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Research article

Global boundedness of a quasilinear predator-prey model with nonlinear indirect signal production mechanism

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Abstract: This paper studies the following Neumann initial-boundary problem of a quasilinear predator-prey model with nonlinear indirect signal production mechanism:

$$\begin{cases} u_{1t} = \nabla \cdot (\phi_1(u_1)\nabla u_1 + \varphi_1(u_1)\nabla v) + \alpha_1 u_1 \left(1 - u_1^{k_1-1} - \beta_1 u_2\right), & x \in \Omega,\ t > 0, \\ u_{2t} = \nabla \cdot (\phi_2(u_2)\nabla u_2 - \varphi_2(u_2)\nabla v) + \alpha_2 u_2 \left(1 - u_2^{k_2-1} + \beta_2 u_1\right), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + w^{\rho}, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - w + u_1^{\vartheta} + u_2^{\eta}, & x \in \Omega,\ t > 0, \end{cases}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$. Assume that the parameters $\alpha_i, \beta_i, \rho, \vartheta, \eta > 0$, ϕ_i and φ_i (i=1, 2) are nonlinear functions satisfying $\phi_i(s) \geq \chi(1+s)^{\delta}$ and $|\varphi_i(s)| \leq \xi s(1+s)^{\sigma-1}$, respectively, for all $s \geq 0$ with $\chi, \xi > 0$ and $\delta, \sigma \in \mathbb{R}$. The main result shows that for all nonnegative initial data with appropriate regularity, if the power exponents satisfy $k_1 > 1, k_2 > 2$ and $\sigma + \rho(\vartheta + \eta) < \frac{n+2}{n} + \delta$, the model possesses a bounded global classical solution.

Keywords: indirect signal; boundedness; predator-prey model; density-dependent diffusion
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1. Introduction

The process by which cells and bacteria undergo directed movement in response to gradients of chemical signal concentration is known as chemotaxis. It is widely recognized that chemotaxis plays a crucial role in biological survival. Biological processes are extremely complex, and the signal may either come from external substances or be produced indirectly. The indirect signal production mechanism was proposed by Fujie and Senba [1] in a two-chemical substance chemotaxis system and independently in a slightly different context in [2]. Their work established the foundational framework

for indirect chemotactic dynamics. The indirect signal production mechanism was studied in the following one-species model:

$$\begin{cases} u_t = \nabla \cdot (\Psi_1(u)\nabla u - \Psi_2(u)\nabla v_1) + g(u), & x \in \Omega,\ t > 0, \\ \tau v_{1t} = \Delta v_1 - v_1 + v_2, & x \in \Omega,\ t > 0, \\ \tau v_{2t} = \Delta v_2 - v_2 + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, u is cell density, and v_1 and v_2 represent chemical signal concentrations. The growth and death dynamics of the cell population are represented by the function $g(u)$, and the nonlinear functions $\Psi_1(u), \Psi_2(u)$ describe the strengths of diffusion. Suppose that $\Psi_1(u) = 1$, $\Psi_2(u) = u$, and $g(u) = \eta(u - u^\theta)$. Zhang et al. [3] proved that the system (1.1) admits a global classical solution if $\tau = 1$ and $\theta > \frac{n+2}{4}$ $(n \geq 2)$. Meanwhile, they established the convergence of solutions for sufficiently large η. Additionally, for $\tau = 0$, under the conditions that $\Psi_1(s) \geq \rho(s+1)^l$ and $|\Psi_2(s)| \leq \delta s(1+s)^{m-1}$ $(\forall s \geq 0,\ \rho, \delta > 0,\ l, m \in \mathbb{R})$, the authors in [4] obtained bounded results and resolved long-term behavior. Further results on system (1.1) can be found in [5, 6].

As is widely recognized, the two-species chemotaxis model serves as a crucial bridge connecting predator and prey, and studying its dynamic behavior has important biological significance. For predator-prey models with single-signal production, optimized conditions for the global boundedness of solutions were established [7, 8]. Under suitable parameter conditions, these works also provided an in-depth analysis of the global asymptotic behavior in two-species chemotaxis models with density-dependent motility. In parallel, other researchers investigated two-species attraction-repulsion chemotaxis systems [9–13]. They proved the existence of a unique global and uniformly bounded classical solution and established global asymptotic stability via Lyapunov functions. Since nonlinear density-dependent diffusion and nonlinear signal secretion are critical factors in the dynamics of two-species chemotaxis models, Gnanasekaran et al. [14] conducted a detailed study of the following system:

$$\begin{cases} u_{1t} = \lambda_1 \Delta u_1 + \chi \nabla \cdot (u_1^\vartheta \nabla v) + a_1 u_1(1 - b_1 u_2 - u_1), & x \in \Omega,\ t > 0, \\ u_{2t} = \lambda_2 \Delta u_2 - \xi \nabla \cdot (u_2^\vartheta \nabla v) + a_2 u_2(1 + b_2 u_1 - u_2), & x \in \Omega,\ t > 0, \\ \tau v_t = \lambda_3 \Delta v - \mu v + f(u_1, u_2), & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $f(s_1, s_2)$ is a nonlinear function satisfying $f(s_1, s_2) \leq \sigma s_1^l + \eta s_2^l$ for $\sigma, \eta, l > 0$ and $s_1, s_2 \geq 0$. For any $p > \frac{n}{2}$ $(n \geq 2)$ and $C > 0$ satisfying $\frac{a_2}{a_1 b_2^{p+1}} < C$, if $l + \vartheta < 1 + \frac{2}{n}$ with $\vartheta \geq 1$, they established the uniqueness of the classical solution and proved that this solution remains bounded within the domain $\Omega \times (0, \infty)$. In addition, the mechanism of indirect signal production also plays an important role in two-species systems. A similar idea of two-chemical indirect signal production from Fujie and Senba's work [1] was independently proposed by Tello and Wrzosek [2] to describe predator-prey model with diffusion and indirect prey-taxis. Later, Xiang et al. [15] proposed the following model of indirect chemotaxis:

$$\begin{cases} u_{1t} = \lambda_1 \Delta u_1 - \varphi_1 \nabla \cdot (u_1 \nabla v) + a_1 u_1(1 - b_1 u_2 - u_1), & x \in \Omega,\ t > 0, \\ u_{2t} = \lambda_2 \Delta u_2 - \varphi_2 \nabla \cdot (u_2 \nabla v) + a_2 u_2(1 - b_2 u_1 - u_2), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + w, & x \in \Omega,\ t > 0, \\ \tau w_t = \Delta w - w + u_1 + u_2, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

where $\tau \in \{0, 1\}$ and the parameters $\lambda_i, \varphi_i, a_i, b_i$ $(i = 1, 2)$ are positive. When $\tau = 0$ and $n = 2$, based on a priori estimates, they showed that regardless of the size of the initial data, this system possessed a globally bounded classical solution. Moreover, when $\tau = 1$ and $n \leq 2$, they studied the global boundedness of classical solutions for this system. Furthermore, if $a_i, b_i (i = 1, 2)$ satisfy some certain conditions, the asymptotic stability of the solutions was studied. Recently, Wang et al. [16] studied a version of system (1.3) with singular sensitivity. They showed that the existence and boundedness of classical solutions were determined by the system coefficients and the spatial dimension n. Furthermore, predator-prey models that involve two chemical signals are studied in [17–21].

Inspired by the above studies, we consider the following quasilinear predator-prey chemotaxis model with nonlinear indirect signal generation mechanisms:

$$\begin{cases} u_{1t} = \nabla \cdot (\phi_1(u_1)\nabla u_1 + \varphi_1(u_1)\nabla v) + \alpha_1 u_1 \left(1 - u_1^{k_1 - 1} - \beta_1 u_2\right), & x \in \Omega,\ t > 0, \\ u_{2t} = \nabla \cdot (\phi_2(u_2)\nabla u_2 - \varphi_2(u_2)\nabla v) + \alpha_2 u_2 \left(1 - u_2^{k_2 - 1} + \beta_2 u_1\right), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + w^{\rho}, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - w + u_1^{\vartheta} + u_2^{\eta}, & x \in \Omega,\ t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x, 0) = u_{10}(x),\ u_2(x, 0) = u_{20}(x),\ v(x, 0) = v_0(x), & x \in \Omega,\ t > 0, \end{cases} \tag{1.4}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$, where ν is the unit outward normal. The parameters of system (1.4) satisfy $\alpha_i, \beta_i, \rho, \vartheta, \eta > 0$ and $k_i > 1$ $(i = 1, 2)$. Here, u_1 stands for saprophytic bacteria and u_2 denotes *Paramecium caudatum*. The kinetic terms $\alpha_1 u_1(1 - u_1^{k_1 - 1} - \beta_1 u_2)$ and $\alpha_2 u_2(1 - u_2^{k_2 - 1} + \beta_2 u_1)$ constitute a nonlinear extension of the classical Lotka-Volterra predator-prey dynamics. The intraspecific competition is governed by the exponents k_1 and k_2, α_1, α_2 denote the growth rates of two species, and β_1, β_2 measure interaction between two species. The primary signal w (extracellular enzymes and metabolites) is secreted by both populations u_1 and u_2, while the secondary signal v (chemotactic factor) arises from the transformation of w. This specific signaling mechanism (a shared precursor leading to opposite taxis) represents a novel theoretical construct that has recently been formalized in two-species models, such as those proposed by Wang and Ke [21]. The terms $\nabla \cdot (\phi_i(u_i)\nabla u_i)$ $(i = 1, 2)$ characterize the density-dependent self-diffusion of the populations. The term $\nabla \cdot (\varphi_1(u_1)\nabla v)$ denotes chemorepulsion, meaning the prey moves directionally away from the chemical attractant v. In contrast, the term $-\nabla \cdot (\varphi_2(u_2)\nabla v)$ signifies chemoattraction, describing the predators' directional movement toward the chemical attractant v. Compared to traditional models with direct or linear indirect signal mechanisms, our work considers a nonlinear indirect signal generation mechanism, combined with density-dependent self-diffusion and cross-diffusion. These mechanisms together offer a more accurate representation of natural microbial predator-prey dynamics.

The nonlinear functions $\phi_i(u_i)$ and $\varphi_i(u_i)$ $(i = 1, 2)$ belong to $C^2[0, \infty)$ and satisfy

$$\phi_i(s) \geq \chi(1 + s)^{\delta} \text{ and } |\varphi_i(s)| \leq \xi s(1 + s)^{\sigma - 1}, \quad \forall\ s \geq 0, \tag{1.5}$$

where $\chi, \xi > 0$ and $\delta, \sigma \in \mathbb{R}$. The assumption that φ_1 and φ_2 share the same functional form simplifies the mathematical analysis and facilitates proving solution boundedness through a priori estimates and Moser iteration. This is a common strategy in initial studies of complex chemotaxis systems (see [14]).

We give the main result of this work as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary. Suppose that the condition* (1.5) *holds and the parameters of system* (1.4) *satisfy $\alpha_i, \beta_i, \rho, \vartheta, \eta > 0$ $(i = 1, 2)$. If $k_1 > 1, k_2 > 2$ and $\sigma + \rho(\vartheta + \eta) < \frac{n+2}{n} + \delta$, then for any nonnegative initial data $u_{10}, u_{20} \in C^\gamma(\overline{\Omega})$ with $0 < \gamma < 1$ and $v_0 \in W^{1,\infty}(\Omega)$, subject to the conditions $\frac{\partial u_{10}}{\partial \nu} = \frac{\partial u_{20}}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0$ on $\partial\Omega$ (where ν denotes the unit outward normal), there exists a global classical solution*

$$(u_1, u_2, v, w) \in \Big(C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))\Big)^3 \times C^{2,0}(\overline{\Omega} \times (0, \infty)).$$

In addition, there exists $C_0 > 0$ such that the solution remains bounded in $\Omega \times (0, \infty)$, that is,

$$\|u_1(\cdot, t)\|_{L^\infty(\Omega)} + \|u_2(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_0, \quad \forall\, t > 0.$$

Comparing with [14], we remove the restrictions on the coefficients of the system and our global boundedness results (see Theorem 1.1) rely only on system's power exponents $\delta, \sigma, \rho, \vartheta, \eta, k_i$ $(i = 1, 2)$ and spatial dimension n. To our knowledge, the conventional approaches for deriving boundedness criteria of fully parabolic or parabolic-elliptic-elliptic indirect chemotaxis systems primarily involve the variation-of-constants formula and heat semigroup theory. Nevertheless, these conventional strategies are not directly applicable to the parabolic-parabolic-parabolic-elliptic framework proposed in this work. Herein, we employ maximal Sobolev regularity (refer to Lemma 2.3) and a priori estimates of w (refer to Lemma 2.2) to handle the term $\int_\Omega |\Delta v|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}}$ generated by Young's inequality in (3.9). This enables us to investigate how nonlinear diffusion functions, the nonlinear indirect signal production mechanism and superlinear intraspecific competition impact solution boundedness.

2. Preliminaries

Lemma 2.1. *(cf. [5, 22–24]) Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary. For any nonnegative initial data $u_{10}, u_{20} \in C^\gamma(\overline{\Omega})$ with $0 < \gamma < 1$ and $v_0 \in W^{1,\infty}(\Omega)$, there exists $T_{\max} \in (0, \infty]$ ensuring that system* (1.4) *has a classical solution*

$$(u_1, u_2, v, w) \in \Big(C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))\Big)^3 \times C^{2,0}(\overline{\Omega} \times (0, T_{\max})).$$

Furthermore, if $T_{\max} < \infty$, it follows that

$$\|u_1(\cdot, t)\|_{L^\infty(\Omega)} + \|u_2(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty \quad as \;\; t \to T_{\max}.$$

Lemma 2.2. *(cf. [25, Lemma 2.2]) For $\zeta > 0$ and $l > 1$, we can find $C_1(\vartheta, \eta, l, \zeta) > 0$, where $\vartheta, \eta > 0$, satisfying*

$$\int_\Omega w^l \leq \zeta \int_\Omega \Big((u_1 + 1)^{l\vartheta} + (u_2 + 1)^{l\eta}\Big) + C_1, \;\; \forall\, t \in (0, T_{\max}). \tag{2.1}$$

Lemma 2.3. *(cf. [26, Lemma 2.5]) Assume that $m \in (1, \infty)$ and $s_0 \in [0, T)$ with $T \in (0, \infty]$. Consider the following equation*

$$\begin{cases} v_t = \Delta v - v + w^\rho, & x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases}$$

where $\rho > 0$. Suppose that the initial data $v_0 \in W^{2,m}(\Omega)$ satisfies $\frac{\partial v_0}{\partial \nu}|_{\partial\Omega} = 0$, and let w be a nonnegative function such that $w^\rho \in L^m((0,T); L^m(\Omega))$. Then, this equation admits a unique solution

$$v \in W^{1,m}((0,T); L^m(\Omega)) \cap L^m((0,T); W^{2,m}(\Omega)).$$

Additionally, there exists $C_m = C_m(m, \rho, \Omega) > 0$ such that whenever $v(\cdot, s_0) \in W^{2,m}(\Omega)$ with $\frac{\partial v(\cdot,s_0)}{\partial \nu}|_{\partial\Omega} = 0$, it follows that

$$\int_{s_0}^{T}\int_\Omega e^{ms}|\Delta v|^m \le C_m \int_{s_0}^{T}\int_\Omega e^{ms} w^{m\rho} + C_m e^{ms_0}(\|v(\cdot, s_0)\|^m_{L^m(\Omega)} + \|\Delta v(\cdot, s_0)\|^m_{L^m(\Omega)}).$$

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary. If the condition* (1.5) *holds, and the parameters of system* (1.4) *satisfy $\alpha_i, \beta_i, \rho, \vartheta, \eta > 0$ and $k_i > 1$ $(i = 1, 2)$, then there exist constants $C_2, C_3 > 0$ such that for all $t \in (0, T_{\max})$,*

$$\|u_1(\cdot, t)\|_{L^1(\Omega)} \le C_2 = \max\{\|u_{10}\|_{L^1(\Omega)}, |\Omega|\} \tag{2.2}$$

and

$$\|u_2(\cdot, t)\|_{L^1(\Omega)} \le C_3. \tag{2.3}$$

Proof. Due to $k_1 > 1$, we can infer by integrating the first equation in system (1.4) and using Hölder's inequality that

$$\frac{d}{dt}\int_\Omega u_1 \le \alpha_1 \int_\Omega u_1 - \frac{\alpha_1}{|\Omega|^{k_1-1}}\left(\int_\Omega u_1\right)^{k_1}$$

for all $t \in (0, T_{\max})$. Thus, by ODE comparison principle, the estimate (2.2) follows. We next multiply the first two equations of system (1.4) by $\alpha_2\beta_2$ and $\alpha_1\beta_1$, respectively, and then take their sum and integrate it over Ω to obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (\alpha_2\beta_2 u_1 + \alpha_1\beta_1 u_2) \\ &= \int_\Omega \alpha_1\alpha_2\beta_2 u_1 - \int_\Omega \alpha_1\alpha_2\beta_2 u_1^{k_1} + \int_\Omega \alpha_1\alpha_2\beta_1 u_2 - \int_\Omega \alpha_1\alpha_2\beta_1 u_2^{k_2} \end{aligned} \tag{2.4}$$

for all $t \in (0, T_{\max})$. Since $k_1, k_2 > 1$, Young's inequality implies that there exist constants $c_1, c_2 > 0$ satisfying

$$\int_\Omega (\alpha_1\alpha_2\beta_2 + \alpha_2\beta_2)\, u_1 \le \int_\Omega \alpha_1\alpha_2\beta_2 u_1^{k_1} + c_1 \tag{2.5}$$

and

$$\int_\Omega (\alpha_1\alpha_2\beta_1 + \alpha_1\beta_1)\, u_2 \le \int_\Omega \alpha_1\alpha_2\beta_1 u_2^{k_2} + c_2 \tag{2.6}$$

for all $t \in (0, T_{\max})$. Substituting (2.5) and (2.6) into (2.4), we obtain

$$\frac{d}{dt}\int_\Omega (\alpha_2\beta_2 u_1 + \alpha_1\beta_1 u_2) + \int_\Omega (\alpha_2\beta_2 u_1 + \alpha_1\beta_1 u_2) \le c_3$$

for all $t \in (0, T_{\max})$, where $c_3 = c_1 + c_2 > 0$. By ODE comparison principle and (2.2), we obtain (2.3).

3. Global existence and boundedness

In this section, we first establish the L^p-estimate for u_1 and u_2 and then utilize the Moser iteration [27, Lemma A.1] to prove Theorem 1.1. We can deduce from Lemma 2.1 that

$$u_1(\cdot, s_0), u_2(\cdot, s_0), v(\cdot, s_0) \in C^2(\overline{\Omega})$$

with $\frac{\partial u_1(\cdot,s_0)}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial u_2(\cdot,s_0)}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial v(\cdot,s_0)}{\partial \nu}\Big|_{\partial\Omega} = 0$ for any $s_0 \in (0, T_{\max})$ satisfying $s_0 < 1$. Moreover, there exists $C_4 > 0$ satisfying

$$\begin{cases} \sup_{0\le s\le s_0} \|u_1(\cdot, s)\|_{L^\infty(\Omega)} \le C_4, \quad \sup_{0\le s\le s_0} \|u_2(\cdot, s)\|_{L^\infty(\Omega)} \le C_4, \\ \sup_{0\le s\le s_0} \|v(\cdot, s)\|_{L^\infty(\Omega)} \le C_4, \quad \|\Delta v(\cdot, s_0)\|_{L^\infty(\Omega)} \le C_4. \end{cases} \tag{3.1}$$

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary. Suppose that the condition* (1.5) *holds and the parameters of system* (1.4) *satisfy $\alpha_i, \beta_i, \rho, \vartheta, \eta > 0$ $(i = 1, 2)$. If $k_1 > 1, k_2 > 2$ and $\sigma + \rho(\vartheta + \eta) < \frac{n+2}{n} + \delta$ are valid, then for any $p > \max\{1, 1-\sigma, 1-\sigma+(1-\rho)(\vartheta+\eta), \frac{(2-k_1)(k_2-1)}{k_2-2}\}$, there exists $C_5 > 0$ satisfying*

$$\int_\Omega (u_1 + 1)^p + \int_\Omega (u_2 + 1)^p \le C_5 \tag{3.2}$$

for all $t \in (0, T_{\max})$.

Proof. **The L^p-estimate for prey population** u_1 For any $p > 1$, we multiply the first equation of system (1.4) by $(u_1 + 1)^{p-1}$ and integrate it by parts, and then we arrive at

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega (u_1+1)^p &\le -(p-1)\int_\Omega (u_1+1)^{p-2}\phi_1(u_1)|\nabla u_1|^2 \\ &\quad - (p-1)\int_\Omega (u_1+1)^{p-2}\varphi_1(u_1)\nabla u_1 \cdot \nabla v \\ &\quad + \alpha_1 \int_\Omega u_1(u_1+1)^{p-1} - \alpha_1 \int_\Omega u_1^{k_1}(u_1+1)^{p-1} \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{3.3}$$

for all $t \in (0, T_{\max})$. By (1.5), we can deal with the term I_1 as follows

$$I_1 = -(p-1)\int_\Omega (u_1+1)^{p-2}\phi_1(u_1)|\nabla u_1|^2 \le -\frac{4\chi(p-1)}{(p+\delta)^2}\int_\Omega \left|\nabla (u_1+1)^{\frac{p+\delta}{2}}\right|^2 \tag{3.4}$$

for all $t \in (0, T_{\max})$. By Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \int_\Omega (u_1+1)^{p+\delta+\frac{2}{n}} &= \left\|(u_1+1)^{\frac{p+\delta}{2}}\right\|^{\frac{2(p+\delta+\frac{2}{n})}{p+\delta}}_{L^{\frac{2(p+\delta+\frac{2}{n})}{p+\delta}}(\Omega)} \\ &\le c_4 \left\|\nabla (u_1+1)^{\frac{p+\delta}{2}}\right\|^{\frac{2(p+\delta+\frac{2}{n})}{p+\delta}\iota_0}_{L^2(\Omega)} \cdot \left\|(u_1+1)^{\frac{p+\delta}{2}}\right\|^{\frac{2(p+\delta+\frac{2}{n})}{p+\delta}(1-\iota_0)}_{L^{\frac{2}{p+\delta}}(\Omega)} \\ &\quad + c_4 \left\|(u_1+1)^{\frac{p+\delta}{2}}\right\|^{\frac{2(p+\delta+\frac{2}{n})}{p+\delta}}_{L^{\frac{2}{p+\delta}}(\Omega)} \end{aligned} \tag{3.5}$$

for all $t \in (0, T_{\max})$, where $\iota_0 = \frac{\frac{p+\delta}{2} - \frac{p+\delta}{2(p+\delta+\frac{2}{n})}}{\frac{p+\delta}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$ and $c_4 = c_4(p,\delta,n,\Omega) > 0$. Since $\frac{(p+\delta+\frac{2}{n})}{p+\delta}\iota_0 = 1$, we are able to derive from (2.2) that

$$\int_\Omega (u_1+1)^{p+\delta+\frac{2}{n}} \le c_4 C_2^{\frac{2}{n}} \left\| \nabla (u_1+1)^{\frac{p+\delta}{2}} \right\|^2_{L^2(\Omega)} + c_4 C_2^{p+\delta+\frac{2}{n}} \tag{3.6}$$

for all $t \in (0, T_{\max})$. By (3.4), (3.5), and (3.6), one may obtain

$$I_1 \le -c_5 \int_\Omega (u_1+1)^{p+\delta+\frac{2}{n}} + \frac{4\chi(p-1)}{(p+\delta)^2} C_2^{p+\delta} \tag{3.7}$$

for all $t \in (0, T_{\max})$, where $c_5 = \frac{4\chi(p-1)}{(p+\delta)^2 c_4 C_2^{\frac{2}{n}}} > 0$. Let $\Phi_1(\tau) = \int_0^\tau (\varsigma+1)^{p-2}\varphi_1(\varsigma)d\varsigma$. Then,

$$\nabla \Phi_1(u_1) = (u_1+1)^{p-2}\varphi_1(u_1)\nabla u_1 \ \text{ and } \ |\Phi_1(u_1)| \le \frac{\xi}{p+\sigma-1}(u_1+1)^{p+\sigma-1} \tag{3.8}$$

for all $t \in (0, T_{\max})$. From (3.8) and Young's inequality, we obtain

$$\begin{aligned} I_2 &= -(p-1)\int_\Omega (u_1+1)^{p-2}\varphi_1(u_1)\nabla u_1 \cdot \nabla v \\ &= -(p-1)\int_\Omega \nabla \Phi_1(u_1)\cdot \nabla v \\ &\le c_6 \int_\Omega (u_1+1)^{p+\sigma-1}|\Delta v| \\ &\le \int_\Omega (u_1+1)^{p+\sigma+\rho(\vartheta+\eta)-1} + c_7 \int_\Omega |\Delta v|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}} \end{aligned} \tag{3.9}$$

for all $t \in (0, T_{\max})$, where $c_6 = \frac{\xi(p-1)}{p+\sigma-1} > 0$ and $c_7 = c_6^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}} > 0$. Invoking the inequality $(u_1+1)^{k_1} \le 2^{k_1}(u_1^{k_1}+1)$ for $u_1 \ge 0$ and $k_1 \ge 1$, we obtain

$$\begin{aligned} I_3 &= \alpha_1 \int_\Omega u_1(u_1+1)^{p-1} - \alpha_1 \int_\Omega u_1^{k_1}(u_1+1)^{p-1} \\ &\le \alpha_1 \int_\Omega u_1(u_1+1)^{p-1} - \frac{\alpha_1}{2^{k_1}} \int_\Omega (u_1+1)^{p+k_1-1} + \alpha_1 \int_\Omega (u_1+1)^{p-1} \\ &= \alpha_1 \int_\Omega (u_1+1)^p - \frac{\alpha_1}{2^{k_1}} \int_\Omega (u_1+1)^{p+k_1-1} \end{aligned} \tag{3.10}$$

for all $t \in (0, T_{\max})$. Collecting (3.7), (3.9), and (3.10), we are led to

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega (u_1+1)^p \le & -c_5 \int_\Omega (u_1+1)^{p+\delta+\frac{2}{n}} + \int_\Omega (u_1+1)^{p+\sigma+\rho(\vartheta+\eta)-1} \\ & + c_7 \int_\Omega |\Delta v|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}} + \alpha_1 \int_\Omega (u_1+1)^p \\ & - \frac{\alpha_1}{2^{k_1}} \int_\Omega (u_1+1)^{p+k_1-1} + c_8 \end{aligned} \tag{3.11}$$

for all $t \in (0, T_{\max})$, where $c_8 = \frac{4\chi(p-1)}{(p+\delta)^2} C_2^{p+\delta} > 0$. Adding $\frac{p+\sigma+\rho(\vartheta+\eta)-1}{p\rho(\vartheta+\eta)} \int_\Omega (u_1+1)^p$ to both sides of (3.11), and then testing $e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}t}$ on both sides of it, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}t}\frac{1}{p}\int_\Omega (u_1+1)^p\right] \\
&\le -c_5 e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}t}\int_\Omega (u_1+1)^{p+\delta+\frac{2}{n}} + e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}t}\int_\Omega (u_1+1)^{p+\sigma+\rho(\vartheta+\eta)-1} \\
&\quad + c_7 e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}t}\int_\Omega |\Delta v|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}} - \frac{\alpha_1}{2^{k_1}} e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}t}\int_\Omega (u_1+1)^{p+k_1-1} \\
&\quad + \left(\alpha_1 + \frac{p+\sigma+\rho(\vartheta+\eta)-1}{p\rho(\vartheta+\eta)}\right) e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}t}\int_\Omega (u_1+1)^p + c_8 e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}t}
\end{aligned} \tag{3.12}$$

for all $t \in (0, T_{\max})$. Then, integrating the inequality (3.12) from s_0 to t yields

$$\begin{aligned}
\frac{1}{p}\int_\Omega (u_1+1)^p \le &- c_5 \int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_1+1)^{p+\delta+\frac{2}{n}} \\
&+ \int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_1+1)^{p+\sigma+\rho(\vartheta+\eta)-1} \\
&+ c_7 \int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega |\Delta v|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}} \\
&- \frac{\alpha_1}{2^{k_1}}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_1+1)^{p+k_1-1} \\
&+ \left(\alpha_1 + \frac{p+\sigma+\rho(\vartheta+\eta)-1}{p\rho(\vartheta+\eta)}\right)\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_1+1)^p + c_9
\end{aligned} \tag{3.13}$$

for all $t \in (s_0, T_{\max})$, where $c_9 = \frac{c_8\rho(\vartheta+\eta)}{p+\sigma+\rho(\vartheta+\eta)-1} + \frac{1}{p}\int_\Omega (u_1(\cdot, s_0)+1)^p$. Let $m = \frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}$. Since $p > 1-\sigma$, it follows that $m > 1$. We can directly derive from Lemma 2.3 that

$$\begin{aligned}
&\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega |\Delta v|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}} \\
&\le C_m \int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega w^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\vartheta+\eta}} \\
&\quad + C_m e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}s_0}\left(\|v(\cdot, s_0)\|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}}_{L^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}}(\Omega)} + \|\Delta v(\cdot, s_0)\|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}}_{L^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}}(\Omega)}\right) \\
&\le C_m \int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega w^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\vartheta+\eta}} \\
&\quad + C_m e^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}s_0}\|v(\cdot, s_0)\|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}}_{W^{2,\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}}(\Omega)}
\end{aligned} \tag{3.14}$$

for all $t \in (s_0, T_{\max})$. Taking $l = \frac{p+\sigma+\rho(\vartheta+\eta)-1}{\vartheta+\eta} > 1$, the Lemma 2.2 and Young's inequality enable us to

infer that

$$\begin{aligned}&\int_\Omega w^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\vartheta+\eta}}\\&\le \zeta\int_\Omega\left((u_1+1)^{\frac{\vartheta(p+\sigma+\rho(\vartheta+\eta)-1)}{\vartheta+\eta}}+(u_2+1)^{\frac{\eta(p+\sigma+\rho(\vartheta+\eta)-1)}{\vartheta+\eta}}\right)+C_1\\&\le \int_\Omega\left((u_1+1)^{p+\sigma+\rho(\vartheta+\eta)-1}+(u_2+1)^{p+\sigma+\rho(\vartheta+\eta)-1}\right)+c_{10}\end{aligned}\tag{3.15}$$

for all $t\in(0,T_{\max})$, where $c_{10}>0$. Collecting (3.13), (3.14), and (3.15), it follows that

$$\begin{aligned}\frac{1}{p}\int_\Omega(u_1+1)^p\le&-c_5\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_1+1)^{p+\delta+\frac{2}{n}}\\&+c_{11}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_1+1)^{p+\sigma+\rho(\vartheta+\eta)-1}\\&+c_7C_m\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_2+1)^{p+\sigma+\rho(\vartheta+\eta)-1}\\&-\frac{\alpha_1}{2^{k_1}}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_1+1)^{p+k_1-1}\\&+\left(\alpha_1+\frac{p+\sigma+\rho(\vartheta+\eta)-1}{p\rho(\vartheta+\eta)}\right)\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_1+1)^p+c_{12}\end{aligned}\tag{3.16}$$

for all $t\in(s_0,T_{\max})$, where

$$c_{12}=c_9+c_7C_me^{ms_0}\left\|v(\cdot,s_0)\right\|^{\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}}_{W^{2,\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}}(\Omega)}+\frac{c_7c_{10}C_m\rho(\vartheta+\eta)}{p+\sigma+\rho(\vartheta+\eta)-1}$$

and $c_{11}=1+c_7C_m$. Due to $\sigma+\rho(\vartheta+\eta)<\frac{n+2}{n}+\delta$, Young's inequality implies that

$$\begin{aligned}&c_{11}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_1+1)^{p+\sigma+\rho(\vartheta+\eta)-1}\\&\le\frac{c_5}{4}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_1+1)^{p+\delta+\frac{2}{n}}+c_{13}\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}&c_7C_m\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_2+1)^{p+\sigma+\rho(\vartheta+\eta)-1}\\&\le\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_2+1)^{p+\delta+\frac{2}{n}}+c_{14}\end{aligned}\tag{3.18}$$

as well as

$$\begin{aligned}&\left(\alpha_1+\frac{p+\sigma+\rho(\vartheta+\eta)-1}{p\rho(\vartheta+\eta)}\right)\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_1+1)^p\\&\le\frac{c_5}{4}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega(u_1+1)^{p+\delta+\frac{2}{n}}+c_{15}\end{aligned}\tag{3.19}$$

for all $t \in (s_0, T_{\max})$, with $c_{13}, c_{14}, c_{15} > 0$. By virtue of (3.16)-(3.19), it follows that

$$\begin{aligned}\frac{1}{p}\int_\Omega (u_1+1)^p \le &-\frac{c_5}{2}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_1+1)^{p+\delta+\frac{2}{n}}\\ &+\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_2+1)^{p+\delta+\frac{2}{n}}\\ &-\frac{\alpha_1}{2^{k_1}}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_1+1)^{p+k_1-1}+c_{16}\end{aligned} \tag{3.20}$$

for all $t \in (s_0, T_{\max})$, where $c_{16} > 0$.

The L^p-estimate for predator population u_2 Multiplying the second equation in system (1.4) by $(u_2+1)^{p-1}$ for any $p > 1$, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u_2+1)^p \le &-\frac{4\chi(p-1)}{(p+\delta)^2}\int_\Omega \left|\nabla (u_2+1)^{\frac{p+\delta}{2}}\right|^2-\frac{\xi(p-1)}{p+\sigma-1}\int_\Omega (u_2+1)^{p+\sigma-1}\Delta v\\ &+\alpha_2\int_\Omega u_2(u_2+1)^{p-1}-\alpha_2\int_\Omega u_2^{k_2}(u_2+1)^{p-1}\\ &+\alpha_2\beta_2\int_\Omega u_2(u_2+1)^{p-1}u_1\end{aligned} \tag{3.21}$$

for all $t \in (0, T_{\max})$. Similar to (3.10), we have

$$\begin{aligned}&\alpha_2\int_\Omega u_2(u_2+1)^{p-1}-\alpha_2\int_\Omega u_2^{k_2}(u_2+1)^{p-1}\\ &\le -\frac{\alpha_2}{2^{k_2}}\int_\Omega (u_2+1)^{p+k_2-1}+\alpha_2\int_\Omega (u_2+1)^p\end{aligned} \tag{3.22}$$

for all $t \in (0, T_{\max})$. Applying Young's inequality, it follows that

$$\begin{aligned}&\alpha_2\beta_2\int_\Omega u_2(u_2+1)^{p-1}u_1\\ &\le \frac{\alpha_2}{2^{k_2+1}}\int_\Omega (u_2+1)^{p+k_2-1}+c_{17}\int_\Omega (u_1+1)^{\frac{p+k_2-1}{k_2-1}}\end{aligned} \tag{3.23}$$

for all $t \in (0, T_{\max})$, where $c_{17} > 0$. Collecting (3.21)-(3.23), Young's inequality shows that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (u_2+1)^p\\ &\le -\frac{4\chi(p-1)}{(p+\delta)^2}\int_\Omega \left|\nabla (u_2+1)^{\frac{p+\delta}{2}}\right|^2-\frac{\xi(p-1)}{p+\sigma-1}\int_\Omega (u_2+1)^{p+\sigma-1}\Delta v\\ &\quad +c_{17}\int_\Omega (u_1+1)^{\frac{p+k_2-1}{k_2-1}}+c_{18}\end{aligned} \tag{3.24}$$

for all $t \in (0, T_{\max})$, where $c_{18} > 0$. As in the case of dealing with the equation for u_1, it follows that

$$\begin{aligned}\frac{1}{p}\int_\Omega (u_2+1)^p \le &-\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_2+1)^{p+\delta+\frac{2}{n}}\\ &+\frac{c_5}{2}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_1+1)^{p+\delta+\frac{2}{n}}\\ &+c_{17}\int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)}\int_\Omega (u_1+1)^{\frac{p+k_2-1}{k_2-1}}+c_{19}\end{aligned} \tag{3.25}$$

for all $t \in (0, T_{\max})$, where $c_{19} > 0$. We collect (3.20) and (3.25) to arrive at

$$\begin{aligned}
&\frac{1}{p}\left(\int_\Omega (u_1+1)^p + \int_\Omega (v_1+1)^p\right) \\
&\quad \le -\frac{\alpha_1}{2^{k_1}} \int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)} \int_\Omega (u_1+1)^{p+k_1-1} \\
&\qquad + c_{17} \int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)} \int_\Omega (u_1+1)^{\frac{p+k_2-1}{k_2-1}} + c_{20}
\end{aligned} \tag{3.26}$$

for all $t \in (s_0, T_{\max})$, where $c_{20} > 0$. For $p > \max\{1, 1-\sigma, 1-\sigma+(1-\rho)(\vartheta+\eta), \frac{(2-k_1)(k_2-1)}{k_2-2}\}$, and $k_2 > 2$, we have

$$\begin{aligned}
p + k_1 - 1 - \frac{p+k_2-1}{k_2-1} &= \frac{(p+k_1-1)(k_2-1)-(p+k_2-1)}{k_2-1} \\
&= \frac{p(k_2-2)+(k_1-1)(k_2-1)-(k_2-1)}{k_2-1} \\
&= \frac{p(k_2-2)}{k_2-1} + k_1 - 2 > 0.
\end{aligned}$$

Then, by using Young's inequality, we obtain

$$\begin{aligned}
&c_{17} \int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)} \int_\Omega (u_1+1)^{\frac{p+k_2-1}{k_2-1}} \\
&\le \frac{\alpha_1}{2^{k_1}} \int_{s_0}^t e^{-\frac{p+\sigma+\rho(\vartheta+\eta)-1}{\rho(\vartheta+\eta)}(t-s)} \int_\Omega (u_1+1)^{\frac{p+k_2-1}{k_2-1}\cdot q} + c_{21}
\end{aligned} \tag{3.27}$$

for all $t \in (s_0, T_{\max})$, where $q = (p+k_1-1)\cdot\frac{k_2-1}{p+k_2-1} > 1$ and $c_{21} > 0$. In light of (3.26) and (3.27), it follows that

$$\frac{1}{p}\left(\int_\Omega (u_1+1)^p + \int_\Omega (u_2+1)^p\right) \le c_{22} \tag{3.28}$$

for all $t \in (s_0, T_{\max})$, where $c_{22} > 0$. This, along with (3.1), results in the desired Lemma 3.1.

We now combine the L^p-estimates of u_1 and u_2 with regularity theories and Moser iteration [27, Lemma A.1] to complete the proof of Theorem 1.1.

The proof of Theorem 1.1 For any $p > \max\{1, n\vartheta, n\eta, 1-\sigma, 1-\sigma+(1-\rho)(\vartheta+\eta), \frac{(2-k_1)(k_2-1)}{k_2-2}\}$, Lemma 3.1 indicates that $\|u_1(\cdot,t)\|^p_{L^p(\Omega)} + \|u_2(\cdot,t)\|^p_{L^p(\Omega)} \le C_5$. By elliptic L^p-estimates, we can derive from the fourth equation in system (1.4) that $\|w(\cdot,t)\|_{W^{2,\frac{p}{\max\{\vartheta,\eta\}}}(\Omega)} \le c_{23}$ for all $t \in (0, T_{\max})$, where $c_{23} > 0$. Subsequently, by the Sobolev embedding theorem, we arrive at $\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_{24}$ for all $t \in (0, T_{\max})$, where $c_{24} > 0$. By parabolic regularity theory, the third equation in system (1.4) implies that $\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_{25}$ for all $t \in (0, T_{\max})$, where $c_{25} > 0$. Invoking Moser iteration [27, Lemma A.1] and (3.2), we can infer that there exists $c_{26} > 0$ satisfying

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)} \le c_{26}$$

for all $t \in (0, T_{\max})$. The Lemma 2.1 indicates that $T_{\max} = \infty$. We thus complete the proof of this Theorem.

4. Conclusions and outlook

This study investigates the interaction between saprophytic bacteria (prey u_1) and *Paramecium caudatum* (predator u_2). In model (1.4), the nonlinear exponents $k_1, k_2, \sigma, \rho, \vartheta, \eta$, and δ are crucial to the global boundedness of the solution. The main goal is to overcome the constraints on the spatial dimensions and coefficients of the system and explore how nonlinear diffusion, nonlinear indirect signal production mechanisms, and superlinear population density constraints affect the global boundedness of solutions. For $n = 1, 2$, replacing the model's third equation with $0 = \Delta v - v + w^{\rho}$ allows further improvement of boundedness conditions via a priori estimates and elliptic regularity, which will be addressed in future work. The predator u_2 obtains energy from u_1 and exhibits chemoattraction to the indirect chemotactic signal v, while u_1 suffers predation losses from u_2 and shows chemorepulsion to v. Thus, u_2 requires a strong superlinear density restriction $k_2 > 2$ to prevent population blow-up, whereas u_1 only needs $k_1 > 1$ for stability. The parameters $\delta, \sigma \in \mathbb{R}$ characterize a wide range of diffusion and chemotaxis mechanisms, encompassing both enhancement and inhibition types. All these cases are uniformly covered by our global boundedness analysis in Theorem 1.1. The key conditions $k_1 > 1$, $k_2 > 2$ and $\sigma + \rho(\vartheta + \eta) < \frac{n+2}{n} + \delta$ in Theorem 1.1 essentially quantify the balance between chemotactic aggregation, nonlinear diffusion, and intraspecific competition. These conditions are inherently required for model stability and are technically indispensable for establishing global boundedness via a priori estimates and Moser iteration.

Regarding the long-time behavior of solutions, reference [28] showed that a specific indirect signal-driven one-species model is exponentially stable. Analogously, for special cases of (1.4), the Lyapunov functional can be used to prove that the solution converges exponentially to the steady state. In future work, we will further investigate this specific case and draw more general conclusions about the long-term dynamics of solutions.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declare there is no conflict of interest.

References

1. K. Fujie, T. Senba, Application of an adams-type inequality to a two-chemical substance chemotaxis system, *J. Differ. Equ.*, **263** (2017), 88–148. https://doi.org/10.1016/j.jde.2017.02.031

2. J. Tello, D. Wrzosek, Predator-prey model with diffusion and indirect prey-taxis, *Math. Models Methods Appl. Sci.*, **26** (2016), 2129–2162. https://doi.org/10.1142/S0218202516400108
3. W. Zhang, P. Niu, S. Liu, Large time behavior in a chemotaxis model with logistic growth and indirect signal production, *Nonlinear Anal. Real World Appl.*, **50** (2019), 484–497. https://doi.org/10.1016/j.nonrwa.2019.05.002
4. D. Li, Z. Li, Asymptotic behavior of a quasilinear parabolic-elliptic-elliptic chemotaxis system with logistic source, *Z. Angew. Math. Phys.*, **73** (2022), 22. https://doi.org/10.1007/s00033-021-01655-y
5. M. Ding, W. Wang, Global boundedness in a quasilinear fully parabolic chemotaxis system with indirect signal production, *Discrete Contin. Dyn. Syst. Ser. B*, **24** (2019), 4665–4684. https://doi.org/10.3934/dcdsb.2018328
6. C. Wang, J. Zhu, Global boundedness in an attraction-repulsion chemotaxis system involving nonlinear indirect signal mechanism, *J. Math. Anal. Appl.*, **531** (2024), 127876. https://doi.org/10.1016/j.jmaa.2023.127876
7. Y. Li, C. Mu, X. Pan, Boundedness and asymptotic stability in a predator-prey system with density-dependent motilities, *Discrete Contin. Dyn. Syst., Ser. B*, **29** (2024), 2192–2212. https://doi.org/10.3934/dcdsb.2023173
8. L. Yang, L. Xu, X. Pan, Improvement of conditions for global boundedness in a predator-prey system with pursuit-evasion interaction, *Discrete Contin. Dyn. Syst., Ser. B*, **29** (2024), 1187–1205. https://doi.org/10.3934/dcdsb.2023128
9. R. Hu, P. Zheng, Global stability in a two-species attraction-repulsion system with competitive and nonlocal kinetics, *J. Dyn. Differ. Equ.*, **36** (2024) 2555–2592. https://doi.org/10.1007/s10884-022-10215-5
10. J. Tello, M. Winkler, Stabilization in a two-species chemotaxis system with a logistic source, *Nonlinearity*, **25** (2012), 1413–1425. https://doi.org/10.1088/0951-7715/25/5/1413
11. C. Wan, P. Zheng, W. Shan, Global stability of a quasilinear predator-prey model with indirect pursuit-evasion interaction, *Int. J. Biomath.*, **17** (2023), 2350076. https://doi.org/10.1142/S1793524523500766
12. P. Zheng, R. Hu, W. Shan, On a two-species attraction-repulsion chemotaxis system with nonlocal terms, *J. Nonlinear Sci.*, **33** (2023), 57. https://doi.org/10.1007/s00332-023-09912-2
13. P. Zheng, Y. Xiang, J. Xing, On a two-species chemotaxis system with indirect signal production and general competition terms, *Math. Models Methods Appl. Sci.*, **32** (2022), 1385–1430. https://doi.org/10.1142/S0218202522500312
14. S. Gnanasekaran, A. Gurusamy, A. Andr, N. Nagarajan, Global existence of solutions to a two species predator-prey parabolic chemotaxis system, *Int. J. Biomath.*, **15** (2022), 2250054. https://doi.org/10.1142/S1793524522500541
15. Y. Xiang, P. Zheng, J. Xing, Boundedness and stabilization in a two-species chemotaxis-competition system with indirect signal production, *J. Math. Anal. Appl.*, **507** (2022), 125825. https://doi.org/10.1016/j.jmaa.2021.125825
16. D. Wang, F. Zeng, M. Jiang, Global existence and boundedness of solutions to a two-species chemotaxis-competition system with singular sensitivity and indirect signal production, *Z. Angew. Math. Phys.*, **74** (2023), 33. https://doi.org/10.1007/s00033-022-01921-7

17. E. Akkoyunlu, Global existence in a predator-prey model with nonlinear indirect chemotaxis mechanism, *J. Inst. Sci. Technol.*, **14** (2024), 1705–1716. https://doi.org/10.21597/jist.1550265
18. P. Amorim, B. Telch, A chemotaxis predator-prey model with indirect pursuit-evasion dynamics and parabolic signal, *J. Math. Anal. Appl.*, **500** (2021), 125128. https://doi.org/10.1016/j.jmaa.2021.125128
19. G. Li, Y. Tao, M. Winkler, Large time behavior in a predator-prey system with indirect pursuit-evasion interaction, *Discrete Contin. Dyn. Syst., Ser. B*, **25** (2020), 4383–4396. https://doi.org/10.3934/dcdsb.2020102
20. X. Liu, J. Zheng, Convergence rates of solutions in a predator-prey system with indirect pursuit-evasion interaction in domains of arbitrary dimension, *Discrete Contin. Dyn. Syst., Ser. B*, **28** (2023), 2269–2293. https://doi.org/10.3934/dcdsb.2022168
21. C. Wang, C. Ke, Global classical solutions to a predator-prey model with nonlinear indirect chemotaxis mechanism, *Acta Appl. Math.*, **190** (2024), 11. https://doi.org/10.1007/s10440-024-00648-z
22. D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Differ. Equ.*, **215** (2005), 52–107. https://doi.org/10.1016/j.jde.2004.10.022
23. O. Ladyzhenskaya, V. Solonnikov, N. Uralceva, *Linear and quasilinear equations of parabolic type,* American Mathematical Society, Providence, RI, 1968.
24. J. Tello, M. Winkler, A chemotaxis system with logistic source, *Commun. Partial Differ. Equ.*, **32** (2007), 849–877. https://doi.org/10.1080/03605300701319003
25. L. Hong, M. Tian, S. Zheng, An attraction-repulsion chemotaxis system with nonlinear productions, *J. Math. Anal. Appl.*, **484** (2020), 123703. https://doi.org/10.1016/j.jmaa.2019.123703
26. X. Cao, Boundedness in a three-dimensional chemotaxis-haptotaxis model, *Z. Angew. Math. Phys.*, **67** (2016), 11. https://doi.org/10.1007/s00033-015-0601-3
27. N. Alikakos, L^p bounds of solutions of reaction-diffusion equations, *Commun. Partial Differ. Equ.*, **4** (1979), 827–868. https://doi.org/10.1080/03605307908820113
28. C. Wang, P. Wang, X. Zhu, Global dynamics in a chemotaxis system involving nonlinear indirect signal secretion and logistic source, *Z. Angew. Math. Phys.*, **74** (2023), 237. https://doi.org/10.1007/s00033-023-02126-2

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