m \leq S(m)$ for every $m \in \mathbb{N}$, where $S(t)$ is a solution of the differential equation

$$\begin{cases} \dfrac{dS}{dt}(t) + q(S(t)) = 0,\ t > 0 \\ S(0) = s_0. \end{cases} \tag{3.6}$$

Moreover, if p satisfies $p(s) > 0$ for $s > 0$, then

$$\lim_{t\to\infty} S(t) = 0. \tag{3.7}$$

The main result of this paper is given by the following Theorem.

Theorem 3.2. *Let (u, ϕ) and $\mathcal{E}(t)$ be the solution and the energy of System (1.16), respectively. Assume that* **Assumptions 2.1** *on $g_i,\ i = 1, 2$, is satisfied. Then, there exists a positive constant $T_0 > 0$ such that*

$$\mathcal{E}(t) \leq S\left(\frac{t}{T_0} - 1\right), \quad t > T_0 \tag{3.8}$$

where S is the solution of the ordinary differential equation (ODE) system (3.6) that satisfies (3.7) with the initial data $s_0 = \mathcal{E}(0)$.

Proof. Multiplying Equations. $(1.16)_1$ and $(1.16)_2$ by u_t and ϕ_t, respectively, integrating by parts, and using the boundary conditions, one has that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= -\alpha_1 g_1(u_t(L,t))u_t(L,t) - \beta_1 u_x(L,t)u_t(L,t) \\ &- \alpha_2 g_2(\phi_t(L,t))\phi_t(L,t) - \beta_2 \phi_x(L,t)\phi_t(L,t). \end{aligned}$$

From Remark 2.3, we have the following:

$$u_x(L,t)u_t(L,t) \geq \frac{1}{2}[u_x(L,t)]^2 - \frac{1}{2}[u_t(L,t)]^2,$$
$$\phi_x(L,t)\phi_t(L,t) \geq \frac{1}{2}[\phi_x(L,t)]^2 - \frac{1}{2}[\phi_t(L,t)]^2.$$

Thus, it follows that

$$\begin{aligned} -\beta_1 u_x(L,t)u_t(L,t) &\leq -\frac{\beta_1}{2}[u_x(L,t)]^2 + \frac{\beta_1}{2}[u_t(L,t)]^2, \\ -\beta_2 \phi_x(L,t)\phi_t(L,t) &\leq -\frac{\beta_2}{2}[\phi_x(L,t)]^2 + \frac{\beta_2}{2}[\phi_t(L,t)]^2. \end{aligned} \tag{3.9}$$

From (3.9) and using Assumptions 2.1 (III), we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) \;&\leq\; -\left(\alpha_1 M_1 - \frac{\beta_1}{2}\right)|u_t(L,t)|^2 - \left(\alpha_2 M_2 - \frac{\beta_2}{2}\right)|\phi_t(L,t)|^2 \\ &-\; \frac{\beta_1}{2}|u_x(L,t)|^2 - \frac{\beta_2}{2}|\phi_x(L,t)|^2 \end{aligned}$$

for $\alpha_1 > \frac{\beta_1}{2M_1}$ and $\alpha_2 > \frac{\beta_2}{2M_2}$. Again, using the Assumptions 2.1 (III) and choosing

$$\alpha_1 > \frac{K_1^2 + \beta_1}{2M_1} > \frac{\beta_1}{2M_1} > 1, \quad \alpha_2 > \frac{K_2^2 + \beta_2}{2M_2} > \frac{\beta_2}{2M_2} > 1$$

we infer that

$$\mathcal{E}(t_2) + \int_{t_1}^{t_2} u_t(L,t)g_1(u_t(L,t))\,dt + \int_{t_1}^{t_2} \phi_t(L,t)g_2(\phi_t(L,t))\,dt \leq \mathcal{E}(t_1) \tag{3.10}$$

with $0 < t_1 < t_2 < T$.

Multiplying Equations $(1.16)_1$ and $(1.16)_2$ by xu_x and $x\phi_x$, respectively, integrating by parts over $[0, L]$, and summing the result, we have the following:

$$\begin{aligned} &\frac{\rho}{2}\int_0^L |u_t|^2\,dx + \frac{J}{2}\int_0^L |\phi_t|^2\,dx + \frac{\mu}{2}\int_0^L |u_x|^2\,dx\,dx + \frac{\delta}{2}\int_0^L |\phi_x|^2\,dx \\ &= \frac{L\rho}{2}|u_t(L,t)|^2 + \frac{L\mu}{2}|u_x(L,t)|^2 - \rho\int_0^L [u_t x u_x]_t\,dx - J\int_0^L [\phi_t x\phi_x]_t\,dx \\ &+ \frac{LJ}{2}|\phi_t(L,t)|^2 + \frac{L\delta}{2}|\phi_x(L,t)|^2 - \frac{L\xi}{2}|\phi(L,t)|^2 + \frac{\xi}{2}\int_0^L |\phi|^2\,dx. \end{aligned} \tag{3.11}$$

Adding and subtracting the terms $\frac{Lb}{2}\phi(L,t)u_x(L,t)$, $b\int_0^L u_x\phi\,dx$, and $\frac{\xi}{2}\int_0^L |\phi|^2\,dx$ in the equality above, we arrive the following

$$\begin{aligned} \mathcal{E}(t) \;&=\; \frac{L\rho}{2}|u_t(L,t)|^2 + \frac{L}{2}(\mu u_x(L,t) + b\phi(L,t))u_x(L,t) - \frac{Lb}{2}\phi(L,t)u_x(L,t) \\ &+\; \frac{LJ}{2}|\phi_t(L,t)|^2 + \frac{L\delta}{2}|\phi_x(L,t)|^2 - \frac{L\xi}{2}|\phi(L,t)|^2 + \xi\int_0^L |\phi|^2\,dx \end{aligned}$$

$$- \quad \rho \int_0^L [u_t x u_x]_t \, dx - J \int_0^L [\phi_t x \phi_x]_t \, dx + b \int_0^L u_x \phi \, dx. \tag{3.12}$$

Taking the boundary conditions into account, one has that

$$\begin{aligned} \mathcal{E}(t) &= \frac{L\rho}{2}|u_t(L,t)|^2 - \frac{L}{2}\alpha_1 g_1(u_t(L,t))u_x(L,t) - \frac{L\beta_1}{2}|u_x(L,t)|^2 \\ &+ \frac{LJ}{2}|\phi_t(L,t)|^2 - \frac{L\alpha_2}{2}g_2(\phi_t(L,t))\phi_x(L,t) - \frac{L\beta_2}{2}|\phi_x(L,t)|^2 \\ &- \frac{Lb}{2}\phi(L,t)u_x(L,t) - \frac{L\xi}{2}|\phi(L,t)|^2 + \xi \int_0^L |\phi|^2 \, dx \\ &- \int_0^L [\rho u_t x u_x + J\phi_t x \phi_x]_t \, dx + b \int_0^L u_x \phi \, dx. \end{aligned} \tag{3.13}$$

Let us analyze some terms of (3.13). To start from Equation (3.13), we have

$$\begin{aligned} &\left| -\frac{L}{2}\alpha_1 g_1(u_t(L,t))u_x(L,t) - \frac{Lb}{2}\phi(L,t)u_x(L,t) \right| \\ &\le \frac{L\alpha_1^2}{4\beta_1}g_1^2(u_t(L,t)) + \frac{L\beta_1}{2}|u_x(L,t)|^2 + \frac{Lb^2}{4\beta_1}|\phi(L,t)|^2 \end{aligned} \tag{3.14}$$

and

$$\left| -\frac{L\alpha_2}{2}g_2(\phi_t(L,t))\phi_x(L,t) \right| \le \frac{L\alpha_2^2}{8\beta_2}g_2^2(\phi_t(L,t)) + \frac{L\beta_2}{2}|\phi_x(L,t)|^2. \tag{3.15}$$

Substituting inequalities (3.14) and (3.15) into (3.13), gives the following

$$\begin{aligned} \mathcal{E}(t) &\le \frac{L\rho}{2}|u_t(L,t)|^2 + \frac{L\alpha_1^2}{4\beta_1}g_1^2(u_t(L,t)) \\ &+ \frac{LJ}{2}|\phi_t(L,t)|^2 + \frac{L\alpha_2^2}{8\beta_2}g_2^2(\phi_t(L,t)) \\ &+ \frac{Lb^2}{4\beta_1}|\phi(L,t)|^2 + \xi \int_0^L |\phi|^2 \, dx \\ &- \int_0^L [\rho u_t x u_x + J\phi_t x \phi_x]_t \, dx + b \int_0^L u_x \phi \, dx. \end{aligned} \tag{3.16}$$

From (3.16) and (3.10), for $T_0 > 0$, we get the following:

$$\begin{aligned} &\left| \int_0^{T_0} \int_0^L [\rho u_t x u_x + J\phi_t x \phi_x]_t \, dx \, dt \right| \\ &\le \left| \int_0^L [\rho u_t(x,T_0) x u_x(x,T_0) + J\phi_t(x,T_0) x \phi_x(x,T_0)] \, dx \right| \\ &+ \left| \int_0^L [\rho u_t(x,0) x u_x(x,0) + J\phi_t(x,0) x \phi_x(x,0)] \, dx \right| \\ &\le C_1(\mathcal{E}(T_0) + \mathcal{E}(0)) \le 2C_1\mathcal{E}(0), \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&\frac{Lb^2}{4\beta_1}|\phi(L,t)|^2 \le \frac{L^2b^2}{4\beta_1}\int_0^L |\phi_x|^2\,dx \le \frac{L^2b^2}{2\beta_1\delta}\frac{\delta}{2}\int_0^L |\phi_x|^2\,dx \\
&\implies \frac{Lb^2}{4\beta_1}\int_0^{T_0} |\phi(L,t)|^2\,dt \le C_1\mathcal{E}(0) \qquad (3.18)
\end{aligned}
$$

and

$$
\xi\int_0^{T_0}\int_0^L |\phi(x,t)|^2\,dx\,dt + b\int_0^{T_0}\int_0^L u_x(x,t)\phi(x,t)\,dx\,dt \le 3C_1T_0\mathcal{E}(0) \qquad (3.19)
$$

where $C_1 = \max\left\{L\rho, LJ, \frac{L}{\left(\mu-\frac{b^2}{\xi}\right)}, \frac{L}{\delta}, \frac{L^2b^2T_0}{2\beta_1\delta}, \frac{2L\xi}{\delta}, \frac{b}{\left(\mu-\frac{b^2}{\xi}\right)}, \frac{bL}{\delta}\right\}$. Now, simultaneously integrating both sides of (3.16) from 0 to T, $(0 < T)$, and substituting inequalities (3.17), (3.18), and (3.19), leads to the following

$$
\begin{aligned}
\int_0^T \mathcal{E}(t)\,dt &\le C_2\mathcal{E}(0) + C_3\int_0^T (|u_t(L,t)|^2 + g_1^2(u_t(L,t)))\,dt \\
&+ C_3\int_0^T (|\phi_t(L,t)|^2 + g_2^2(\phi_t(L,t)))\,dt \qquad (3.20)
\end{aligned}
$$

where $C_2 = \max\{3C_1, 3C_1T_0\}$, and $C_3 = \max\left\{\frac{L\rho}{2}, \frac{LJ}{2}, \frac{L\alpha_1^2}{4\beta_1}, \frac{L\alpha_2^2}{8\beta_2}\right\}$. Due to (3.10), it follows from (3.20) that for $T > T_1 := \max\{T_0, C_2\}$,

$$
\mathcal{E}(T) \le C_T\int_0^T (|u_t(L,t)|^2 + g_1^2(u_t(L,t)) + |\phi_t(L,t)|^2 + g_2^2(\phi_t(L,t)))\,dt \qquad (3.21)
$$

where $C_T > 0$ is a constant that depends on T.

Let us define $\Sigma_u = \{t \in (0,T) : |u_t(L,t)| > N\}$, $\Sigma_\phi = \{t \in (0,T) : |\phi_t(L,t)| > N\}$, and $\Gamma_u = (0,T)\setminus\Sigma_N$, $\Gamma_\phi = (0,T)\setminus\Sigma_\phi$. Here, our strategy here is to estimate the integral at the right side of (3.21).

By Assumption 2.1 (III), it is inferred that

$$
\int_{\Sigma_u} (|u_t(L,t)|^2 + |g_1(u_t(L,t))|^2)\,dt \le \left(\frac{1+K_1^2}{M_1}\right)\int_{\Sigma_u} u_t(L,t)g_1(u_t(L,t))\,dt. \qquad (3.22)
$$

By (3.2) and recalling that h_1 is concave and increasing, we infer the following:

$$
\int_{\Gamma_u} (|u_t(L,t)|^2 + |g_1(u_t(L,t))|^2)\,dt \le \int_{\Gamma_u} h_1(g_1(u_t(L,t))u_t(L,t))\,dt. \qquad (3.23)
$$

Based on Jensens inequality, it holds that

$$
\begin{aligned}
\int_{\Gamma_u} h_1(g_1(u_t(L,t))u_t(L,t))\,dt &\le Th_1\left(\int_0^T \frac{u_t(L,t)g_1(u_t(L,t))}{T}\,dt\right) \\
&\le Tr_1\left(\int_0^T u_t(L,t)g_1(u_t(L,t))\,dt\right), \qquad (3.24)
\end{aligned}
$$

where $r_1 = h_1(\frac{s}{T})$.

Thus, it follows from (3.22) and (3.24) that

$$\begin{aligned}\int_0^T (|u_t(L,t)|^2 + |g_1(u_t(L,t))|^2)\,dt \;&\leq\; \left(\frac{1+K_1^2}{M_1}\right)\int_0^T u_t(L,t)g_1(u_t(L,t))\,dt \\ &+\; Tr_1\left(\int_0^T u_t(L,t)g_1(u_t(L,t))\,dt\right). \end{aligned} \tag{3.25}$$

By using the same arguments on h_2, we find that

$$\begin{aligned}\int_0^T (|\phi_t(L,t)|^2 + |g_2(\phi_t(L,t))|^2)\,dt \;&\leq\; \left(\frac{1+K_2^2}{M_2}\right)\int_0^T \phi_t(L,t)g_2(\phi_t(L,t))\,dt \\ &+\; Tr_2\left(\int_0^T \phi_t(L,t)g_2(\phi_t(L,t))\,dt\right), \end{aligned} \tag{3.26}$$

where $r_2 = h_2(\frac{s}{T})$.

Substituting (3.25) and (3.26) into (3.21) leads to the following:

$$\begin{aligned}&\frac{1}{C_T}\mathcal{E}(T) \leq \left(\frac{1+K_1^2}{M_1}\right)\int_0^T u_t(L,t)g_1(u_t(L,t))\,dt + Tr_1\left(\int_0^T u_t(L,t)g_1(u_t(L,t))\,dt\right) \\ &+\left(\frac{1+K_2^2}{M_2}\right)\int_0^T \phi_t(L,t)g_2(\phi_t(L,t))\,dt + Tr_2\left(\int_0^T \phi_t(L,t)g_2(\phi_t(L,t))\,dt\right). \end{aligned} \tag{3.27}$$

Therefore, we get the following

$$\beta\mathcal{E}(T) \leq (\delta I + r)\left(\int_0^T (u_t(L,t)g_1(u_t(L,t)) + \phi_t(L,t)g_2(\phi_t(L,t)))\,dt\right) \tag{3.28}$$

where $\delta = \frac{1}{C_T T}$, $\beta = \frac{M_1^{-1}(1+K_1^2)+M_2^{-1}(1+K_2^2)}{T}$, and $r(\cdot) = r_1(\cdot) + r_2(\cdot)$.

Then, by noting that r is an increasing function, $\delta I + r$ is invertible. By denoting

$$p(x) = (\delta I + r)^{-1}(\beta x),$$

we deduce from (3.28) that

$$p(\mathcal{E}(t)) \leq \int_0^T (u_t(L,t)g_1(u_t(L,t)) + \phi_t(L,t)g_2(w_t(L,t)))\,dt. \tag{3.29}$$

Then, from (3.10) and (3.29), we obtain the following

$$p(\mathcal{E}(T)) + \mathcal{E}(T) \leq \mathcal{E}(0). \tag{3.30}$$

Due to the dissipativity of $\mathcal{E}(t)$ and $S(t)$, for $t = mt + \theta$ and $\theta \in (0,T)$, the following is obtained:

$$\mathcal{E}(t) \leq \mathcal{E}(mt) \leq S(m) \leq S\left(\frac{t-\theta}{T}\right) \leq S\left(\frac{t}{T} - 1\right), \quad \text{for } t > T.$$

Hence, this proves **Theorem 3.2**. □

4. Examples of applications

In this Section, we assume that $M_i x^2 \leq g_i(x)x \leq K_i x^2,\ |x| \geq 1$.

Example 4.1. *Here, we assume that the dissipations $g_i(x),\ i = 1,2$ are linearly bounded at the origin and also assume that $m_0 x^2 \leq g_i(x)x \leq M_0 x^2$ for all $|x| < 1$. Then, $h(x) = 2(M_0 + m_0^{-1})x$ and the function q given in (3.5) can be explicitly calculated and the ODE (3.6) becomes the following:*

$$\begin{aligned} &\frac{dS}{dt} + \widehat{C}S = 0, \\ &S(0) = \mathcal{E}(0), \end{aligned} \tag{4.1}$$

where $\widehat{C} = \frac{K^{-1}med(Q_T)}{med(Q_T)(2K^{-1}) + M_0 + m_0^{-1}}$, $K = [C(T, med(Q_T))]$, and $Q_T = [0, L] \times [0, T]$. This gives us

$$S(t) = S(0)e^{-\widehat{C}t} \implies \mathcal{E}(t) \leq \mathcal{E}(0)e^{-\widehat{C}(\frac{t}{T_0}-1)},$$

for all $t \geq T_0$.

Example 4.2. *In this example, we assume that the dissipations are superlinear and polynomial near the origin. Let's assume that $m_0 x^{p+1} \leq g_i(x)x \leq M_0 x^{p+1},\ i = 1,2,$ for all $|x| < 1$ and for some $p > 1$. Since the function $x \to \sqrt{x}g_i(\sqrt{x}) = x^{\frac{p+1}{2}}$ is convex for $p \geq 1$, then the ODE (3.6) become the following*

$$\begin{aligned} &\frac{dS}{dt} + \widehat{C}S^{\frac{p+1}{2}} = 0, \\ &S(0) = \mathcal{E}(0), \end{aligned} \tag{4.2}$$

where $\widehat{C} = [K]^{\frac{p+1}{2}}$. This equation can be directly integrated; however, to illustrate the general formula, we obtain the following:

$$G(y, \mathcal{E}(0)) = \int_{\sqrt{\mathcal{E}(0)}}^{\sqrt{y}} z^{-p}\, dz = \frac{1}{p-1}[y^{\frac{-p+1}{2}} - [\mathcal{E}(0)]^{\frac{-p+1}{2}}],$$

with $S(t) = G^{-1}(t)$. Thus, it follows that $G^{-1}(t) = [[\mathcal{E}(0)]^{\frac{-p+1}{2}} + t(1-p)]^{\frac{2}{-p+1}}$. Therefore,

$$\mathcal{E}(t) \leq S\left(\frac{t}{T_0} - 1\right) = G^{-1}\left(\frac{t}{T_0} - 1\right) = \left[[\mathcal{E}(0)]^{\frac{-p+1}{2}} + \frac{(p-1)\widehat{C}}{2}\left(\frac{t}{T_0} - 1\right)\right]^{-\frac{2}{p-1}},$$

for all $t \geq T_0$.

Example 4.3. *If we now consider $g_i(s) = s^3 e^{\frac{-1}{s^2}},\ i = 1,2,$ for s at the origen, then since the function $s^2 e^{\frac{-1}{s}}$ is convex in a neighborhood of the origin. The decay of energy is determined by solving the following ODE*

$$\begin{aligned} &\frac{dS}{dt} + \sqrt{\widehat{C}S}\, g_i[(\widehat{C}S)^{\frac{1}{2}}] = 0, \\ &S(0) = \mathcal{E}(0), \end{aligned} \tag{4.3}$$

where $\widehat{C} = K^{-1}$. *In this case,* $G(y, \mathcal{E}(0) = -\frac{1}{2}\left[e^{\frac{1}{\widehat{C}y}} - e^{\frac{1}{\widehat{C}\mathcal{E}(0)}}\right]$ *and*
$G^{-1}(t, \mathcal{E}(0)) = \frac{1}{\widehat{C}}\left[\ln\left(e^{\frac{1}{\widehat{C}\mathcal{E}(0)}} - 2t\right)\right]^{-1}$. *Therefore,*

$$\mathcal{E}(t) \leq \frac{1}{\widehat{C}}\left[\ln\left(\widehat{C}\left(\frac{t}{T_0} - 1\right) + e^{\frac{1}{\widehat{C}\mathcal{E}(0)}}\right)\right]^{-1}, \quad \textit{for all } t \geq T_0$$

where C is a positive constant.

Example 4.4. *Take* $g_i(s) = s|s|e^{\frac{-1}{|s|}}$, $i = 1, 2$, *for s near the origin. Then, since the function* $s^{\frac{3}{2}}e^{-\frac{1}{\sqrt{s}}}$ *is convex in* $[0, s_0]$ *for some small* s_0, *then, using the same procedure as above, the energy satisfies the following:*

$$\mathcal{E}(t) \leq \frac{1}{\widehat{C}} \frac{1}{\ln^2\left(e^{\frac{1}{\sqrt{\widehat{C}\mathcal{E}(0)}}} + \frac{\widehat{C}}{2}\left(\frac{t}{T_0} - 1\right)\right)}, \quad \textit{for all } t \geq T_0.$$

4.1. Final comment

Due to their wide applicability, as can be seen in geomechanics, petroleum engineering, acoustics and vibration, bioengineering and the medical field, structural and materials engineering, and in environmental and transportation applications, porous elastic materials have attracted considerable attention (see [27]). The novelty of this paper, namely nonlinear damping at a boundary point combined with scheduled control gains at that point, is not limited to simply uniformly stabilizing the model. The method used to obtain such stabilization provides a way of how we should design these controls, that is, how to construct damping functions g_i for this purpose. Examples of such functions can be seen in Section 4.

It is important to mention that the application of nonlinear damping at specific points on the boundary, combined with scheduled control gains, is an advanced control strategy, frequently applied to distributed systems (partial differential equations(PDE)) and complex mechanical systems to improve the stabilization and energy efficiency. This technique is useful when the system exhibits a nonlinear behavior dependent on the vibration amplitude, dead zone, or when it operates under different load conditions that require adjustments to the control gain.

Some applications are as follows:

- Vibrations in Panels and Beams: the use of boundary actuators to stabilize nonlinear panels subjected to aerodynamic loads (flutter), where boundary damping is adjusted to handle the non-linear behavior of large deflections.
- Space Structures: the application of dampers (active damping) at the boundary of robotic arms or flexible antennas to ensure high precision and safety, thereby adjusting the gains as the structure configuration changes.
- Stabilization of beams with friction at the boundary, where control gains are scheduled as a function of vibration amplitude to mitigate impact nonlinearities.

Some benefits of the combination are as follows:

- Energy Efficiency: instead of continuous active control throughout the structure, damping is localized and its intensity is only adjusted (scheduled) only when necessary (e.g., high amplitudes or load changes).
- Robustness: combines the ability of scheduled control to handle changes in the operating point with the ability of nonlinear damping to dissipate energy in regimes where linear dampers would be ineffective.

In summary, the combination allows overcoming the limitations of conventional linear controls, thus offering a robust solution for systems with high nonlinear complexity.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author of this paper declare no conflict of interest.

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