



Research article

Existence and uniqueness of the solution to Caputo-Hadamard differential equations with delay and nonlocal conditions

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Abstract: In this paper, we established the existence conditions for initial value problems involving the Caputo-Hadamard derivative operator with delay and nonlocal conditions. The problems were analyzed for various source functions. In a special case, we obtained a unique solution. These results were derived by combining the concept of the measure of non-compactness with appropriate applications of fixed point theorems. Additionally, we provided illustrative examples to support the theoretical findings.

Keywords: Caputo-Hadamard derivative; fractional differential; fractional equation; measure; noncompactness

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1. Introduction

Throughout this paper, let E be a separable Banach space with norm $|\cdot|_E$. Let J be a closed and bounded interval of $\mathbb{R}$. We denote by $C(J, E)$ the Banach space of continuous functions on J with values in E . In this paper, we study two initial value problems (fractional IVPs) for fractional differential equations. The first issue is the form

$${}_H^C D_{a^+}^p x(t) = f(t, x_t), \quad t \in (a, T], \quad (1.1)$$

$$x|_{[a-r, a]} = \varphi \in C([a-r, a], E), \quad (1.2)$$

and the second issue is the form

$${}_H^C D_{a^+}^p u(t) = f(t, u(t)), \quad t \in (a, T], \quad (1.3)$$

$$u(a) = -h(u) + u_a, \quad (1.4)$$

where ${}^C_H D_{a^+}^p$ is the Caputo-Hadamard derivative operator of order $p \in (0, 1)$, $a > 0$, $T > a$ (see Section 2).

For the problem (1.1)–(1.2), let $r > 0$, and $x_t \in C := C([-r, 0], E)$ be defined by $x_t(s) = x(t + s)$, $s \in [-r, 0]$, $x|_{[a-r, a]}$ is the restriction of x to $[a - r, a]$, and $f : (a, \infty) \times C \rightarrow E$ is a given function satisfying some assumptions:

(A1): There exists $T > 0$ such that the function $f(t, \cdot) : C \rightarrow E$ is continuous for almost all $t \in (a, \infty)$ and for $x \in C$, the function $f(\cdot, x) : (a, T] \rightarrow E$ is strongly measurable;

(A2): There is $T \in (a, \infty)$ such that for every $\tau > 0$, there exist constant $p_1 \in [0, p)$ and $\beta_\tau \in L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)$ such that $|f(t, x)|_E \leq \beta_\tau(t)$ for all $x \in C$ with $\|x\|_{C([-r, 0], E)} \leq \tau$ and a.e. $t \in [a, T]$ (here $\frac{1}{0} = \infty$).

(A3): There is $T \in (a, \infty)$ such that for every $\tau > 0$, there exist $p_1 \in [0, p)$, $\gamma \in (-\infty, p - p_1)$, and $\beta_\tau \in L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)$ such that

$$\left| \left(\log \frac{t}{a} \right)^\gamma f(t, x) \right|_E \leq \beta_\tau(t)$$

for all $x \in C$ with $\|x\|_* := \sup_{s \in [-r, 0]} |x(s)|_E \leq \tau$, and a.e. $t \in [a, T]$.

(A4): There exists $T > a$ such that f is the uniformly Lipschitz with respect to the second variable on $[a, T]$, i.e., there exists $L > 0$ such that

$$|f(t, u) - f(t, v)|_E \leq L\|u - v\|_{C([-r, 0], E)} \quad \text{for all } u, v \in C, t \in [a, T].$$

For problem (1.3)–(1.4), we make the following assumptions: u_a is an element of E , $f : (a, T] \times E \rightarrow E$ and $h : C([a, T], E) \rightarrow E$ are given functions satisfying the following conditions:

(H1): The function $f(t, \cdot) : E \rightarrow E$ is continuous for almost all $t \in [a, T]$ and for $x \in E$, the function $f(\cdot, x) : [a, T] \rightarrow E$ is strongly measurable;

(H2): There are $C_f, \mathcal{M}_f > 0$, $p_f \in [0, p)$, $\mu_f \in [0, 1)$, and $\beta_f \in L^{\frac{1}{p_f}}([a, T], \mathbb{R}_+)$ such that

$$|f(t, u(t))|_E \leq (C_f \|u\|^{\mu_f} + \mathcal{M}_f) \beta_f(t) \quad \text{a.e. on } [a, T]$$

and for all $u \in C([a, T], E)$;

(H3): There is a constant $\mathcal{K}_h \in [0, 1)$ such that

$$|h(u) - h(v)|_E \leq \mathcal{K}_h \|u - v\| \quad \text{for all } u, v \in C([a, T], E);$$

(H4): There are $C_h, \mathcal{M}_h > 0$, $q_2 \in [0, 1)$ such that

$$|h(u)|_E \leq C_h \|u\|^{q_2} + \mathcal{M}_h \quad \text{for all } u \in C([a, T], E)$$

(resp., there exists $C_h \in (0, 1)$ such that

$$|h(u)|_E \leq C_h (\|u\| + 1) \quad \text{for all } u \in C([a, T], E),$$

where $\|\cdot\| = \|\cdot\|_{C([a, T], E)}$.

Fractional differential equations have attracted considerable attention due to their numerous applications in various fields, such as biology, microchemistry, mechanics, computer science, and engineering (see [1–5] and references therein). Here, we highlight some studies that are more closely related to our topic, particularly those involving models with Riemann-Liouville, Grunwald-Letnikov, Riesz, Caputo, and Hadamard derivatives (e.g., [1, 4]).

We focus on the Hadamard derivative. Unlike classical fractional derivatives, the Hadamard derivative and integral involve logarithmic functions of arbitrary order, specifically $(\log t - \log s)$ instead of $(t - s)$. The kernel of the Hadamard derivative operator contains the logarithmic function $(\log t - \log s)^{\alpha-1}$ rather than $(t - s)^{\alpha-1}$. In recent years, the concept of the Caputo-Hadamard derivative has been introduced in several works (see [6–12]), proposing a modification that replaces the Hadamard derivative with the Caputo derivative. We summarize some relevant contributions to this research direction. Ma and Li [13] established fundamental properties of Hadamard-type fractional operators, including semigroup and reciprocal properties. The authors formulated well-posed conditions for Hadamard-type fractional differential equations (HTFDEs) of fractional order $p \in (0, 1)$:

$${}_H D_{a^+}^p x(t) = f(t, x)$$

with the initial condition $u(a) = u_a$. The problem was solved using the generalized weighted space $C_{\mu, \gamma, \log}([a, b], \mathbb{R})$.

C. Li and Z. Li, in [7], investigated the stability and logarithmic decay of solutions to fractional differential equations, where derivatives are considered in the sense of Hadamard and Caputo-Hadamard. Specifically, they examined the following forms:

$$\begin{aligned} {}_H D_{a^+}^p x(t) &= f(x), \quad t > a > 0; \\ {}_H D_{a^+}^{p-1} x(t)|_{t=a} &= x_a; \end{aligned}$$

and

$$\begin{aligned} {}^C_H D_{a^+}^p x(t) &= f(x), \quad t > a > 0; \\ x|_{t=a} &= x_a, \end{aligned}$$

where $0 < p < 1$, $x(t) \in \mathbb{R}^d$. Here, ${}_H D_{a^+}^p$ and ${}^C_H D_{a^+}^p$ denote the Hadamard fractional derivative of order p and the Caputo-Hadamard fractional derivative, respectively.

In [6], Li et al. investigated the blow-up and global existence of solutions to the space-time fractional diffusion equation involving the Caputo-Hadamard derivative:

$$\begin{aligned} {}^C_H D_{a^+}^p u(t, x) + (-\Delta)^s u(t, x) &= |u(t, x)|^{q-1}, \quad x \in \mathbb{R}^d, t > a > 0; \\ u(t, x) &= u_a(x), \quad x \in \mathbb{R}^d, \end{aligned}$$

where $p \in (0, 1)$, $0 < s < 1$, and $q > 1$.

A notable common feature of the aforementioned problems is that their solutions are considered over the interval $[a, T]$ with $a > 0$. In this paper, we aim to extend the analysis to equations involving the Caputo-Hadamard derivative on the domain $[a, T]$ with $a \geq 0$, meaning that $[a, T]$ may include 0. Building upon the aforementioned studies, we establish the existence and uniqueness of solutions to equations with deviating arguments (1.1)–(1.2), where the Caputo-Hadamard derivatives are of order

$p \in (0, 1)$. To achieve these results, we employ the concept of the measure of noncompactness, assuming that the source function satisfies the Carathéodory condition. In addition to the theoretical findings, we provide illustrative examples and discuss mild solutions under weaker assumptions on the source function.

Let us discuss some key ideas and techniques employed in this paper.

- The right-hand derivative at 0 plays an important role in applications such as computing the instantaneous velocity or acceleration of a motion. However, addressing the existence of the right-sided Caputo-Hadamard derivative at 0 presents a challenge. This difficulty arises from estimating the convergence of the integral $\int_0^t K(t, s)u(s)ds$, where the kernel $K(t, s)$ involves the function $\left(\log \frac{t}{s}\right)^{p-1}$. In this paper, we do not fully resolve this issue; however, we identify a special case in which this derivative exists (see Remark 3.10). Additionally, a common assumption on the source function is that $f : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous in both variables. In contrast, we consider a more general setting where $f : (a, b] \times E \rightarrow E$ is only continuous with respect to the second variable, which takes values in a Banach space. Some previous studies have imposed the condition $f(\cdot, u) \in C_{\gamma, \log}([a, b], \mathbb{R})$; we replace this with the weaker assumption $f(\cdot, u) \in L_{\gamma, \log}^p([a, b], E)$ for some $p \in [1, \infty]$. As a result, our approach is more adaptable to infinite systems of equations, as illustrated in Section 4.
- When applying fixed point theorems (such as Banach's or Schauder's fixed point theorem) to an operator F mapping a subset $B \subset E$ into the Banach space E , it is often challenging to verify the following conditions: B is a closed set, $F(B) \subset B$, and F is a compact operator. To address these difficulties, we utilize the fixed point theorem for α -condensing operators.

To the best of our knowledge, this problem has not yet been studied. The main contribution of this paper lies in the application of the measure of noncompactness, combined with appropriate analytical techniques, to establish the existence and uniqueness of solutions to the Cauchy problem involving fractional derivatives.

The rest of this paper is organized as follows. In the next section, we recall the definitions of the Hadamard integral, Hadamard derivative, and Caputo-Hadamard derivative, along with the properties that will be used later. Section 3 presents the existence and uniqueness of local solutions to the problem (1.1)–(1.2) and establishes the existence of a non-local solution to the problem (1.3)–(1.4). Finally, we provide illustrative examples.

2. Preliminaries

In this section, we review Caputo-Hadamard derivatives and several related properties that will be used in the next section. For the convenience of the reader, some concepts from functional analysis are also recalled here. In addition to the notations introduced in Section 1, we introduce the following: the family of bounded subsets of the normed space X is denoted by $b(X)$; the space $C(J, E)$ of continuous functions on J is always considered with the norm

$$\|x\|_{C(J, E)} = \sup\{|x(t)|_E : t \in J\},$$

unless otherwise stated. The space $C^n(J, E)$ (with $n \in \mathbb{N} \setminus \{0\}$) consists of all functions u such that $u \in C^{n-1}(J, E)$, where $C^0(J, E) := C(J, E)$. The space $AC(J, E)$ denotes the set

of functions that are absolutely continuous on J . For $n \in \mathbb{N} \setminus \{0\}$, we define $AC_\delta^n(J, E) = \{f : J \rightarrow E \mid \delta^{n-1}f \in AC(J, E)\}$, here $\delta = t \frac{d}{dt}$. For $q \in [1, \infty)$, we denote

$$L^q(J, E) = \left\{ f : J \rightarrow E \mid \|f\|_{L^q} := \left(\int_J |f(s)|_E^q \right)^{\frac{1}{q}} < \infty \right\}, \quad 1 \leq q < \infty;$$

$$L^\infty(J, E) = \{f : J \rightarrow E \mid \|f\|_{L^\infty} := \text{esssup} \{|f(s)|_E : s \in J\} < \infty\}.$$

Definition 2.1. [6] Let $a \geq 0$ and u be a locally integrable function on (a, ∞) . The Hadamard integral and Hadamard derivative of order $p > 0$ at $t > a$ of the function u are defined by

$${}_H D_{a^+}^{-p} u(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{1}{\tau} \left(\log \frac{t}{\tau} \right)^{p-1} u(\tau) d\tau$$

and

$${}_H D_{a^+}^p u(t) = \frac{1}{\Gamma(n-p)} \delta^n \left(\int_a^t \frac{1}{\tau} \left(\log \frac{t}{\tau} \right)^{n-p-1} u(\tau) d\tau \right).$$

respectively, where $n = [p] + 1$, $\delta = t \frac{d}{dt}$ and the gamma function Γ is defined by

$$\Gamma(p) = \int_0^\infty s^{p-1} e^{-s} ds, \quad p > 0.$$

Definition 2.2. [6] The Caputo-Hadamard derivative of order $p > 0$ is defined by

$${}_H^C D_{a^+}^p u(t) = {}_H D_{a^+}^p \left[u(t) - \sum_{j=0}^{n-1} \frac{\delta^j u(a)}{j!} \left(\log \frac{t}{a} \right)^j \right], \quad t > a > 0,$$

here $n = [p] + 1$, $(\delta f)(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{\log(t+h) - \log(t)} = t \frac{d}{dt} u(t)$. Particularly, $u \in C([a, \infty), E)$,

$${}_H^C D_{a^+}^p u(t) = {}_H D_{a^+}^p [u(t) - u(a)], \quad {}_H^C D_{0^+}^p u(t) = {}_H D_{0^+}^p [u(t) - u(0)] \text{ if } p \in (0, 1).$$

We need to use the following properties, which is a special case in [14].

Lemma 2.3. [14, Page 116, Property 2.28] If $p > 0$ and $u \in L^q((a, b), E)$ for some $q \in [1, \infty]$, then,

$${}_H D_{a^+}^p \left({}_H D_{a^+}^{-p} u(t) \right) = u(t).$$

Lemma 2.4. [14, Page 117, Lemma 2.35,] Let $p > 0$, $0 < a < b < \infty$. If $x \in \mathcal{J}_{a^+}^p(L^q((a, b), E))$ for some $q \in [1, \infty]$, then

$${}_H D_{a^+}^{-p} ({}_H D_{a^+}^p x)(t) = x(t),$$

where $\mathcal{J}_{a^+}^p(L^q((a, b), E)) = \{ {}_H D_{a^+}^{-p} \psi : \psi \in L^q((a, b), E) \}$.

Lemma 2.5. [14, 15, Banach fixed point theorem] Let W be a non-empty closed subset of Banach space X and a sequence $\{q_n\}$ with $q_n \rightarrow 0$. Assume that the map $A : W \rightarrow W$ satisfies

$$\|A^n(u) - A^n(v)\|_X \leq q_n \|u - v\|_X \text{ for all } n \in \mathbb{N}.$$

Then, A has a unique fixed point in $x_* \in W$, i.e., $Ax_* = x_*$. Further, $x_n \rightarrow x_*$, here $x_{n+1} = Ax_n$, $x_0 \in W$.

Lemma 2.6. (Schauder fixed point theorem). *Let X be a Banach space and $\mathcal{B}$ be a convex, closed, bounded subset of X . If $A : \mathcal{B} \rightarrow X$ is a continuous operator such that $A(\mathcal{B}) \subset \mathcal{B}$ and $A(\mathcal{B})$ is relatively compact, then A has at least one fixed point in $\mathcal{B}$.*

Let B be a bounded subset of the Banach space X , and α be a measure of noncompactness defined by

$$\alpha(B) = \inf \{ \epsilon > 0 : B \text{ has a finite } \epsilon\text{-net} \}.$$

The measure of noncompactness α has the following properties that will be used later:

- (i) If $B_1, B_2 \in b(X)$, $B_1 \subset B_2$, $\alpha(B_1) \subset \alpha(B_2)$;
- (ii) If $B \in b(X)$ and $x \in X$, then $\alpha(\{x\} \cup B) = \alpha(B)$;
- (iii) If $B \in b(X)$, then $\alpha(B) = 0$ if and only if B is relatively compact in X .
- (iv) If $B_1, B_2 \in b(X)$, then $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$.
- (v) If $B \in b(X)$ and $\lambda \in \mathbb{R}$, then $\alpha(\lambda B) \leq |\lambda| \alpha(B)$.

Definition 2.7. [16, Definition 1.10, p.15] A continuous bounded mapping $\mathcal{F} : \Omega \subset X \rightarrow X$ is said to be α -Lipschitz if there exists $\mathcal{K} \geq 0$ such that

$$\alpha(\mathcal{F}(B)) \leq \mathcal{K} \alpha(B) \text{ for all } B \in b(\Omega). \quad (2.1)$$

In addition, if $\mathcal{K} < 1$, then $\mathcal{F}$ is called a strict α -contraction. We say that $\mathcal{F}$ is α -condensing if

$$\alpha(\mathcal{F}(B)) < \alpha(B) \text{ for all } B \in b(\Omega) \text{ with } \alpha(B) > 0.$$

We observe that if $\mathcal{F}$ is a compact mapping, then (2.1) is satisfied with $\mathcal{K} = 0$. Hence, $\mathcal{F}$ is α -Lipschitz and, in particular, α -condensing.

The following lemma recalls a result concerning the existence of fixed points for α -condensing mappings.

Lemma 2.8. [16, Theorem 1.2, p.17] *Let $\mathcal{F}$ be α -condensing. If the set*

$$S = \{x \in X : \exists \lambda \in [0, 1], x = \lambda \mathcal{F}(x)\}$$

is bounded, then $\mathcal{F}$ has at least one fixed point. In addition, there exists $r > 0$ such that $\|x\| \leq r$ for all $x \in S$.

Lemma 2.9. (Arzela-Ascoli theorem). *Let J be a compact metric space and $(E, \|\cdot\|)$ be a normed space. Then, set $\mathcal{M} \subset C(J, E)$ is relatively compact if and only if the following conditions are fulfilled:*

- (i) $\mathcal{M}$ is pointwise bounded, i.e., for every $x \in J$, the set $\{f(x) : f \in \mathcal{M}\}$ is bounded;
- (ii) $\mathcal{M}$ is equicontinuous, i.e., for every $x \in J$ and $\epsilon > 0$, has a neighborhood $\mathcal{U}_x$ such that

$$\|f(x) - f(y)\| < \epsilon \text{ for all } y \in \mathcal{U}_x, f \in \mathcal{M}.$$

Lemma 2.10. *Let $\{G_n\}$ be a sequence in $L^q(J, \mathbb{R}_+)$ ($q \geq 1$), and $G \in L^q(J, \mathbb{R}_+)$ such that*

(i) $G_1(s) \leq G_2(s) \leq \dots \leq G_n(s) \leq \dots$ (resp., $G_1(s) \geq G_2(s) \geq \dots \geq G_n(s) \geq \dots$) for a.e. $s \in J$;

(ii) $\lim_{n \rightarrow \infty} G_n(s) = G(s)$ for a.e. $s \in J$.

Then $\int_J |G_n(s) - G(s)|^q ds \rightarrow 0$, as $n \rightarrow \infty$.

Proof. Define $\Phi_n = |G_n - G|^q$, $\varphi_n = \Phi_1 - \Phi_n$. Then, $\Phi_n, \varphi_n \in L^1(J, \mathbb{R}_+)$, and $\varphi_n(s) \nearrow \Phi_1(s)$ for a.e. $s \in J$. Using Beppo Levi's theorem, we get $\int_J \varphi_n(s) ds \rightarrow \int_J \Phi_1(s) ds$, as $n \rightarrow \infty$. Therefore

$$\int_J |\Phi_n(s)| ds = \int_J |\varphi_n(s) - \Phi_1(s)| ds = \int_J (\Phi_1(s) - \varphi_n(s)) ds \rightarrow 0.$$

The lemma is proven. $\square$

Lemma 2.11. Let $0 < p < 1$, $p_1 \in [0, p)$, $\infty > t > b \geq a > 0$ and $G(t, \cdot) : (b, \infty) \rightarrow \mathbb{R}$, defined by $G(t, s) = \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} \chi_{[a,t]}(s)$ (here $\chi_A(s) = 1$ if $s \in A$, and $\chi_A(s) = 0$ if $s \notin A$). Then

(i) $G(t, \cdot) \in L^{\frac{1}{1-p_1}}((b, t), \mathbb{R}_+)$ for all $t > b \geq a > 0$ and

$$\|G(t, \cdot)\|_{L^{\frac{1}{1-p_1}}([b,t], \mathbb{R})}^{\frac{1}{1-p_1}} = \int_b^t (G(t, s))^{\frac{1}{1-p_1}} ds \leq C(t, b, p, p_1),$$

where

$$C(t, b, p, p_1) = \frac{(1 - p_1)(t - b)^{\frac{p-p_1}{1-p_1}}}{(p - p_1)b^{\frac{p}{1-p_1}}}.$$

(ii) For $a < x \leq T$, we have

$$\lim_{t \rightarrow x} \int_a^T |G(t, s) - G(x, s)|^{\frac{1}{1-p_1}} ds = 0.$$

Proof. (i): Using variable substitution $z = \log \frac{t}{s}$, we get

$$\begin{aligned} \int_b^t \left(\frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} \right)^{\frac{1}{1-p_1}} ds &= \int_b^t \left(\frac{1}{s} \right)^{q_1} \left(\log \frac{t}{s} \right)^{b_1} \frac{1}{s} ds \\ &= \int_0^{\log \frac{t}{b}} \left(\frac{e^z}{t} \right)^{q_1} z^{b_1} dz \\ &\leq \left(\frac{1}{b} \right)^{q_1} \frac{1}{b_1 + 1} \left(\log \frac{t}{b} \right)^{b_1+1}, \end{aligned}$$

where $q_1 = \frac{p_1}{1-p_1} > 0$, $b_1 = \frac{p-1}{1-p_1} \in (-1, 0)$. The final evaluation is based on the inequality $\log(1 + \eta) \leq \eta$ for all $\eta \geq 0$. Hence, (i) holds.

(ii): For $x \in (a, T]$ and $T \geq t \geq x$ (resp., $x > t$), we define $\Phi_t(x, s) = |G(t, s) - G(x, s)|^{\frac{1}{1-p_1}}$. We have

$$|\Phi_t(x, s)| \leq (|G(t, s)| + |G(x, s)|)^{\frac{1}{1-p_1}}$$

$$\leq 2G(x, s)^{\frac{1}{1-p_1}} \quad (\text{resp., } |\Phi_t(s)| \leq 2G(t, s)^{\frac{1}{1-p_1}}).$$

Since $G(t, \cdot) \in L^{\frac{1}{1-p_1}}((a, T), \mathbb{R}_+)$ (resp., $G(x, \cdot) \in L^{\frac{1}{1-p_1}}((a, T), \mathbb{R}_+)$), we get $\Phi_t(x, \cdot) \in L^1((a, T), \mathbb{R})$. Since $G(t, s) \nearrow G(x, s)$, as $t \searrow x^+$ (resp., $G(t, s) \searrow G(x, s)$, as $t \nearrow x^-$) for a.e. $s \in (a, T)$. According to Lemma 2.10 with $t := t_n \rightarrow x^+$, we get

$$\lim_{t \rightarrow x^+} \int_a^T \Phi_t(x, s) ds = 0 \quad \left(\text{resp., } \lim_{t \rightarrow x^-} \int_a^T \Phi_t(x, s) ds = 0 \right).$$

Hence, (ii) holds. $\square$

Lemma 2.12. Assume that (A1) and (A3) are satisfied. Let $u \in C([a-r, T], E)$ with $\|u_s\|_* \leq \tau$. We have

(i) if $T \geq x > b \geq a > 0$, then

$$\frac{1}{\Gamma(p)} \left| \int_b^x \frac{1}{s} \left(\log \frac{x}{s} \right)^{p-1} f(s, u_s) ds \right|_E \leq C(\gamma, b, p, p_1) \|\beta_\tau\|_{L^{\frac{1}{p_1}}((a, T), \mathbb{R})} (x-b)^{p-\gamma-p_1},$$

$$\text{where } C(\gamma, b, p, p_1) = \frac{1}{\Gamma(p)} \left(\frac{1}{b} \right)^{p-\gamma} \left(\frac{\Gamma(1+b_1)\Gamma(1+c_1)}{\Gamma(2+b_1+c_1)} \right)^{1-p_1}, \quad b_1 = \frac{p-1}{1-p_1}, \quad c_1 = \frac{-\gamma}{1-p_1};$$

(ii) the function $Fu(t) = \frac{1}{\Gamma(p)} \int_a^t G(t, s) f(s, u_s) ds$ is continuous on $(a, T]$.

Proof. (i): Define $\phi(x, s) := G(x, s) \left(\log \frac{x}{s} \right)^{-\gamma}$ for $x, s \in (a, T]$. We rewrite

$$\begin{aligned} \left| \int_b^x \frac{1}{s} \left(\log \frac{x}{s} \right)^{p-1} f(s, u_s) ds \right|_E &= \left| \int_b^x \phi(x, s) \left(\log \frac{s}{a} \right)^\gamma f(s, u_s) ds \right|_E \\ &\leq C_0 \left(\int_b^x \left(\frac{1}{s} \right)^{\frac{p_1}{1-p_1}} \left(\log \frac{x}{s} \right)^{\frac{p-1}{1-p_1}} \left(\log \frac{s}{a} \right)^{\frac{-\gamma}{1-p_1}} \left(\frac{1}{s} \right) ds \right)^{1-p_1}, \\ &\leq C_0 \left(\int_b^x \left(\frac{1}{s} \right)^{q_1} \left(\log \frac{x}{s} \right)^{b_1} \left(\log \frac{s}{b} \right)^{c_1} \left(\frac{1}{s} \right) ds \right)^{1-p_1}, \end{aligned}$$

where $C_0 = \|\beta_\tau\|_{L^{\frac{1}{p_1}}((a, T), E)}$, $q_1 = \frac{p_1}{1-p_1}$. Since $\gamma < p - p_1 < 1 - p_1$, we have $b_1 + c_1 + 1 > 0$ and $1 + c_1 > 0$.

Using variable substitution $z = \frac{\log \frac{x}{b}}{\log \frac{s}{b}}$, we have

$$\begin{aligned} \left| \int_b^x \frac{1}{s} \left(\log \frac{x}{s} \right)^{p-1} f(s, u_s) ds \right|_E &\leq C_0 \left(\left(\frac{1}{b} \right)^{q_1} \left(\log \frac{x}{b} \right)^{b_1+c_1+1} \int_0^1 (1-z)^{b_1} z^{c_1} dz \right)^{1-p_1} \\ &\leq C_0 \left(\frac{1}{b} \right)^{p_1} \left(\log \frac{x}{b} \right)^{p-\gamma-p_1} \left(\frac{\Gamma(1+b_1)\Gamma(1+c_1)}{\Gamma(2+b_1+c_1)} \right)^{1-p_1}. \end{aligned}$$

Thus, (i) holds.

(ii): Assume $t, t_0 \in (a, T]$. We have

$$|Fu(t) - Fu(t_0)|_E = \frac{1}{\Gamma(p)} \left| \int_a^t G(t, s) f(s, u_s) ds - \int_a^{t_0} G(t_0, s) f(s, u_s) ds \right|_E.$$

• Case $t_0 < t$. We write

$$\int_a^t G(t, s) f(s, u_s) ds = \int_a^{t_0} G(t, s) f(s, u_s) ds + \int_{t_0}^t G(t, s) f(s, u_s) ds,$$

$$|Fu(t) - Fu(t_0)|_E \leq A(t, t_0) + B(t, t_0),$$

where

$$A(t, t_0) = \frac{1}{\Gamma(p)} \int_{t_0}^t |G(t, s) f(s, u_s)|_E ds$$

$$\leq C(\gamma, a, p, p_1) \|\beta_\tau\|_{L^{\frac{1}{p_1}}((a, T), \mathbb{R})} (t - t_0)^{p - \gamma - p_1} \rightarrow 0, \text{ as } t \rightarrow t_0^+;$$

and

$$B(t, t_0) = \frac{1}{\Gamma(p)} \int_a^{t_0} |\phi(t, s) - \phi(t_0, s)| \left| \left(\log \frac{s}{a} \right)^{-\gamma} f(s, u_s) \right|_E ds$$

$$\leq \frac{1}{\Gamma(p)} \|\beta_\tau\|_{L^{\frac{1}{p_1}}((a, T), \mathbb{R})} \left(\int_a^{t_0} |\phi(t, s) - \phi(t_0, s)|^{\frac{1}{1-p_1}} ds \right)^{1-p_1}.$$

Using Lemma 2.10 with $G_{t_n}(s) := \phi_{t_n}(t_0, s)$, we get $B(t_n, t_0) \rightarrow 0$, as $t_n \searrow t_0^+$. Hence

$$|Fu(t) - Fu(t_0)|_E \rightarrow 0, \text{ as } t \rightarrow t_0^+.$$

• Case $t < t_0$. By a similar argument, we have $|Fu(t) - Fu(t_0)|_E \rightarrow 0$, as $t \rightarrow t_0^-$. $\square$

3. Main results

3.1. Local existence and uniqueness

We begin this section with the definition of a solution to the problem (1.1)–(1.2). Let $a > 0$, $r > 0$, $T > a$, $\varphi \in C([a - r, a], E)$, and let $f : [a, \infty) \times C \rightarrow E$.

Definition 3.1. A function $x : [a - r, T] \rightarrow E$ is said to be a solution for fractional IVP (1.1)–(1.2) on $[a - r, T]$ for some $T \in [a, \infty)$ if

- (i) x is continuous on $[a, T]$;
- (ii) $x|_{[a-r, a]} = \varphi$;
- (iii) x satisfies the equation (1.1).

Proposition 3.2. Assume (A1)–(A2), $p \in (0, 1)$ and $R > a > 0$. A function $x \in C([a - r, T], E)$, with $x|_{[a, T]} \in AC([a, T], E)$, is an integral solution of the fractional IVP (1.1)–(1.2) on $[a - r, T]$ for some $T \in [a, R)$ if and only if x satisfies the following conditions

$$\begin{cases} x(\tau) = \varphi(\tau), & \text{for } \tau \in [a - r, a], \\ x(t) = \varphi(a) + \frac{1}{\Gamma(p)} \int_a^t \frac{1}{\tau} \left(\log \frac{t}{\tau} \right)^{p-1} f(\tau, x_\tau) d\tau, & \text{for } t \in (a, T]. \end{cases} \quad (3.1)$$

Proof. Assume that $x \in C([a-r, T], E)$ satisfies (3.1). Since the function $t \mapsto x_t$ is continuous on $[a, T]$, according to (A1), the function $f(\cdot, x_t)$ is measurable in $[a, \infty)$.

From assumption (A2) and using Holder's inequality and Lemma 2.11, we obtain

$$\begin{aligned} \int_a^t \left| \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} f(s, x_s) \right|_E ds &\leq \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \left(\int_a^t \left(\log \frac{t}{s} \right)^{\frac{p-1}{1-p_1}} s^{-\frac{1}{1-p_1}} ds \right)^{1-p_1} \\ &\leq C^{1-p_1}(T, a, p, p_1) \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} < \infty. \end{aligned}$$

Hence, the function $s \mapsto \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} f(s, x_s)$ is Bochner integrable on $[a, t]$ for $R > t > a$.

We now prove that x satisfies (1.1). Indeed, noting $f(t, x_t) \in L^{\frac{1}{p_1}}((a, T), E)$ and by using Lemma 2.3, we have

$$\begin{aligned} {}^C_H D_{a^+}^p x(t) &= {}_H D_{a^+}^p [x(t) - x(a)] \\ &= {}_H D_{a^+}^p [x(t) - \varphi(a)] \\ &= {}_H D_{a^+}^p \left({}_H D_{a^+}^{-p} f(t, x_t) \right) \\ &= f(t, x_t) \quad (t > a). \end{aligned}$$

Therefore, x is a solution of fractional IVP (1.1)–(1.2). In contrast, assume that $x \in C([a-r, T])$ is a solution to (1.1)–(1.2). Then, ${}_H D_{a^+}^p x(t) = f(t, x_t)$. Since $x \in AC([a, T], E)$, means that

$$\phi(\cdot) := {}_H D_{a^+}^p (x - x(a))(\cdot) = f(\cdot, x(\cdot)) \in L^{\frac{1}{p_1}}((a, T), E),$$

and ${}_H D_{a^+}^{-p} \phi(\cdot) \in \mathcal{J}_{a^+}^p(L^{\frac{1}{p_1}}(a, T), E)$. By using Lemma 2.4, we have ${}_H D_{a^+}^{-p} ({}_H D_{a^+}^p (x - x(a))) = x - x(a)$. Hence

$$x(t) - x(a) = {}_H D_{a^+}^{-p} f(t, x_t) = \frac{1}{\Gamma(p)} \int_a^t \frac{1}{\tau} \left(\log \frac{t}{\tau} \right)^{p-1} f(\tau, x_\tau) d\tau.$$

It is obvious that x satisfies the equation (3.1), which completes the proof. $\square$

Theorem 3.3. Assume (A1)–(A2). Then, for any $\varphi \in C([a-r, a], E)$, there is $T \in (a, \infty)$ such that the fractional IVP (1.1)–(1.2) has at least one integral solution $x \in C([a-r, T], E)$ represented by

$$x(t) = \varphi(a) + \frac{1}{\Gamma(p)} \int_a^t \frac{1}{\tau} \left(\log \frac{t}{\tau} \right)^{p-1} f(\tau, x_\tau) d\tau, t \in (a, T].$$

Proof. Let h be any positive number. Similar to the demonstration of Proposition 3.2, we can find T mentioned in (A1)–(A2), and condition

$$\frac{C^{1-p_1}(T, a, p, p_1)}{\Gamma(p)} \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \leq h, \quad (3.2)$$

is satisfied. Consider the set $\mathcal{B}_h$ defined as follows

$$\mathcal{B}_h = \left\{ x \in C([a-r, T], E) : x|_{[a-r, a]} = \varphi, \sup_{s \in [a, T]} |x(s) - \varphi(a)|_E \leq h \right\}.$$

First, we see that $\mathcal{B}_h \ni \varphi^0$, φ^0 is defined by

$$\varphi^0(t) = \begin{cases} \varphi(t), & t \in [a-r, a], \\ \varphi(a), & t \in [a, T]. \end{cases}$$

Let $\{x^{(n)}\} \subset \mathcal{B}_h$ with $x^{(n)} \rightarrow x$. Using evaluation

$$\begin{aligned} \sup_{s \in [a, T]} |x(s) - \varphi(a)|_E &\leq \sup_{s \in [a, T]} |x(s) - x^{(n)}(s)|_E + \sup_{s \in [a, T]} |x^{(n)}(s) - \varphi(a)|_E \\ &\leq \sup_{s \in [a-r, T]} |x(s) - x^{(n)}(s)|_E + h \\ &\leq \|x - x^{(n)}\|_{C([a-r, T], E)} + h \rightarrow h \text{ (as } n \rightarrow \infty), \end{aligned}$$

we get $x \in \mathcal{B}_h$. It is easy to check the convexity of $\mathcal{B}_h$. Thus, $\mathcal{B}_h$ is a convex, closed subset of $C([a-r, T], E)$. Define the operator A as follows:

$$Ax(t) = \begin{cases} \varphi(t), & \text{if } t \in [a-r, a], \\ \varphi(a) + \frac{1}{\Gamma(p)} \int_a^t \frac{1}{\tau} \left(\log \frac{t}{\tau}\right)^{p-1} f(\tau, x_\tau) d\tau, & \text{if } t \in (a, T]. \end{cases}$$

Next, we show that $Ax \in C([a-r, T], E)$ for $x \in \mathcal{B}_h$. It just remains to prove that $Ax \in C([a, T], E)$. Let t_0 be a fixed point in $[a, T]$. We consider the following case:

- $t_0 = a$: Using Holder's inequality and applying Lemma 2.11, we get

$$\begin{aligned} |Ax(t) - Ax(t_0)|_E &= \frac{1}{\Gamma(p)} \left| \int_a^t G(t, s) f(s, x_s) ds \right|_E \\ &\leq \frac{1}{\Gamma(p)} \left| \int_a^t (G(t, s))^{\frac{1}{1-p_1}} \beta_\tau(s) ds \right|_E \\ &\leq \frac{\|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)}}{\Gamma(p)} \left(\int_a^t (G(t, s))^{\frac{1}{1-p_1}} ds \right)^{1-p_1} \\ &\leq \frac{\|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)}}{\Gamma(p)} \frac{(1-p_1)^{1-p_1} (t-a)^{p-p_1}}{a^p (p-p_1)^{1-p_1}} \rightarrow 0, \text{ as } t \rightarrow a^+. \end{aligned} \quad (3.3)$$

- $t_0 \in (a, T]$: If $t > t_0$ (resp., $t < t_0$), by representing

$$\begin{aligned} \int_a^t G(t, s) f(s, x_s) ds &= \int_a^{t_0} G(t, s) f(s, x_s) ds + \int_{t_0}^t G(t, s) f(s, x_s) ds \\ \left(\text{resp., } \int_a^{t_0} G(t_0, s) f(s, x_s) ds &= \int_a^t G(t_0, s) f(s, x_s) ds + \int_t^{t_0} G(t_0, s) f(s, x_s) ds \right), \end{aligned}$$

we obtain

$$|Ax(t) - Ax(t_0)|_E \equiv \frac{1}{\Gamma(p)} (\mathcal{A}(t, t_0) + \mathcal{B}(t, t_0)), \quad (3.4)$$

where

$$\begin{aligned}\mathcal{A}(t, t_0) &:= \left| \int_{t_0}^t (G(t, s) f(s, x_s)) ds \right|_E, \\ \mathcal{B}(t, t_0) &:= \left| \int_a^{t_0} (G(t_0, s) - G(t, s)) f(s, x_s) ds \right|_E,\end{aligned}\quad (3.5)$$

and

$$\begin{aligned}\mathcal{A}(t, t_0) &\leq \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \left(\int_{t_0}^t (G(t, s))^{\frac{1}{1-p_1}} ds \right)^{1-p_1} \\ &\leq \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \frac{(1-p_1)^{1-p_1} (t-t_0)^{p-p_1}}{a^p (p-p_1)^{1-p_1}} \rightarrow 0, \text{ as } t \rightarrow t_0^+.\end{aligned}\quad (3.6)$$

In addition, from (ii) of Lemma 2.11, we have the following assessment:

$$\begin{aligned}\mathcal{B}(t, t_0) &\leq \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \left(\int_a^{t_0} |G(t_0, s) - G(t, s)|^{\frac{1}{1-p_1}} ds \right)^{1-p_1} \\ &\leq \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \left(\int_a^T |G(t_0, s) - G(t, s)|^{\frac{1}{1-p_1}} ds \right)^{1-p_1} \rightarrow 0,\end{aligned}\quad (3.7)$$

as $t \rightarrow t_0^+$. From (3.4)–(3.7), it follows that $|Ax(t) - Ax(t_0)|_E \rightarrow 0$, as $t \rightarrow t_0^+$.

If $t < t_0$, by a similar argument, we also have $|Ax(t) - Ax(t_0)|_E \rightarrow 0$, as $t \rightarrow t_0^-$. So we conclude $Ax \in C([a-r, T], E)$.

We will prove that A has at least one fixed point in $\mathcal{B}_h$. This is demonstrated by the following steps:

Step 1: We will show that $A\mathcal{B}_h \subset \mathcal{B}_h$. For $x \in \mathcal{B}_h$, the function $t \mapsto Ax(t)$ is continuous on $[a-r, T]$. For $t \in [a, T]$, by Holder's inequality and (3.2), we have

$$\begin{aligned}|Ax(t) - \varphi(a)|_E &\leq \frac{1}{\Gamma(p)} \int_a^t \left| \left(\log \frac{t}{\tau} \right)^{p-1} \frac{f(\tau, x_\tau)}{\tau} \right|_E d\tau \\ &\leq \frac{1}{\Gamma(p)} \left(\int_a^t \left(\log \frac{t}{\tau} \right)^{\frac{p-1}{1-p_1}} \tau^{-\frac{1}{1-p_1}} d\tau \right)^{1-p_1} \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \\ &\leq \frac{C^{1-p}(T, a, p, p_1)}{\Gamma(p)} \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \\ &\leq h.\end{aligned}\quad (3.8)$$

Thus, $\sup_{t \in [a, T]} |Ax(t) - \varphi(a)|_E \leq h$, this implies $A(\mathcal{B}_h) \subset \mathcal{B}_h$.

Step 2: Prove that A is a continuous operator on $\mathcal{B}_h$. Assume sequence $\{x^{(n)}\} \subset \mathcal{B}_h$ with $x^{(n)} \rightarrow x$ on $\mathcal{B}_h$. By the hypothesis (A1) and the fact that $x_t^{(n)} \rightarrow x_t$, $t \in [a, T]$, we get

$$f(\tau, x_\tau^{(n)}) \rightarrow f(\tau, x_\tau) \quad \text{a.e. } \tau \in [a, T] \quad \text{as } n \rightarrow \infty.$$

This leads to the estimate that

$$|f(\tau, x_\tau^{(n)}) - f(\tau, x_\tau)|_E \leq 2\beta_\tau(\tau).$$

Using Holder's inequality and applying Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} |Ax^{(n)}(t) - Ax(t)|_E &\leq \frac{1}{\Gamma(p)} \int_a^t \left| \frac{1}{\tau} \left(\log \frac{t}{\tau} \right)^{p-1} \right| |f(\tau, x_\tau^{(n)}) - f(\tau, x_\tau)|_E d\tau \\ &\leq \frac{1}{\Gamma(p)} \int_a^T |G(t, \tau)| |f(\tau, x_\tau^{(n)}) - f(\tau, x_\tau)|_E d\tau \\ &\leq \frac{c}{\Gamma(p)} \left(\int_a^T |f(\tau, x_\tau^{(n)}) - f(\tau, x_\tau)|_E^{\frac{1}{p_1}} d\tau \right)^{p_1} \rightarrow 0 \end{aligned}$$

(where $c = C^{1-p_1}(T, a, p, p_1) < \infty$) independently of t , as $n \rightarrow \infty$. It implies that $\|Ax^{(n)} - Ax\|_{C([a-r, T], E)} \rightarrow 0$, thus $Ax^{(n)} \rightarrow Ax$ in $\mathcal{B}_h$.

Step 3: Shows that A is a compact operator from $\mathcal{B}_h$ to $C([a-r, T], E)$.

Let $\{x^{(n)}\}$ be a sequence in $\mathcal{B}_h$. We will show that $\mathcal{M} = \{A(x^{(n)}) : n \geq 1\}$ is a relatively compact subset of $C([a, T], E)$ by checking the hypothesis of the Azela-Ascoli theorem.

• For $t \in [a-r, T]$. It is clear that

$$|Ax^{(n)}(t)|_E \leq \|\varphi\|_{C([a-r, a], E)} + \frac{1}{\Gamma(p)} \frac{(1-p_1)^{1-p_1} (t-a)^{p-p_1}}{a^p (p-p_1)^{1-p_1}} \quad \text{for all } n \geq 1.$$

It implies that $\mathcal{M}$ is uniformly bounded at t .

- Put $y^{(n)} = A(x^{(n)})$. For $t, t_0 \in [a-r, T]$ with $t \neq t_0$ (t is near t_0), we consider the following cases:
- If $a-r \leq t < t_0 \leq a$ or $a-r \leq t_0 < t \leq a$, we have

$$|y^{(n)}(t_0) - y^{(n)}(t)|_E = |\varphi(t_0) - \varphi(t)|_E. \quad (3.9)$$

- If $a < t_0 < t \leq T$. Then, by representing

$$\int_a^t G(t, s) f(s, x_s^{(n)}) ds = \int_a^{t_0} G(t, s) f(s, x_s^{(n)}) ds + \int_{t_0}^t G(t, s) f(s, x_s^{(n)}) ds,$$

and using Holder inequality, we have

$$|y^{(n)}(t) - y^{(n)}(t_0)|_E \leq \frac{1}{\Gamma(p)} \mathcal{A}(t, t_0) + \frac{1}{\Gamma(p)} \mathcal{B}(t, t_0), \quad (3.10)$$

where

$$\begin{aligned} \mathcal{A}(t, t_0) &= \left| \int_{t_0}^t G(t, s) f(s, x_s^{(n)}) ds \right|_E, \\ \mathcal{B}(t, t_0) &= \left| \int_a^{t_0} (G(t_0, s) - G(t, s)) f(s, x_s^{(n)}) ds \right|_E. \end{aligned}$$

Using Lemma 2.11, we have

$$\mathcal{A}(t, t_0) \leq \frac{(1-p_1)^{1-p_1} (t-t_0)^{p-p_1}}{a^p (p-p_1)^{1-p_1}} = c_0 (t-t_0)^{p-p_1}. \quad (3.11)$$

Noting $0 < \left(\log \frac{t}{s}\right)^{p-1} < \left(\log \frac{t_0}{s}\right)^{p-1}$, we have

$$\begin{aligned} \mathcal{B}(t, t_0) &\leq \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \left(\int_a^{t_0} |G(t_0, s) - G(t, s)|^{\frac{1}{1-p_1}} ds \right)^{1-p_1} \\ &\leq \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)} \left(\int_a^T |G(t_0, s) - G(t, s)|^{\frac{1}{1-p_1}} ds \right)^{1-p_1}. \end{aligned}$$

Therefore

$$\begin{aligned} &|y^{(n)}(t) - y^{(n)}(t_0)|_E \\ &\leq \frac{1}{\Gamma(p)} \left\{ c_0(t - t_0)^{p-p_1} + C \left(\int_a^T |G(t_0, s) - G(t, s)|^{\frac{1}{1-p_1}} ds \right)^{1-p_1} \right\}, \end{aligned} \quad (3.12)$$

here $C = \|\beta_\tau\|_{L^{\frac{1}{p_1}}([a, T], \mathbb{R}_+)}$.

• If $a < t < t_0 \leq T$. Similar to the preceding case, we have

$$\begin{aligned} &|y^{(n)}(t) - y^{(n)}(t_0)|_E \\ &\leq \frac{1}{\Gamma(p)} \left\{ c_0(t_0 - t)^{p-p_1} + C \left(\int_a^T (G(t, s) - G(t_0, s))^{\frac{1}{1-p_1}} ds \right)^{1-p_1} \right\}, \end{aligned} \quad (3.13)$$

From (3.9)–(3.13), noting that $\varphi \in C([a - r, T], E)$, $|G(t, s) - G(t_0, s)| \rightarrow 0$ as $t \rightarrow t_0$, and the function $s \mapsto |G(t, s) - G(t_0, s)|$ belongs to $L^{\frac{1}{1-p_1}}([a, T], \mathbb{R}_+)$, we apply item (ii) of Lemma 2.11 to obtain that

$$|y^{(n)}(t) - y^{(n)}(t_0)| \rightarrow 0$$

uniformly in $y^{(n)}$. This implies that the set $\mathcal{M}$ is equicontinuous at t_0 . Therefore, we conclude that $\mathcal{M}$ is a relatively compact subset of $C([a, T], E)$.

Step 4: Applying Lemma 2.6 (Schauder fixed point theorem), there exists $x^* \in \mathcal{B}_h$ such that $A(x^*) = x^*$, i.e.,

$$x^*(t) = \varphi(a) + \frac{1}{\Gamma(p)} \int_a^t \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} f(s, x_s^*) ds, \quad t \in (a, T].$$

According to Proposition 3.2, we complete the proof of the theorem. $\square$

We propose to replace hypothesis (A2) with hypothesis (A3) and also get the same result in the following theorem.

Theorem 3.4. Assume that (A1) and (A3) are satisfied. Then, for any $\varphi \in C([a - r, a], E)$, there exists $T \in (a, \infty)$ such that the fractional initial value problem (1.1)–(1.2) admits at least one integral solution $x \in C([a - r, T], E)$, given by

$$x(t) = \varphi(a) + \frac{1}{\Gamma(p)} \int_a^t \frac{1}{\tau} \left(\log \frac{t}{\tau} \right)^{p-1} f(\tau, x_\tau) d\tau, \quad t \in (a, T].$$

Proof. The proof of this result is based on Lemma 2.12 and the same arguments as in Theorem 3.3. The difference is that the computations are more difficult, as presented in Lemma 2.12. $\square$

Theorem 3.5. Assume (A4). Then, for every $\varphi \in C([a-r, a], E)$, there exists $T \in (a, \infty)$ such that the fractional IVP (1.1)–(1.2) has a unique integral solution $x \in C([a-r, T], E)$.

Proof. We first see that the assumption (A4) implies (A1) and (A2). Let $h > 0$ and, $T \in (a, \infty)$ be a positive number mentioned in (A2) and (A4), and conditions (3.2) are satisfied. Firstly, similar to the proof of Theorem 3.3, we introduce the set again:

$$\mathcal{B}_h = \left\{ x \in C([a-r, T], E) : x|_{[a-r, a]} = \varphi, \sup_{s \in [a, T]} |x(s) - \varphi(a)|_E \leq h \right\}$$

and the operator A is defined by

$$A(x)(t) = \begin{cases} \varphi(t), & \text{if } t \in [a-r, a], \\ \varphi(a) + \frac{1}{\Gamma(p)} \int_a^t \frac{1}{\tau} \left(\log \frac{t}{\tau} \right)^{p-1} f(\tau, x_\tau) d\tau, & \text{if } t \in (a, T]. \end{cases} \quad (3.14)$$

To facilitate presentation, we reuse symbols: $G(t, s) = \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} \chi_{[a, t)}(s)$.

By similar arguments as in the proof of Theorem 3.3, we see that $\mathcal{B}_h$ is closed and $A\mathcal{B}_h \subset \mathcal{B}_h$. Consider the operator $A : \mathcal{B}_h \rightarrow C([a-r, T], E)$. For $t > a$, by induction on n , we will prove the following statement:

$$|A^n u(t) - A^n v(t)|_E \leq \frac{\left(L \left(\log \frac{t}{a} \right)^p \right)^n}{\Gamma(np+1)} \|u - v\|_{C([a-r, t], E)}, \quad u, v \in \mathcal{B}_h \quad (3.15)$$

for all $n \in \mathbb{N}$. Indeed, it is clear that (3.15) is true for $n = 0$. Assume that (3.15) is right for $n = k$. We see that

$$Au(t) - Av(t) = \begin{cases} 0, & t \in [a-r, a]; \\ \frac{1}{\Gamma(p)} \int_a^t G(t, s) (f(s, u_s) - f(s, v_s)) ds, & t \in (a, T] \end{cases}$$

for all $u, v \in \mathcal{B}_h$. Therefore, for $\tau > a$, we have

$$\begin{aligned} |Au(\tau) - Av(\tau)|_E &\leq \|Au - Av\|_{C([a-r, \tau], E)} = \|Au - Av\|_{C([a, \tau], E)} \text{ and} \\ |u(\tau) - v(\tau)|_E &\leq \|u - v\|_{C([a-r, \tau], E)} = \|u - v\|_{C([a, \tau], E)} \text{ if } u, v \in \mathcal{B}_h. \end{aligned}$$

We also see that

$$\begin{aligned} \|u_s - v_s\|_{C([-r, 0], E)} &= \sup_{\theta \in [-r, 0]} |u(s + \theta) - v(s + \theta)|_E \\ &= \sup_{\theta \in [s-r, s]} |u(\theta) - v(\theta)|_E \\ &= \|u - v\|_{C([s-r, s], E)} \\ &\leq \|u - v\|_{C([a-r, s], E)} \\ &\leq \|u - v\|_{C([a, s], E)} \text{ for all } u, v \in \mathcal{B}_h. \end{aligned}$$

By the Lipschitz condition, we derive

$$|A^{k+1}u(t) - A^{k+1}v(t)|_E \leq \frac{1}{\Gamma(p)} \int_a^t G(\tau, s) |f(s, (A^k u)_s) - f(s, (A^k v)_s)|_E ds$$

$$\leq \frac{L}{\Gamma(p)} \int_a^t G(t, s) \|A^k(u)_s - A^k(v)_s\|_{C([-r, a], E)} ds. \quad (3.16)$$

For $s \leq t$, by the induction hypothesis, we get

$$\begin{aligned} \|A^k(u)_s - A^k(v)_s\|_{C([-r, a], E)} &= \sup_{\tau \in [-r, 0]} |A^k u(s + \tau) - A^k v(s + \tau)|_E \\ &= \sup_{\tau \in [s-r, s]} |A^k u(\tau) - A^k v(\tau)|_E \\ &\leq \sup_{\tau \in [s-r, s]} \frac{L^k \left(\log \frac{\tau}{a}\right)^{kp}}{\Gamma(kp + 1)} \|u - v\|_{C([a-r, \tau], E)} \\ &\leq \frac{L^k \left(\log \frac{s}{a}\right)^{kp}}{\Gamma(kp + 1)} \|u - v\|_{C([a-r, t], E)}. \end{aligned} \quad (3.17)$$

From (3.16) and (3.17), it follows that

$$|A^{k+1}u(t) - A^{k+1}v(t)|_E \leq \frac{L^{k+1} \|u - v\|_{C([a-r, t], E)}}{\Gamma(kp + 1)\Gamma(p)} \int_a^t \frac{1}{s} \left(\log \frac{t}{s}\right)^{p-1} \left(\log \frac{s}{a}\right)^{kp} ds.$$

Using the variable substitution $x = \frac{\log \frac{s}{a}}{\log \frac{t}{a}}$, we obtain

$$\int_a^t \frac{1}{s} \left(\log \frac{t}{s}\right)^{p-1} \left(\log \frac{s}{a}\right)^{kp} ds = \left(\log \frac{t}{a}\right)^{(k+1)p} \beta(p, kp + 1),$$

where the beta function $\beta(p, kp + 1) = \int_0^1 (1-x)^{p-1} x^{kp} dx = \frac{\Gamma(p)\Gamma(kp+1)}{\Gamma(p+1+kp)}$. It implies that

$$|A^{k+1}u(t) - A^{k+1}v(t)|_E \leq \frac{L^{k+1} \left(\log \frac{t}{a}\right)^{(k+1)p}}{\Gamma((k+1)p + 1)} \|u - v\|_{C([a-r, t], E)}$$

for all $t \in [a, T]$. This finishes the induction process. Thus (3.15) is true for all $n \in \mathbb{N}$. Hence,

$$\|A^n u - A^n v\|_{C([a-r, T], E)} \leq \gamma_n \|u - v\|_{C([a-r, T], E)}, \quad u, v \in \mathcal{B}_h,$$

where $\gamma_n := \frac{(L^n (\log \frac{T}{a})^p)^n}{\Gamma(np+1)}$. It is well-known that the series $\sum_{n=0}^{\infty} \frac{(L^n (\log \frac{T}{a})^p)^n}{\Gamma(np+1)}$ converges. Hence $\gamma_n \rightarrow 0$. By the Banach fixed point theorem, A has a unique fixed point in $\mathcal{B}_h$. This completes the proof. $\square$

3.2. Nonlocal existence

In this subsection, we briefly present similar results for the nonlocal problem (1.3)–(1.4), based on the techniques used in the previous section. In addition to establishing the existence of a solution to the initial value problem with nonlocal conditions, we use a fixed point theorem for the α -condensing operator.

Definition 3.6. A function $u \in C([a, T], E)$ is said to be a solution of the nonlocal IVP (1.3)–(1.4) if u satisfies ${}_H^C D_{a^+}^p u(t) = f(t, u(t))$ a.e. $t \in [a, T]$ and initial condition $u(a) = -h(u) + u_a$.

Arguing similarly to the proof of Proposition 3.2, we obtain the following lemma:

Lemma 3.7. *A function $u \in C([a, T], E)$ is a solution of the fractional IVP (1.3)–(1.4) if and only if u satisfies the fractional integral equation*

$$\begin{cases} u(t) = -h(u) + u_a + \frac{1}{\Gamma(p)} \int_a^t \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} f(s, u(s)) ds, & t \in (a, T]; \\ u(a) = -h(u) + u_a. \end{cases}$$

We define operators

$$G : C([a, T], E) \rightarrow C([a, T], E),$$

$$G(u)(t) = -h(u) + u_a, \quad u \in C([a, T], E), t \in [a, T];$$

$$A : C([a, T], E) \rightarrow C([a, T], E),$$

$$A(u)(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} f(s, u(s)) ds \text{ if } t \in (a, T], \text{ and } A(u)(a) = 0.$$

$$F : C([a, T], E) \rightarrow C([a, T], E), F(u) = G(u) + A(u).$$

Obviously, the three operators are well-defined.

Equation (1.3)–(1.4) has a solution if and only if F has a fixed point, that is, there exists $u \in C([a, T], E)$ such that $u = F(u)$. In the rest of the article, we define $\|u\| = \sup_{s \in [a, T]} |u(s)|_E$.

Lemma 3.8. *Assume (H1)–(H2).*

(i) *Let $b \geq a > 0$. We have*

$$\frac{1}{\Gamma(p)} \left| \int_b^t \frac{1}{s} \left(\log \frac{t}{b} \right)^{p-1} f(s, u(s)) ds \right|_E \leq c_0 (C_f \|u\|^{\mu_f} + \mathcal{M}_f) (t - b)^{p-p_f} \quad (3.18)$$

for all $u \in C([a, T], E)$ and $t \in (b, T]$, where

$$c_0 = \frac{(1 - p_f)^{1-p_f}}{\Gamma(p) a^p (p - p_f)^{1-p_f}} \|\beta_f\|_{L^{\frac{1}{p_f}}([a, T], \mathbb{R}_+)}.$$

(ii) *The operator $A : C([a, T], E) \rightarrow C([a, T], E)$ is well-defined and compact. In addition,*

$$\|A(u)\| \leq c_0 (C_f \|u\|^{\mu_f} + \mathcal{M}_f) (T - a)^{p-p_f}$$

for all $u \in C([a, T], E)$.

Proof. (i): It is clear by Lemma 2.11.

(ii): We can see that A is well-defined by using a similar argument as in proof (ii) of Lemma 2.11. Similar to Step 2 in proving Theorem 3.3, we get that A is continuous. Let $r > 0$, define $\mathcal{S}_r := \{u \in C([a, T], E) : \|u\| \leq r\}$. We will prove that $\mathcal{A} := A(\mathcal{S}_r)$ is relatively compact in $C([a, T], E)$.

• Let $t \in [a, T]$. From the definition of A and the evaluation of (3.18), we have

$$\|Au\| \leq c_0 (C_f \|u\|^{\mu_f} + \mathcal{M}_f) (T - a)^{p-p_f} \equiv C < \infty,$$

for all $u \in \mathcal{S}_r$. Thus, $\mathcal{A}$ is pointwise bounded.

• Let $t_0 \in [a, T]$. If $t > t_0 = a$, we have

$$|Au(t) - Au(t_0)|_E = |Au(t)|_E \leq c_0 \left(C_f \|u\|^{\mu_f} + \mathcal{M}_f \right) \|\beta_f\|_{L^{\frac{1}{p_f}}([a, T], \mathbb{R}_+)} (t - t_0)^{p-p_f} \rightarrow 0$$

independently of $u \in \mathcal{S}_r$, as $t \rightarrow t_0^+$.

If $t > t_0 > a$ (resp., $t_0 > t > a$), we have

$$|Au(t) - Au(t_0)|_E \leq \frac{1}{\Gamma(p)} (R_1(t, t_0) + R_2(t, t_0));$$

where

$$\begin{aligned} R_1(t, t_0) &:= \left| \int_{t_0}^t \frac{1}{s} \left(\log \frac{t}{a} \right)^{p-1} f(s, u(s)) ds \right| \\ &\leq c_0 \left(C_f \|u\|^{\mu_f} + \mathcal{M}_f \right) (t - t_0)^{p-p_f} \rightarrow 0 \text{ independently of } u, \text{ as } t \rightarrow t_0^+; \end{aligned}$$

and

$$\begin{aligned} R_2(t, t_0) &= \left| \int_a^{t_0} (G(t, s) - G(t_0, s)) f(s, u(s)) ds \right| \\ &\leq c_0 \left(C_f \|u\|^{\mu_f} + \mathcal{M}_f \right) \|\beta_f\|_{L^{\frac{1}{p_f}}([a, T], \mathbb{R}_+)} \int_a^{t_0} |G(t, s) - G(t_0, s)|^{\frac{1}{1-p_f}} ds \\ &\leq c_0 \left(C_f \|u\|^{\mu_f} + \mathcal{M}_f \right) \|\beta_f\|_{L^{\frac{1}{p_f}}([a, T], \mathbb{R}_+)} \int_a^T |G(t, s) - G(t_0, s)|^{\frac{1}{1-p_f}} ds. \end{aligned}$$

Using Lemma 2.10, we have $R_2(t, t_0) \rightarrow 0$ independently of u , as $t \rightarrow t_0^+$. Thus, $|Au(t) - Au(t_0)| \rightarrow 0$ independently of u , as $t \rightarrow t_0^+$ (resp., $t \rightarrow t_0^-$). We conclude that $\mathcal{A}$ is equicontinuous. The Arzela-Ascoli theorem is used to complete the proof. $\square$

Theorem 3.9. Assume (H1)–(H4). Then the nonlocal problem (1.3)–(1.4) has at least one solution $x \in C([a, T], E)$ represented by

$$u(t) = -h(u) + u_a + \frac{1}{\Gamma(p)} \int_a^t \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} f(s, u(s)) ds, \quad t \in (a, T].$$

Proof. Let the operators $G, A, F : C([a, T], E) \rightarrow C([a, T], E)$ be as defined. They are continuous and bounded. Assume $W \in b(C([a, T], E))$. Since

$$\begin{aligned} \|G(u) - G(v)\| &= |h(u) - h(v)|_E \\ &\leq \mathcal{K}_h \|u - v\| \text{ for all } u, v \in C([a, T], E), \end{aligned}$$

we have $\alpha(G(W)) \leq \mathcal{K}_h \alpha(W)$. Thus

$$\begin{aligned} \alpha(F(W)) &= \alpha(G(W) + A(W)) \\ &\leq \alpha(G(W)) + \alpha(A(W)) \\ &= \alpha(G(W)) \end{aligned}$$

$$\leq \mathcal{K}_h \alpha(W).$$

It implies that F is a strict α -contraction, so it is also an α -condensing. In addition, a consequence of (H4) leads to

$$\begin{aligned} \|G(u)\| &\leq |u_a|_E + C_h \|u\|^{q_2} + \mathcal{M}_h \\ (\text{resp., } \|G(u)\| &\leq |u_a|_E + C_h (\|u\| + 1)). \end{aligned}$$

Define

$$\mathcal{S} = \{u \in C([a, T], E) : \exists \lambda \in [0, 1], u = \lambda F(u)\}.$$

For $u \in \mathcal{S}$, we have

$$\begin{aligned} \|u\| &= \lambda \|F(u)\| \leq \|G(u)\| + \|A(u)\| \\ &\leq |u_a|_E + C_h \|u\|^{q_2} + \mathcal{M}_h + c_0 (C_f \|u\|^{\mu_f} + \mathcal{M}_f) \end{aligned} \quad (3.19)$$

$$(\text{resp., } \|u\| \leq |u_a|_E + C_h (\|u\| + 1) + c_0 (C_f \|u\|^{\mu_f} + \mathcal{M}_f)), \quad (3.20)$$

where $c_0 = \frac{(1-p_f)^{1-p_f}}{\Gamma(p)a^p(p-p_f)^{1-p_f}} \|\beta_f\|_{L^{\frac{1}{p_f}}([a, T], \mathbb{R}_+)} (T-a)^{p-p_f}$, and $u \in C([a, T], E)$. If $\mathcal{S}$ is not bounded, there exists a sequence $\{\rho_n\}$, $\rho_n = \|u_n\|$ with $u_n \in \mathcal{S}$ such that $\rho_n \rightarrow \infty$. From (3.19) (resp., (3.20)), we have a contradiction, that is,

$$\begin{aligned} 1 &\leq \frac{|u_a|_E + C_h (\rho_n)^{q_2} + \mathcal{M}_h + c_0 (C_f (\rho_n)^{\mu_f} + \mathcal{M}_f)}{\rho_n} \rightarrow 0, \text{ as } n \rightarrow \infty \\ (\text{resp., } 1 &\leq \frac{|u_a|_E + C_h (\rho_n + 1) + c_0 (C_f (\rho_n)^{\mu_f} + \mathcal{M}_f)}{\rho_n} \rightarrow C_h < 1, \text{ as } n \rightarrow \infty). \end{aligned}$$

The proof is completed by employing Lemma 2.8. $\square$

Remark 3.10. In Problem (1.1)–(1.2), for $a = 0$, we replace condition (A2) with the condition $|f(t, x)| \leq t^p$ for all $t \in (a, T]$. In this case, the result of Theorem 3.3 remains valid. Indeed, we can easily verify that

$$\begin{aligned} \left| \int_0^t \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} f(s, x_s) ds \right| &< \int_0^t \frac{1}{s} \left(\log \frac{t}{s} \right)^{p-1} s^p ds \\ &= t^p \int_0^\infty \frac{\tau^{p-1}}{e^{p\tau}} d\tau = t^p \left[\int_0^1 \frac{\tau^{p-1}}{e^{p\tau}} d\tau + \int_1^\infty \frac{\tau^{p-1}}{e^{p\tau}} d\tau \right] < \infty. \end{aligned}$$

The proof is argued similarly to the proof of Theorem 3.3.

4. Illustrative examples

In this section, we give examples to illustrate the application of the above results.

Example 4.1. Let $a, r > 0, T > a$. Consider the following infinite system of fractional equations

$$\begin{cases} {}^C_H D_{a^+}^{\frac{2}{3}} x_n(t) = \frac{x_n^2(t-r)}{n(t-a)^{\frac{1}{3}}}, & t \in (a, T], \\ x_n(\theta) = \frac{\theta}{n}, & \theta \in [a-r, a], n = 1, 2, \dots \end{cases} \quad (4.1)$$

Let $E = \{\xi = \{\xi_j\}_{j=1}^\infty : \xi_j \in \mathbb{R}, \forall j \geq 1, \lim_{n \rightarrow \infty} \xi_n = 0\}$ be a Banach space with norm $|\xi|_E = \sup_{n \geq 1} |\xi_n|$. For $\mathbf{x} \in C([a-r, \infty), E)$, $\mathbf{x}(t) = (x_1(t), x_2(t), \dots)$ (write $\mathbf{x} = (x_1, x_2, \dots)$), here $x_j \in C([a-r, \infty), \mathbb{R})$ and $\mathbf{x}_t(s) = (x_1(t+s), x_2(t+s), \dots)$ for $t > a, s \in [-r, 0]$. For $n = 1, 2, \dots$, we define $C_n := C([-r, 0], \mathbb{R})$ and $f_n : (a, \infty) \times C_n \rightarrow \mathbb{R}$ with

$$\begin{aligned} f_n(t, v) &= \frac{(v(-r))^2}{n(t-a)^{\frac{1}{3}}}, \quad v \in C_n, \\ f(t, \mathbf{v}) &= (f_1(t, v_1), f_2(t, v_2), \dots), \end{aligned} \quad (4.2)$$

where $\mathbf{v} \in C := C([-r, 0], E)$, $\mathbf{v}(s) = (v_1(s), v_2(s), \dots) \in E$, $v_j \in C_j$. Hence, for $\mathbf{x} = (x_1, x_2, \dots)$, we have $(x_j)_t \in C_j$ ($t > a, j = 1, 2, \dots$), and

$$f_n(t, (x_n)_t) = \frac{(x_n(t-r))^2}{n(t-a)^{\frac{1}{3}}} \quad \forall n \in \mathbb{N}.$$

Then, we choose $\varphi : [a-r, a] \rightarrow E$ defined by $\varphi(\theta) = (\theta, \frac{\theta}{2}, \frac{\theta}{3}, \dots)$ for $\theta \in [a-r, a]$.

We can easily check that $f((a, \infty) \times C) \subset E$. The problem (4.1) can be seen as a fractional IVP of the form (1.1)–(1.2) with $p = \frac{2}{3}$.

It is clear that the assumption (A1) is satisfied. Next, for $\tau > 0, t > a$ and $\mathbf{v} \in C([-r, 0], E)$, $\mathbf{v} = (v_1, v_2, \dots)$, $\mathbf{v}(s) = (v_1(s), v_2(s), \dots)$ with $\|\mathbf{v}\|_{C([-r, 0], E)} \leq \tau$, we have

$$\begin{aligned} |v_n(-r)| &\leq \sup_{\theta \in [-r, 0]} (|v_n(\theta)|) \\ &\leq \sup_{\theta \in [-r, 0]} \left(\sup_{n \geq 1} |v_n(\theta)| \right) \\ &= \sup_{\theta \in [-r, 0]} |\mathbf{v}(\theta)|_E \\ &= \|\mathbf{v}\|_{C([-r, 0], E)} \\ &\leq \tau \quad \text{for all } n = 1, 2, \dots \end{aligned}$$

It implies that

$$\begin{aligned} |f(t, \mathbf{v})|_E &\leq \sup_{n \geq 1} |f_n(t, v_n)|_E \\ &= \sup_{n \geq 1} \left| \frac{v_n^2(-r)}{n(t-a)^{\frac{1}{3}}} \right| \\ &\leq \sup_{n \geq 1} \frac{\tau^2}{n(t-a)^{\frac{1}{3}}} \leq \frac{\tau^2}{(t-a)^{\frac{1}{3}}}. \end{aligned}$$

Thus, the assumption (A2) is satisfied with $p_1 = \frac{1}{2}$ and $\beta_\tau(t) = \frac{\tau^2}{(t-a)^{\frac{1}{3}}}$. By Theorem 3.3, problem (4.1) has at least one local continuous solution.

Example 4.2. Let $a, r > 0$. Consider the following infinite system of equations

$$\begin{cases} {}^C_H D_{a^+}^{\frac{2}{3}} x_n(t) = \frac{\sin^2(x_n(t-r)) \left(\log \frac{t}{a}\right)^\gamma}{n(t-a)^{\frac{1}{3}}}, & t \in (a, \infty), \\ x_n(\theta) = \frac{\theta}{n} & \theta \in [a-r, a], n = 1, 2, \dots \end{cases}$$

Similar to Example 4.1, with $\mathbf{x} = (x_1, x_2, \dots)$, $f_n(t, (x_n)_t) = \frac{(\log \frac{t}{a})^\gamma \sin^2(x_n(t-r))}{n(t-a)^{1/3}}$, conditions (A1) and (A3) are satisfied (with $\gamma = -\frac{1}{4}$).

Example 4.3. Let $R > a > 0$, $r > 0$, and $0 < n \in \mathbb{N}$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function. Consider the following system of matrix equations

$${}^C_H D_{a^+}^{\frac{2}{3}} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ \dots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} \frac{\phi(x_1(t-r))}{1+(t-a)^{\frac{1}{3}}} \\ \frac{\phi(x_2(t-r))}{1+2(t-a)^{\frac{1}{3}}} \\ \dots \\ \dots \\ \frac{\phi(x_n(t-r))}{1+n(t-a)^{\frac{1}{3}}} \end{pmatrix} \text{ and } \begin{pmatrix} x_1(\theta) \\ x_2(\theta) \\ \dots \\ \dots \\ x_n(\theta) \end{pmatrix} = \begin{pmatrix} \theta/1 \\ \theta/2 \\ \dots \\ \dots \\ \theta/n \end{pmatrix}$$

for all $t \in [a, \infty)$, $\theta \in [a-r, a]$. We rewrite the system as the following finite system of fractional equations

$$\begin{cases} {}^C_H D_{a^+}^{\frac{2}{3}} x_j(t) = \frac{\phi(x_j(t-r))}{1+j(t-a)^{\frac{1}{3}}}, & t \in (a, \infty), \\ x_j(\theta) = \frac{\theta}{j} & \theta \in [a-r, a], j = 1, 2, \dots, n. \end{cases} \quad (4.3)$$

Similar to Example 4.1, where E is replaced by $\mathbb{R}^n$ with norm $|\mathbf{x}|_E = \max_{\{1 \leq j \leq n\}} |x_j|$. Define $f_j : (a, \infty) \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$ and $f : (a, \infty) \times C([-r, 0], E)$ with

$$\begin{aligned} f_j(t, v) &= \frac{\phi(v(-r))}{1+j(t-a)^{\frac{1}{3}}}, \quad 1 \leq j \leq n, \\ f(t, \mathbf{v}) &= (f_1(t, v_1), \dots, f_n(t, v_n)), \quad \mathbf{v} = (v_1, \dots, v_n), v_j \in C([-r, 0], \mathbb{R}). \end{aligned}$$

Therefore, for $\mathbf{x} = (x_1, \dots, x_n) \in C([a-r, T], E)$, we have $x_j \in C([a-r, T], \mathbb{R})$, $(x_j)_t \in C([-r, 0], \mathbb{R})$, and

$$f_j(t, (x_j)_t) = \frac{\phi(x_j(t-r))}{1+j(t-a)^{\frac{1}{3}}}, \quad \text{for } j = 1, 2, \dots, n.$$

Then, the conditions (A1) and (A2) are fulfilled. We now check hypothesis (A4). Assume $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in C([-r, 0], E)$, $x_j, y_j \in C([-r, 0], \mathbb{R})$.

For $j = 1, \dots, n$, since ϕ has the Lipschitz property, there exists L satisfying

$$|\phi(x_j(-r)) - \phi(y_j(-r))| \leq L|x_j(-r) - y_j(-r)|$$

$$\begin{aligned} &\leq L|\mathbf{x}(-r) - \mathbf{y}(-r)|_E \\ &\leq L\|\mathbf{x} - \mathbf{y}\|_{C([-r,0],E)}. \end{aligned}$$

It shows that

$$|f(t, \mathbf{x}) - f(t, \mathbf{y})|_E \leq L\|\mathbf{x} - \mathbf{y}\|_{C([-r,0],\mathbb{R})} \quad \text{for all } \mathbf{x}, \mathbf{y} \in C([-r,0], E).$$

Thus, condition (A4) holds. Equation (4.3) has a unique continuous solution by Theorem 3.5.

Example 4.4. Let $0 < a < T$, $\Omega = [0, 1]$, $\ell \in L^2(\Omega \times \Omega, \mathbb{R})$ satisfying $\|\ell\|_{L^2(\Omega^2)} < \frac{1}{m+1}$ ($m \in \mathbb{N}$). Consider the problem of finding a function $u = u(t, x)$ such that

$$\begin{cases} {}^C_H D_{a^+}^{2/3} u(t, x) = f(t, u(t, x)), & t \in (a, T], & x \in \Omega, \\ u(a, x) = -\sum_{j=0}^m \int_0^1 \ell(x, w) u(t_j, w) dw + \psi_a(x), & x \in \Omega, \end{cases} \quad (4.4)$$

where $a < t_0 < t_1 < \dots < t_m \leq T$, $\psi_a \in L^2(\Omega, \mathbb{R})$.

Denote

$$h(u) = \sum_{j=0}^m \mathcal{L}u(t_j) \quad \text{with } \mathcal{L}v(x) = \int_0^1 \ell(w, x) v(w) dw, \quad (4.5)$$

where $u(t)(x) := u(t, x)$. Let $E := L^2(\Omega, \mathbb{R})$ be a Hilbert space with the norm $|w|_E = \left(\int_\Omega |w(s)|^2 ds\right)^{1/2}$. Problem (4.4) is equivalent to the problem of finding $u \in C([a, T], E)$ that satisfies

$$\begin{cases} {}^C_H D_{a^+}^{1/3} u(t)(x) = f(t, u(t)(x)), & t \in (a, T], \\ u(a) = -h(u) + \psi_a, & x \in \Omega. \end{cases}$$

We consider the problem with the function f defined as follows:

$$f(t, u(t, x)) = \frac{x + \left(\int_0^1 |u(t)(s)|^2 ds\right)^{1/4}}{(t-a)^{1/3}} \quad \text{if } t \in (a, T].$$

Then

$$\begin{aligned} |f(t, u(t)(x))| &\leq \frac{1 + \|u\|^{1/2}}{(t-a)^{1/3}} \\ &\leq \beta_f(t)(1 + \|u\|^{1/2}) \quad \forall x \in \Omega, \end{aligned}$$

where $\beta_f(t) = (t-a)^{-1/3}$, $\beta_f \in L^2((a, T), \mathbb{R}_+)$ (with $p_f = \frac{1}{2}$). Thus, $f(t, u(t)) \in E$ and $|f(t, u(t))|_E \leq \beta_f(t)(1 + \|u\|^{1/2})$ for a.e. $t \in (a, T]$. In addition,

$$\begin{aligned} |h(u)(x) - h(v)(x)| &\leq \sum_{j=0}^m \int_0^1 |\ell(w, x)| |u(t_j)(w) - v(t_j)(w)| dw \\ &\leq \sum_{j=0}^m \int_0^1 |\ell(., x)| |u(t_j) - v(t_j)| \end{aligned}$$

$$\leq (m+1)\|u-v\|\|\ell(\cdot, x)\|,$$

and

$$\begin{aligned} \|h(u) - h(v)\|_{L^2(\Omega, \mathbb{R})} &= \left(\int_0^1 |h(u)(w) - h(v)(w)|^2 \right)^{\frac{1}{2}} \\ &\leq (m+1)\|\ell\|_{L^2(\Omega^2)}\|u-v\|. \end{aligned}$$

Conditions (H1)–(H4) have been met. The existence of a solution to equation (4.4) is established by Theorem 3.9 (with $\mathcal{K}_h = C_h = (m+1)\|\ell\|_{L^2(\Omega^2)} < 1$, $p_f = \frac{1}{2}$).

5. Conclusion

Addressing challenges associated with fractional-order derivatives requires the careful selection of an appropriate function space. The primary objective of this paper is to develop comprehensive function spaces by integrating Lebesgue and Bochner integrals. Additionally, this study aims to derive conditions that ensure both the existence and uniqueness of solutions for equations involving the Caputo-Hadamard derivative operator. These investigations are carried out under reasonable assumptions.

Following the above result, we can consider problems with weaker source function conditions, as well as study the problem in spaces with a weaker topological structure.

Author contributions

All authors equally contributed.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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