



Research article

On the optimal control of viscous Cahn–Hilliard systems with hyperbolic relaxation of the chemical potential

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Abstract: In this paper, we study an optimal control problem for a viscous Cahn–Hilliard system with zero Neumann boundary conditions in which a hyperbolic relaxation term involving the second time derivative of the chemical potential has been added to the first equation of the system. For the initial-boundary value problem of this system, results concerning well-posedness, continuous dependence, and regularity are known. We show Fréchet differentiability of the associated control-to-state operator, study the associated adjoint state system, and derive first-order necessary optimality conditions. Concerning the nonlinearities driving the system, we can include the case of logarithmic potentials. In addition, we perform an asymptotic analysis of the optimal control problem as the relaxation coefficient approaches zero.

Keywords: Optimal control; Cahn–Hilliard system; hyperbolic relaxation; first-order optimality conditions; asymptotic analysis

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We denote by $\mathbf{n}$ the unit outward normal to $\partial\Omega$, with the associated outward normal derivative $\partial_{\mathbf{n}}$. Moreover, let $T > 0$ stand for some final time, and set

$$Q_t := \Omega \times (0, t), \quad \Sigma_t := \partial\Omega \times (0, t), \quad \text{for } t \in (0, T], \quad \text{and} \quad Q := Q_T, \quad \Sigma := \Sigma_T.$$

We then study the following optimal control problem:

(CP) Minimize the tracking-type cost functional

$$\begin{aligned} \mathcal{J}(\varphi, u) &:= \frac{b_1}{2} \iint_Q |\varphi - \varphi_Q|^2 + \frac{b_2}{2} \int_\Omega |\varphi(T) - \varphi_\Omega|^2 + \frac{b_3}{2} \iint_Q |u|^2 + \kappa G(u) \\ &=: J(\varphi, u) + \kappa G(u) \end{aligned} \quad (1.1)$$

subject to the initial-boundary value system

$$\alpha \partial_{tt} \mu + \partial_t \varphi - \Delta \mu = 0 \quad \text{a.e. in } Q, \quad (1.2)$$

$$\tau \partial_t \varphi - \Delta \varphi + f'(\varphi) = \mu + w \quad \text{a.e. in } Q, \quad (1.3)$$

$$\gamma \partial_t w + w = u \quad \text{a.e. in } Q, \quad (1.4)$$

$$\partial_{\mathbf{n}} \mu = \partial_{\mathbf{n}} \varphi = 0 \quad \text{a.e. on } \Sigma, \quad (1.5)$$

$$\mu(0) = \mu_0, \quad (\partial_t \mu)(0) = \nu_0, \quad \varphi(0) = \varphi_0, \quad w(0) = w_0 \quad \text{a.e. in } \Omega, \quad (1.6)$$

and to the control constraint

$$u \in \mathcal{U}_{\text{ad}} = \{u \in \mathcal{U} : \underline{u}(x, t) \leq u(x, t) \leq \bar{u}(x, t) \text{ for a.a. } (x, t) \text{ in } Q\}. \quad (1.7)$$

Here, the control space is specified by

$$\mathcal{U} = L^\infty(Q). \quad (1.8)$$

The given bounds $\underline{u}, \bar{u} \in L^\infty(Q)$ satisfy $\underline{u} \leq \bar{u}$ almost everywhere in Q , the targets $\varphi_Q, \varphi_\Omega$ are given functions, and $b_1 \geq 0, b_2 \geq 0$, and $b_3 > 0$ are constants. Moreover, $G : L^2(Q) \rightarrow [0, +\infty)$ is a convex and continuous functional enhancing the occurrence of *sparsity* of optimal controls, where $\kappa \geq 0$ is a fixed constant (the sparsity parameter); a typical choice for G is

$$G(u) = \|u\|_{L^1(Q)} = \iint_Q |u|. \quad (1.9)$$

The equations (1.2)–(1.3) constitute a variation of the well-studied viscous Cahn–Hilliard system (see, e.g., the recent contributions [1–5] and the references given therein)

$$\partial_t \varphi - \Delta \mu = 0 \quad \text{in } Q, \quad (1.10)$$

$$\tau \partial_t \varphi - \Delta \varphi + f'(\varphi) = \mu + w \quad \text{in } Q, \quad (1.11)$$

in which a hyperbolic relaxation term $\alpha \partial_{tt} \mu$ has been added in the first equation. While hyperbolic relaxations of the viscous Cahn–Hilliard system involving an inertial term of the phase variable φ have been considered in the previous works [6–12], an inertial term like $\alpha \partial_{tt} \mu$ in (1.2) was, to our knowledge, studied only in the recent paper [13].

In the above class of problems (which model, e.g., the coarsening processes in binary metallic alloys), the unknown functions φ and μ usually stand for the *order parameter*, which can represent a scaled density of one of the involved phases, and the *chemical potential* associated with the phase separation process, respectively. The state variables φ and μ are monitored through the input variable w , which is in turn determined by the action of the control u via the linear control equation (1.4). This equation

describes how the “forcing” w is generated by the external control u . We note that equation (1.4) could be replaced by more general differential equations that model the relationship between an L^∞ -control u and the resulting forcing w . For example, in [14], the linear relation given by (1.4) is replaced with a reaction-diffusion equation, where the variable w represents the concentration of a nutrient in a tumor growth model.

In the practical application to the coarsening process of a binary metallic alloy, a typical aim of the control problem **(CP)** is to monitor the system in such a way as to achieve desired distributions φ_Q and φ_Ω of one of the metallic components in the container Ω during the process and at the final time $t = T$, respectively, at minimal cost.

In the state system (1.2)–(1.6), $\gamma > 0$ and $\tau > 0$ are given fixed constants, and μ_0 , ν_0 , φ_0 , and w_0 are given initial data. Moreover, f' denotes the derivative of a double-well potential f and stands for the local part of the thermodynamic force driving the evolution of the system. Typically, f is split into a (possibly nondifferentiable) convex part f_1 and a smooth and concave perturbation f_2 . Typical and physically relevant examples for f are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$f_{\text{reg}}(r) := \frac{1}{4}(r^2 - 1)^2, \quad r \in \mathbb{R}, \quad (1.12)$$

$$f_{\text{log}}(r) := \begin{cases} (1+r) \ln(1+r) + (1-r) \ln(1-r) - c_1 r^2 & \text{if } r \in (-1, 1) \\ 2 \ln(2) - c_1 & \text{if } r \in \{-1, 1\} \\ +\infty & \text{if } r \notin [-1, 1] \end{cases}, \quad (1.13)$$

$$f_{2\text{obs}}(r) := \begin{cases} c_2(1 - r^2) & \text{if } r \in [-1, 1] \\ +\infty & \text{if } r \notin [-1, 1] \end{cases}. \quad (1.14)$$

Here, the constants c_i in (1.13) and (1.14) satisfy $c_1 > 1$ and $c_2 > 0$, so that f_{log} and $f_{2\text{obs}}$ are nonconvex. Notice that for $f = f_{\text{log}}$ the term $f'(\varphi)$ occurring in (1.3) becomes singular as $\varphi \searrow -1$ and $\varphi \nearrow 1$, which forces the order parameter φ to attain its values in the physically meaningful range $(-1, 1)$. In the nonsmooth case (1.14), the convex part f_1 is given by the indicator function of $[-1, 1]$. Accordingly, in such cases one has to replace the derivative of the convex part by the subdifferential ∂f_1 and, consequently, to interpret (1.3) as a differential inclusion or a variational inequality. We admit in this paper only the regular and logarithmic cases.

At this point, we observe that there is a strong difference in the energetic structures in the primary a priori estimates between the system (1.2)–(1.6) and the case when $\alpha = 0$. To see this, assume that $w = 0$. If $\alpha = 0$ (in which case the initial datum for $\partial_t \mu$ is missing), then the standard procedure to find a first a priori estimate is to test (1.2) by μ and (1.3) by $\partial_t \varphi$. Addition of the resulting identities brings a cancellation of the terms involving the expression $\mu \partial_t \varphi$, and we obtain the identity

$$\int_0^t \int_\Omega |\nabla \mu|^2 + \tau \int_0^t \int_\Omega |\partial_t \varphi|^2 + \int_\Omega \left(\frac{1}{2} |\nabla \varphi(t)|^2 + f(\varphi(t)) \right) = \text{constant}$$

for $t \in (0, T)$. Notice that the left-hand side, besides the first two dissipative terms, only involves the total potential energy and thus does not account for possible singularities of the thermodynamic force f' which drives the phase separation. The same is the case if for $\alpha = 0$ we add an inertial term $\beta \partial_{tt} \varphi$ to the phase equation (1.3): indeed, then only the kinetic term $\frac{\beta}{2} |\partial_t \varphi(t)|^2$ additionally appears in the bracket of the third integral on the left, and the constant on the right-hand side changes. A similar

observation can be made for the case when the hyperbolic relaxation $\alpha \partial_{tt}\mu$ is replaced by the parabolic relaxation $\alpha \partial_t \mu$.

In contrast to this, if the inertial term of the chemical potential $\alpha \partial_{tt}\mu$ occurs in (1.2), then the above procedure does not work. Indeed, the natural test function for (1.2) is $\partial_t \mu$, and in order to compensate for the term $\partial_t \varphi \partial_t \mu$, which then occurs, one is forced to differentiate (1.3) with respect to time and then test by $\partial_t \varphi$. We then arrive at the identity

$$\begin{aligned} & \int_0^t \int_{\Omega} |\nabla \partial_t \mu|^2 + \int_0^t \int_{\Omega} f_1''(\varphi) |\partial_t \varphi|^2 \\ & + \int_{\Omega} \left(\frac{\alpha}{2} |\partial_t \mu(t)|^2 + \frac{1}{2} |\nabla \mu(t)|^2 + \frac{\tau}{2} |\partial_t \varphi(t)|^2 \right) \\ & = \text{constant} - \int_0^t \int_{\Omega} f_2''(\varphi) |\partial_t \varphi|^2 \quad \text{for all } t \in (0, T). \end{aligned}$$

Note that in this way, through the dissipative second term on the left-hand side, the delicate interplay between the possible singularity of f_1'' (which occurs, for instance, for the logarithmic potential (1.13) where $f_1''(\varphi) = 2/(1 - \varphi^2)$) and the rate of phase change $\partial_t \varphi$ comes into play from the very beginning.

There exists a large literature concerning the optimal control of viscous Cahn–Hilliard systems that cannot be cited here; in this connection, we refer the interested reader to the references given in the recent papers [1, 15, 16]. On the other hand, optimal control problems like **(CP)** for the system (1.2)–(1.6) have not been investigated before. In particular, a standout reference is paper [1], which deserves special recognition for its in-depth exploration of the optimal control problem applied to system (1.2)–(1.6) with $\alpha = 0$, including the case of the logarithmic potential (1.13). Notably, the resulting viscous Cahn–Hilliard system is far more manageable, as the absence of the inertial term $\alpha \partial_{tt}\mu$ (when $\alpha > 0$) transforms the challenge from a hyperbolic system into a more tractable parabolic one. In contrast, here we take on the more complex scenario, successfully demonstrating convergence results, also for the optimal control problem, as the coefficient α tends to zero.

The paper is organized as follows. In the following section, we formulate the general assumptions and state the main results concerning the system (1.2)–(1.6). In Section 3, we show the Fréchet differentiability of the control-to-state operator associated with the state system. Section 4 then brings an analysis of the control problem **(CP)** for fixed $\alpha \in (0, 1]$. Besides the existence of an optimal control, first-order necessary optimality conditions are shown in terms of a variational inequality and the corresponding adjoint state variables. We also obtain a sparsity result in the case $\kappa > 0$. The final section then deals with asymptotic results as $\alpha \searrow 0$. We show that both the state and adjoint state variables converge in a well-defined sense to their counterparts associated with the nonrelaxed case when $\alpha = 0$, and we also prove a convergence result for the optimal controls as $\alpha \searrow 0$.

Prior to this, let us fix some notation. For any Banach space X , we let X^* denote its dual space, and $\|\cdot\|_X$ stands for the norm in X and any power of X . For two Banach spaces X and Y that are both continuously embedded in some topological vector space Z , the linear space $X \cap Y$ is the Banach space equipped with its natural norm $\|v\|_{X \cap Y} := \|v\|_X + \|v\|_Y$ for $v \in X \cap Y$. The standard Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{m,p}(\Omega)$ are defined on Ω for $1 \leq p \leq \infty$ and $m \in \mathbb{N}$. For the sake of convenience, we denote the norm of $L^p(\Omega)$ by $\|\cdot\|_p$ for $1 \leq p \leq \infty$. If $p = 2$, we employ the usual notation $H^m(\Omega) := W^{m,2}(\Omega)$. We also set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0 \text{ on } \partial\Omega\}. \quad (1.15)$$

Moreover, V^* is the dual space of V , and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between V^* and V . We also denote by $(\cdot, \cdot)$ the natural inner product in H . As usual, H is identified with a subspace of the dual space V^* according to the identity

$$\langle u, v \rangle = (u, v) \quad \text{for every } u \in H \text{ and } v \in V.$$

Note that $W \subset V \subset H \equiv H^* \subset V^*$ with dense and compact embeddings. Moreover, the space W defined above is contained in $C^0(\overline{\Omega})$. This embedding is important for proving boundedness of different terms later in the paper, in particular, in the proof of differentiability properties. Finally, we introduce for functions $v \in L^1(Q)$ the temporal antiderivative $1 * v$ by setting

$$(1 * v)(x, t) := \int_0^t v(x, s) ds \quad \text{for a.e. } x \in \Omega \text{ and all } t \in [0, T]. \quad (1.16)$$

About the constants used in the sequel for estimates, we adopt the rule that C denotes any positive constant that depends only on the given data. The value of such generic constants C may change from formula to formula or even within the lines of the same formula. Finally, the notation C_δ indicates a positive constant that additionally depends on the quantity δ .

2. General assumptions and properties of the state system

In this section, we formulate the general assumptions for the data of the system (1.2)–(1.6) and state known existence, continuous dependence, and regularity results. First, let us remark that the positive parameter α is not listed in the assumptions below, since it is also involved in the related asymptotic analysis. Consequently, we let

$$0 < \alpha \leq 1.$$

On the other hand, throughout the paper we generally assume:

(A1) $\tau > 0$ and $\gamma > 0$ are fixed constants.

(A2) $f = f_1 + f_2$, where $f_1 : \mathbb{R} \rightarrow [0, +\infty]$ is convex and lower semicontinuous with $f_1(0) = 0$, and where $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ has a Lipschitz continuous first derivative f'_2 on $\mathbb{R}$.

Note that **(A2)** implies that the subdifferential ∂f_1 is maximal monotone in $\mathbb{R} \times \mathbb{R}$ and satisfies $0 \in \partial f_1(0)$. Let $D(\partial f_1)$ denote the domain of the subdifferential. An important requirement is the following:

(A3) There are $-\infty \leq r_- < 0 < r_+ \leq +\infty$ such that $D(\partial f_1) = (r_-, r_+)$ and the restrictions of f_1 and f_2 to (r_-, r_+) belong to $C^3(r_-, r_+)$. Hence, for $r \in (r_-, r_+)$, we have $\partial f_1(r) = \{f'_1(r)\}$. Moreover, we assume that

$$\lim_{r \searrow r_-} f'_1(r) = -\infty, \quad \lim_{r \nearrow r_+} f'_1(r) = +\infty. \quad (2.1)$$

Observe that the conditions **(A2)** and **(A3)** are fulfilled in each of the cases considered in (1.12) and (1.13), with the domain $D(\partial f_1)$ given by $\mathbb{R}$ and $(-1, 1)$, respectively.

We further assume:

(A4) $\mu_0 \in W$, $\nu_0 \in V$, $w_0 \in L^\infty(\Omega)$, $\varphi_0 \in W$, and it holds $r_- < \varphi_0(x) < r_+$ for all $x \in \overline{\Omega}$.

Observe that $\varphi_0 \in W$ implies that $\varphi_0 \in C^0(\overline{\Omega})$. Moreover, we obviously have that

$$m_0 := \frac{1}{|\Omega|} \int_{\Omega} \varphi_0$$

lies in the interior of $D(\partial f_1)$. Here, $|\Omega|$ denotes the Lebesgue measure of Ω , and m_0 thus represents the mean value of φ_0 . In the following, we use the general notation $\bar{v}$ to denote the mean value of a generic function $v \in L^1(\Omega)$. If v is in V^* , then we can set

$$\bar{v} := \frac{1}{|\Omega|} \langle v, 1 \rangle \quad (2.2)$$

as well, noting that the constant function 1 is an element of V . Clearly, $\bar{v}$ is the usual mean value of v if $v \in H$, and $m_0 = \bar{\varphi}_0$.

Finally, once and for all, we fix a ball in the control space that contains $\mathcal{U}_{\text{ad}}$:

(A5) The constant $R > 0$ is such that $\mathcal{U}_{\text{ad}} \subset \mathcal{U}_R := \{u \in L^\infty(Q) : \|u\|_{L^\infty(Q)} < R\}$.

In the following theorem, we collect results that have been stated and proved in Theorems 2.2 to 2.5 in [13]. While some of these results may hold under slightly weaker assumptions, the formulation provided here is specifically adapted to the context of the control problem **(CP)**. We now state the result.

Theorem 2.1. *Suppose that (A1)–(A5) are satisfied. Then the state system (1.2)–(1.6) has for any $u \in \mathcal{U}_R$ and every $\alpha \in (0, 1]$ a unique solution triple $(\mu^\alpha, \varphi^\alpha, w^\alpha)$ satisfying the regularity requirements*

$$\mu^\alpha \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; V) \cap L^\infty(0, T; W), \quad (2.3)$$

$$\varphi^\alpha \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W), \quad (2.4)$$

$$w^\alpha \in W^{1,\infty}(0, T; L^\infty(\Omega)). \quad (2.5)$$

Moreover, there is a constant $K_1 > 0$, which depends only on the data of the state system and R and not on $\alpha \in (0, 1]$, such that

$$\begin{aligned} & \alpha \|\mu^\alpha\|_{W^{2,\infty}(0,T;V^*)} + \alpha^{1/2} \|\mu^\alpha\|_{W^{1,\infty}(0,T;H)} + \|\mu^\alpha\|_{L^\infty(0,T;V)} + \|w^\alpha\|_{W^{1,\infty}(0,T;L^\infty(\Omega))} \\ & + \|\varphi^\alpha\|_{W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)} + \|f'_1(\varphi^\alpha)\|_{L^\infty(0,T;H)} \leq K_1. \end{aligned} \quad (2.6)$$

In addition, there holds a uniform separation condition in the following form: for any $\alpha \in (0, 1]$, there are constants $r_*(\alpha), r^*(\alpha)$, which depend only on the data of the state system and R , such that

$$r_- < r_*(\alpha) \leq \varphi^\alpha(x, t) \leq r^*(\alpha) < r_+ \quad \text{for all } (x, t) \in \bar{Q}. \quad (2.7)$$

The above result implies that, for every $\alpha \in (0, 1]$, the control-to-state operator

$$\mathcal{S}^\alpha = (\mathcal{S}_1^\alpha, \mathcal{S}_2^\alpha, \mathcal{S}_3^\alpha) : u \mapsto \mathcal{S}^\alpha(u) = (\mathcal{S}_1^\alpha(u), \mathcal{S}_2^\alpha(u), \mathcal{S}_3^\alpha(u)) := (\mu^\alpha, \varphi^\alpha, w^\alpha) \quad (2.8)$$

is a well-defined mapping between $\mathcal{U}_R$ and the Banach space identified by the regularity properties (2.3)–(2.5).

Remark 2.2. Notice that, due to the compactness of the embedding $W \subset C^0(\bar{\Omega})$, it follows from [17, Sect. 8, Cor. 4] and (2.6) that $\varphi^\alpha \in C^0(\bar{Q})$. By the same token, the compact embedding $V \subset L^p(\Omega)$ for $1 \leq p < 6$ yields that also $\mu^\alpha \in C^1([0, T]; L^p(\Omega))$ for $1 \leq p < 6$. In addition, owing to (2.7) and **(A3)**, there exists for every $\alpha \in (0, 1]$ a constant $K_2(\alpha) > 0$, which depends only on R and the data of the state system, such that

$$\max_{i=1,2} \max_{j=0,1,2,3} \|f_i^{(j)}(\varphi^\alpha)\|_{L^\infty(Q)} \leq K_2(\alpha), \quad (2.9)$$

whenever $\varphi^\alpha = \mathcal{S}_2^\alpha(u)$ for some $u \in \mathcal{U}_R$. Moreover, in the case when $r_- = -\infty$ and $r_+ = +\infty$ so that $D(\partial f_1) = \mathbb{R}$ and $f_1 \in C^3(\mathbb{R})$ in (A3), the values $r_*(\alpha)$ and $r^*(\alpha)$ in (2.7), as well as the constant $K_2(\alpha)$ in (2.9), can be taken independent of α . This is due to the bound (2.6), which ensures that $\|\varphi^\alpha\|_{C^0(\bar{Q})} \leq C$.

We also have the following continuous dependence result.

Theorem 2.3. *Suppose that (A1)–(A5) are fulfilled, and let $\alpha \in (0, 1]$. Then there is a constant $K_3(\alpha) > 0$ that depends only on R and the data of the state system such that the following holds true: whenever $u_i \in \mathcal{U}_R$ are given and $(\mu_i^\alpha, \varphi_i^\alpha, w_i^\alpha) = \mathcal{S}^\alpha(u_i)$, $i = 1, 2$, then*

$$\begin{aligned} & \|\mu_1^\alpha - \mu_2^\alpha\|_{H^2(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)} + \|\varphi_1^\alpha - \varphi_2^\alpha\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \\ & + \|w_1^\alpha - w_2^\alpha\|_{H^1(0,T;H)} \leq K_3(\alpha) \|u_1 - u_2\|_{L^2(0,T;H)}. \end{aligned} \quad (2.10)$$

Proof. First observe that we have $(w_1^\alpha - w_2^\alpha)(0) = 0$ and

$$\gamma \partial_t(w_1^\alpha - w_2^\alpha) + w_1^\alpha - w_2^\alpha = u_1 - u_2 \quad \text{a.e. in } Q.$$

Testing this identity by $\partial_t(w_1^\alpha - w_2^\alpha)$ then immediately yields that

$$\|w_1^\alpha - w_2^\alpha\|_{H^1(0,T;H)} \leq C \|u_1 - u_2\|_{L^2(0,T;H)},$$

where the constant C depends on R and the data of the state system, but not on α . The validity of (2.10) is then a direct consequence of the inequality (2.27) in the statement of [13, Thm. 2.5]. $\square$

3. Differentiability of the solution operator

In this section, we are going to prove differentiability properties for the solution operator $\mathcal{S}^\alpha$, where, throughout this section, we assume that $\alpha \in (0, 1]$ is fixed. To this end, let $u \in \mathcal{U}_R$ be a fixed control with associated state $(\mu^\alpha, \varphi^\alpha, w^\alpha) = \mathcal{S}^\alpha(u)$. We then consider the linearization of the state system (1.2)–(1.6) at u in the direction $h \in L^2(0, T; H)$, which we write in the form

$$\alpha \langle \partial_{tt} \eta, \rho \rangle + \int_{\Omega} \partial_t \psi \rho + \int_{\Omega} \nabla \eta \cdot \nabla \rho = 0 \quad \text{for all } \rho \in V \text{ and a.e. } t \in (0, T), \quad (3.1)$$

$$\tau \partial_t \psi - \Delta \psi + f''(\varphi^\alpha) \psi = \eta + v \quad \text{a.e. in } Q, \quad (3.2)$$

$$\gamma \partial_t v + v = h \quad \text{a.e. in } Q, \quad (3.3)$$

$$\partial_n \psi = 0 \quad \text{a.e. on } \Sigma, \quad (3.4)$$

$$\eta(0) = \partial_t \eta(0) = \psi(0) = v(0) = 0 \quad \text{a.e. in } \Omega. \quad (3.5)$$

Notice that (3.1) is the weak form of the PDE $\alpha \partial_{tt} \eta + \partial_t \psi - \Delta \eta = 0$ together with the boundary condition $\partial_n \eta = 0$.

We have the following result concerning well-posedness.

Theorem 3.1. *Suppose that (A1)–(A5) are fulfilled; let $\alpha \in (0, 1]$ and $u \in \mathcal{U}_R$ be given, as well as $(\mu^\alpha, \varphi^\alpha, w^\alpha) = \mathcal{S}^\alpha(u)$. Then the system (3.1)–(3.5) has for every $h \in L^2(0, T; H)$ a unique solution triple (η_h, ψ_h, v_h) such that*

$$\eta_h \in W^{2,\infty}(0, T; V^*) \cap W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V), \quad (3.6)$$

$$\psi_h \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W), \quad (3.7)$$

$$v_h \in H^1(0, T; H). \quad (3.8)$$

Moreover, there is a constant $K_4(\alpha)$, which depends on R and the data, such that

$$\begin{aligned} & \|\eta_h\|_{W^{2,\infty}(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)} + \|\psi_h\|_{W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)} \\ & + \|v_h\|_{H^1(0,T;H)} \leq K_4(\alpha) \|h\|_{L^2(0,T;H)}. \end{aligned} \quad (3.9)$$

Proof. Obviously, the PDE (3.3), together with the initial condition $v(0) = 0$ a.e. in Ω , has for every $h \in L^2(0, T; H)$ the unique solution

$$v_h(x, t) = \frac{1}{\gamma} \int_0^t e^{-(t-s)/\gamma} h(x, s) ds \quad \text{for a.e. } (x, t) \in Q.$$

It is readily seen that

$$\|v_h\|_{H^1(0,T;H)} \leq C \|h\|_{L^2(0,T;H)}, \quad (3.10)$$

where, here and in the remainder of this proof, $C > 0$ denotes constants that may depend on α , R , and the data of the system, but neither on $n \in \mathbb{N}$ nor on $h \in L^2(0, T; H)$. We thus have to show the existence of a unique pair (η_h, ψ_h) that has zero initial data and solves (3.1), (3.2) with $v = v_h$, and (3.4).

We argue by a Faedo–Galerkin approximation using a special basis. To this end, we take the eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ of the eigenvalue problem

$$-\Delta y = \lambda y \quad \text{in } \Omega, \quad \partial_n y = 0 \quad \text{on } \partial\Omega,$$

and let $\{e_j\}_{j \in \mathbb{N}} \subset W$ be associated eigenfunctions, normalized by $\|e_j\|_H = 1$, $j \in \mathbb{N}$. Then, it turns out that

$$\begin{aligned} 0 = \lambda_1 < \lambda_2 \leq \dots, \quad \lim_{j \rightarrow \infty} \lambda_j = +\infty, \\ \int_{\Omega} e_j e_k = \int_{\Omega} \nabla e_j \cdot \nabla e_k = 0 \quad \text{for } j \neq k, \end{aligned}$$

and it is clear that e_1 is just the constant function $|\Omega|^{-1/2}$. We then define the n -dimensional spaces $V_n := \text{span}\{e_1, \dots, e_n\}$ for $n \in \mathbb{N}$, where V_1 is just the space of constant functions on Ω . It is well known that the union of these spaces is dense in both H and V . The approximating n -dimensional problem is stated as follows: find time-dependent functions η_{nj}, ψ_{nj} , $j = 1, \dots, n$, such that the functions

$$\eta_n(x, t) := \sum_{j=1}^n \eta_{nj}(t) e_j(x), \quad \psi_n(x, t) := \sum_{j=1}^n \psi_{nj}(t) e_j(x), \quad (3.11)$$

satisfy

$$\begin{aligned} & \alpha(\partial_t \eta_n(t), \rho) + (\partial_t \psi_n(t), \rho) + \int_{\Omega} \nabla \eta_n(t) \cdot \nabla \rho = 0 \\ & \text{for all } t \in [0, T] \text{ and every } \rho \in V_n, \end{aligned} \quad (3.12)$$

$$\tau(\partial_t \psi_n(t), \rho) + \int_{\Omega} \nabla \psi_n(t) \cdot \nabla \rho + ((f''(\varphi^\alpha) \psi_n)(t), \rho) = (\eta_n(t) + v_h(t), \rho)$$

for all $t \in [0, T]$ and every $\rho \in V_n$,

(3.13)

$$\eta_n(0) = \partial_t \eta_n(0) = \psi_n(0) = 0 \quad \text{a.e. in } \Omega.$$
(3.14)

We now take $\rho = e_k$ in the equations (3.12) and (3.13), for $k = 1, \dots, n$, obtaining a Cauchy problem for an explicit system of linear ordinary differential equations with zero initial conditions, which is of second order in the variables η_{nk} and of first order in the variables ψ_{nk} . Moreover, the coefficient functions and source terms all belong to $H^1(0, T)$. By Carathéodory's theorem, the Cauchy problem has a unique solution expressed by η_{nk}, ψ_{nk} , with $\eta_{nk} \in H^3(0, T)$ and $\psi_{nk} \in H^2(0, T)$, for $k = 1, \dots, n$. This solution uniquely determines a pair $(\eta_n, \psi_n) \in H^3(0, T; V_n) \times H^2(0, T; V_n)$ that solves (3.12)–(3.14).

We now derive a series of a priori estimates for the finite-dimensional approximations.

First estimate. We take the time derivative of (3.13) and then test it by $\partial_t \psi_n$. Then, we choose $\rho = \partial_t \eta_n$ in (3.12) and add the resulting equations, noting the cancellation of two terms. Next, we integrate with respect to time over $[0, t]$ for $t \in (0, T]$, obtaining the identity

$$\begin{aligned} & \frac{\alpha}{2} \|\partial_t \eta_n(t)\|_H^2 + \frac{1}{2} \|\nabla \eta_n(t)\|_H^2 + \frac{\tau}{2} \|\partial_t \psi_n(t)\|_H^2 + \iint_{Q_t} |\nabla(\partial_t \psi_n)|^2 \\ &= - \iint_{Q_t} f''(\varphi^\alpha) |\partial_t \psi_n|^2 - \iint_{Q_t} f'''(\varphi^\alpha) \partial_t \varphi^\alpha \psi_n \partial_t \psi_n + \iint_{Q_t} \partial_t v_h \partial_t \psi_n \\ &=: I_1 + I_2 + I_3, \end{aligned}$$
(3.15)

with obvious notation. Owing to (2.7), Young's inequality, and (3.10), we have

$$I_1 + I_3 \leq C \iint_{Q_t} |\partial_t \psi_n|^2 + C \|h\|_{L^2(0, T; H)}^2.$$

Now observe that, thanks to the zero initial conditions and Hölder's inequality,

$$\|\eta_n(t)\|_H^2 \leq t \iint_{Q_t} |\partial_t \eta_n|^2, \quad \|\psi_n(t)\|_H^2 \leq t \iint_{Q_t} |\partial_t \psi_n|^2, \quad \text{for all } t \in [0, T].$$
(3.16)

Therefore, by virtue of (2.7), (2.9), Young's inequality, and the continuity of the embedding $V \subset L^4(\Omega)$,

$$\begin{aligned} I_2 &\leq C \int_0^t \|\partial_t \varphi^\alpha(s)\|_{L^4(\Omega)} \|\psi_n(s)\|_H \|\partial_t \psi_n(s)\|_{L^4(\Omega)} ds \\ &\leq \frac{1}{2} \iint_{Q_t} (|\nabla \partial_t \psi_n|^2 + |\partial_t \psi_n|^2) + C \int_0^t \|\partial_t \varphi^\alpha(s)\|_V^2 \|\partial_t \psi_n\|_{L^2(0, T; H)}^2 ds, \end{aligned}$$
(3.17)

where, in view of (2.6), the function $s \mapsto \|\partial_t \varphi^\alpha(s)\|_V^2$ belongs to $L^1(0, T)$. Hence, invoking Gronwall's lemma, it is straightforward to infer from the above estimates that

$$\begin{aligned} & \alpha^{1/2} \|\eta_n\|_{W^{1, \infty}(0, T; H)} + \|\nabla \eta_n\|_{L^\infty(0, T; H)^3} \\ &+ \|\psi_n\|_{W^{1, \infty}(0, T; H) \cap H^1(0, T; V)} \leq C \|h\|_{L^2(0, T; H)}. \end{aligned}$$
(3.18)

Complementary estimates. In the following, let P_n denote the H -orthogonal projection operator onto V_n . It is well known that

$$\|P_n \rho\|_H \leq \|\rho\|_H \quad \text{and} \quad \|P_n \rho\|_V \leq C_\Omega \|\rho\|_V \quad \text{for all } \rho \in V, \quad (3.19)$$

with a constant $C_\Omega > 0$ that depends only on Ω .

Taking now an arbitrary $\rho \in V$ in (3.12) and using (3.19) and the orthogonality of the basis functions, we find that

$$\begin{aligned} \alpha \langle \partial_t \eta_n(t), \rho \rangle &= \alpha(\partial_t \eta_n(t), P_n(\rho)) + \alpha(\partial_t \eta_n(t), \rho - P_n(\rho)) \\ &\leq |\alpha(\partial_t \eta_n(t), P_n(\rho))| \leq \left| (\partial_t \psi_n(t), P_n(\rho)) + \int_\Omega \nabla \eta_n(t) \cdot \nabla P_n(\rho) \right| \\ &\leq C(\|\partial_t \psi_n\|_{L^\infty(0,T;H)} + \|\nabla \eta_n\|_{L^\infty(0,T;H^3)}) \|\rho\|_V \quad \text{for a.e. } t \in (0, T), \end{aligned} \quad (3.20)$$

so that from (3.18) it follows that

$$\alpha \|\partial_t \eta_n\|_{L^\infty(0,T;V^*)} \leq C \|h\|_{L^2(0,T;H)}. \quad (3.21)$$

In addition, we can take $\rho = -\Delta(\psi_n(t))$ in (3.13) and integrate by parts. We then obtain

$$\begin{aligned} \|\Delta \psi_n(t)\|_H^2 &= (\tau \partial_t \psi_n(t) + f''(\varphi^\alpha(t)) \psi_n(t) - v_h(t), \Delta \psi_n(t)) + \int_\Omega \nabla \eta_n(t) \cdot \nabla \psi_n(t) \\ &\leq \frac{1}{2} \|\Delta \psi_n(t)\|_H^2 + C(1 + \|\psi_n\|_{W^{1,\infty}(0,T;H)}^2 + \|v_h\|_{L^\infty(0,T;H)}^2) \\ &\quad + \|\nabla \eta_n\|_{L^\infty(0,T;H^3)} \|\nabla \psi_n\|_{L^\infty(0,T;H^3)} \quad \text{for a.e. } t \in (0, T). \end{aligned} \quad (3.22)$$

Consequently, from (3.18) and the elliptic regularity theory, we find that

$$\|\Delta \psi_n\|_{L^\infty(0,T;H)} + \|\psi_n\|_{L^\infty(0,T;W)} \leq C \|h\|_{L^2(0,T;H)}. \quad (3.23)$$

Conclusion of the proof. By virtue of the estimates shown above, we conclude that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\|\eta_n\|_{W^{2,\infty}(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)} + \|\psi_n\|_{W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)} \\ &\leq C \|h\|_{L^2(0,T;H)}. \end{aligned} \quad (3.24)$$

Hence there exists a pair (η, ψ) such that (possibly on a subsequence, which is again labeled by $n \in \mathbb{N}$)

$$\eta_n \rightarrow \eta \quad \text{weakly star in } W^{2,\infty}(0, T; V^*) \cap W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V), \quad (3.25)$$

$$\psi_n \rightarrow \psi \quad \text{weakly star in } W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W). \quad (3.26)$$

By a standard argument (which needs no repetition here), it then follows that the pair (η, ψ) solves the equations (3.1), (3.2) with $v = v_h$, and (3.4). In addition, we easily conclude that $\eta(0) = \psi(0) = 0$. Moreover, it follows from (3.24) and the lower semicontinuity properties of norms that the triple (η_h, ψ_h, v_h) , where $\eta_h = \eta$ and $\psi_h = \psi$, satisfies the estimate (3.9). From this it is easily seen that the solution is unique: indeed, if (η_i, ψ_i, v_i) , $i = 1, 2$, are two solutions, then the triple $(\eta, \psi, v) := (\eta_1 - \eta_2, \psi_1 - \psi_2, v_1 - v_2)$ solves the system (3.1)–(3.5) with $h = 0$, whence $(\eta, \psi, v) = (0, 0, 0)$ follows. With this, the assertion is completely proved. $\square$

Next, we show a differentiability result for the control-to-state operators $\mathcal{S}^\alpha$, for $\alpha \in (0, 1]$.

Theorem 3.2. *Suppose that (A1)–(A5) are fulfilled, and let $\alpha \in (0, 1]$ be given. Then the control-to-state operator $\mathcal{S}^\alpha$ is Fréchet differentiable in $\mathcal{U}_R$ as a mapping from $L^\infty(Q)$ into the Banach space*

$$\mathcal{X} := C^0([0, T]; H) \times (H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W)) \times H^1(0, T; H). \quad (3.27)$$

Moreover, for every $u \in \mathcal{U}_R$ the Fréchet derivative $D\mathcal{S}^\alpha(u) \in \mathcal{L}(L^\infty(Q), \mathcal{X})$ is obtained as follows: for all $h \in L^\infty(Q)$ it holds that $D\mathcal{S}^\alpha(u)[h] = (\eta_h, \psi_h, v_h)$, where (η_h, ψ_h, v_h) is the unique solution to the linearized system (3.1)–(3.5).

Proof. Let $\alpha \in (0, 1]$ and $u \in \mathcal{U}_R$ be given, and let $(\mu^\alpha, \varphi^\alpha, w^\alpha) = \mathcal{S}^\alpha(u)$. Since $\mathcal{U}_R$ is open, it holds $u + h \in \mathcal{U}_R$ provided $\|h\|_{L^\infty(Q)}$ is sufficiently small. In the following, we only consider such variations h , and we define the quantities

$$\begin{aligned} (\mu_h, \varphi_h, w_h) &:= \mathcal{S}^\alpha(u + h), \quad y_h := \mu_h - \mu^\alpha - \eta_h, \\ z_h &:= \varphi_h - \varphi^\alpha - \psi_h, \quad \omega_h := w_h - w^\alpha - v_h, \end{aligned}$$

where (η_h, ψ_h, v_h) denotes the unique solution to the linearized system (3.1)–(3.5). Notice that

$$\begin{aligned} y_h &\in W^{2,\infty}(0, T; V^*) \cap W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V), \\ z_h &\in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W), \\ \omega_h &\in H^1(0, T; H), \end{aligned}$$

according to Theorem 2.1 and Theorem 3.1. Moreover, from the estimate (2.10) in Theorem 2.3 we conclude that

$$\begin{aligned} &\|\mu_h - \mu^\alpha\|_{H^2(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)} + \|\varphi_h - \varphi^\alpha\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \\ &\leq C\|h\|_{L^2(0,T;H)}, \end{aligned} \quad (3.28)$$

where, here and in the remainder of this proof, $C > 0$ denotes constants that may depend on α , R , and the data of the system, but not on the special choice of h with $u + h \in \mathcal{U}_R$.

Next, we observe that it follows from the estimate (3.9) in Theorem 3.1 that the mapping $h \mapsto (\eta_h, \psi_h, v_h)$ belongs to $\mathcal{L}(L^\infty(Q), \mathcal{X})$, in particular. Hence it suffices to show that

$$\lim_{\|h\|_{L^\infty(Q)} \rightarrow 0} \frac{\|(y_h, z_h, \omega_h)\|_{\mathcal{X}}}{\|h\|_{L^\infty(Q)}} = 0, \quad (3.29)$$

which is certainly satisfied if

$$\begin{aligned} &\|y_h\|_{C^0([0,T];H)} + \|z_h\|_{H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)} \\ &+ \|\omega_h\|_{H^1(0,T;H)} \leq C\|h\|_{L^2(0,T;H)}^2 \end{aligned} \quad (3.30)$$

for all $h \in L^\infty(Q)$ with $u + h \in \mathcal{U}_R$. We are going to show that (3.30) is in fact valid. To this end, observe that the triple (y_h, z_h, ω_h) solves the system

$$\alpha \langle \partial_t y_h, \rho \rangle + (\partial_t z_h, \rho) + \int_{\Omega} \nabla y_h \cdot \nabla \rho = 0 \quad \text{for every } \rho \in V, \text{ a.e. in } (0, T), \quad (3.31)$$

$$\tau \partial_t z_h - \Delta z_h = y_h + \omega_h - (f'(\varphi_h) - f'(\varphi^\alpha) - f''(\varphi^\alpha)\psi_h) \quad \text{a.e. in } Q, \quad (3.32)$$

$$\gamma \partial_t \omega_h + \omega_h = 0 \quad \text{a.e. in } Q, \quad (3.33)$$

$$\partial_{\mathbf{n}} z_h = 0 \quad \text{a.e. on } \Sigma, \quad (3.34)$$

$$y_h(0) = \partial_t y_h(0) = z_h(0) = \omega_h(0) = 0 \quad \text{a.e. in } \Omega. \quad (3.35)$$

Obviously, it results that $\omega_h = 0$ a.e. in Q . Moreover, Taylor's theorem with integral remainder yields almost everywhere in Q the identity

$$\begin{aligned} f'(\varphi_h) - f'(\varphi^\alpha) - f''(\varphi^\alpha)\psi_h &= f''(\varphi^\alpha)z_h + A_h(\varphi_h - \varphi^\alpha)^2, \quad \text{where} \\ A_h &:= \int_0^1 \int_0^1 s f'''(\varphi^\alpha + \tau s(\varphi_h - \varphi^\alpha)) d\tau ds. \end{aligned} \quad (3.36)$$

Owing to **(A3)** and (2.7), we have

$$|A_h| \leq C \quad \text{a.e. in } Q. \quad (3.37)$$

We now integrate (3.31) over $(0, t)$ for $t \in (0, T]$ to obtain the equation

$$\alpha(\partial_t y_h, \rho) + (z_h, \rho) + \int_{\Omega} \nabla(1 * y_h) \cdot \nabla \rho = 0, \quad (3.38)$$

where the expression $1 * y_h$ is defined in (1.16). We test (3.38) by y_h , (3.32) by z_h , add the resulting identities, and integrate with respect to time over $(0, t)$, where $t \in (0, T]$. Noting the cancellation of two terms, we then arrive at the identity

$$\begin{aligned} \frac{\alpha}{2} \|y_h(t)\|_H^2 + \frac{1}{2} \|\nabla(1 * y_h(t))\|_{H \times H \times H}^2 + \frac{\tau}{2} \|z_h(t)\|_H^2 + \iint_{Q_t} |\nabla z_h|^2 \\ = - \iint_{Q_t} f''(\varphi^\alpha) |z_h|^2 - \iint_{Q_t} A_h(\varphi_h - \varphi^\alpha)^2 z_h. \end{aligned} \quad (3.39)$$

Thanks to **(A3)**, (2.7), (3.28), and (3.37), the right-hand side of this equation can be bounded by

$$\begin{aligned} C \iint_{Q_t} |z_h|^2 + C \int_0^t \|z_h(s)\|_H \|\varphi_h(s) - \varphi^\alpha(s)\|_{L^4(\Omega)}^2 ds \\ \leq C \iint_{Q_t} |z_h|^2 + C \int_0^t \|\varphi_h(s) - \varphi^\alpha(s)\|_{L^4(\Omega)}^4 ds \\ \leq C \iint_{Q_t} |z_h|^2 + C \|h\|_{L^2(0,T;H)}^4, \end{aligned} \quad (3.40)$$

where in the last estimate the continuity of the embedding $V \subset L^4(\Omega)$ was used. Combining (3.39) with (3.40), and invoking Gronwall's lemma, we then conclude that

$$\|y_h\|_{C^0([0,T];H)}^2 + \|z_h\|_{C^0([0,T];H) \cap L^2(0,T;V)}^2 \leq C \|h\|_{L^2(0,T;H)}^4. \quad (3.41)$$

At this point we note that z_h solves a linear parabolic initial-boundary value problem of heat conduction type, with zero initial datum and zero Neumann boundary condition, whose right-hand

side $g_h := y_h - f''(\varphi^\alpha)z_h - A_h(\varphi_h - \varphi^\alpha)^2$ has already been shown to satisfy the condition $\|g_h\|_{L^2(0,T;H)}^2 \leq C\|h\|_{L^2(0,T;H)}^4$. It therefore follows from standard linear parabolic theory that

$$\|z_h\|_{H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)}^2 \leq C\|h\|_{L^2(0,T;H)}^4. \quad (3.42)$$

Combining (3.41) with (3.42), and recalling that $\omega_h = 0$, we obtain that (3.30) is valid, which finishes the proof of the assertion. $\square$

Remark 3.3. Note that the value of the constant $R > 0$ did not really matter in the above proof. We therefore conclude that the Fréchet derivatives $D\mathcal{S}^\alpha(u) \in \mathcal{L}(L^\infty(Q), \mathcal{X})$ exist for all $u \in L^\infty(Q)$.

4. The optimal control problem

In this section, we investigate the control problem formulated in the introduction, which, in order to stress its dependence on the parameter $\alpha \in (0, 1]$, will in the following be denoted by $(\mathbf{CP}_\alpha)$. In addition to the general assumptions (A1)–(A5), we postulate:

- (A6) $b_1 \geq 0, b_2 \geq 0, b_3 > 0$, and $\kappa \geq 0$ are given constants.
- (A7) $\varphi_Q \in L^2(Q)$, $\varphi_\Omega \in V$, and $\underline{u}, \bar{u} \in L^\infty(Q)$ are given functions such that $\underline{u} \leq \bar{u}$ a.e. in Q .
- (A8) The mapping $G : L^2(Q) \rightarrow \mathbb{R}$ is convex, continuous, and nonnegative.

4.1. Existence

We begin our analysis with an existence result.

Theorem 4.1. *Suppose that (A1)–(A8) are fulfilled. Then the optimal control problem $(\mathbf{CP}_\alpha)$ has, for every $\alpha \in (0, 1]$, at least one solution.*

Proof. Obviously, the cost functional $\mathcal{J}$ is nonnegative. Therefore, there is a minimizing sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{U}_{\text{ad}}$, i.e., we have

$$\lim_{n \rightarrow \infty} \mathcal{J}(\mathcal{S}_2^\alpha(u_n), u_n) = \inf_{u \in \mathcal{U}_{\text{ad}}} \mathcal{J}(\mathcal{S}_2^\alpha(u), u) \geq 0.$$

Then the triple $(\mu_n^\alpha, \varphi_n^\alpha, w_n^\alpha) := \mathcal{S}^\alpha(u_n)$ solves for $n \in \mathbb{N}$ the state system (1.2)–(1.6) with $u = u_n$ and thus satisfies the global bounds (2.6) and (2.9). We may therefore assume without loss of generality that, with a suitable triple (μ, φ, w) ,

$$\mu_n^\alpha \rightarrow \mu \quad \text{weakly star in } W^{2,\infty}(0, T; V^*) \cap W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V), \quad (4.1)$$

$$\varphi_n^\alpha \rightarrow \varphi \quad \text{weakly star in } W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W), \quad (4.2)$$

$$w_n^\alpha \rightarrow w \quad \text{weakly star in } W^{1,\infty}(0, T; L^\infty(\Omega)), \quad (4.3)$$

$$\varphi_n^\alpha \rightarrow \varphi \quad \text{strongly in } C^0(\overline{Q}), \quad (4.4)$$

where the last convergence result is a consequence of [17, Sect. 8, Cor. 4] and the compactness of the embedding $W \subset C^0(\overline{\Omega})$. In addition, we have $r_*(\alpha) \leq \varphi_n^\alpha \leq r^*(\alpha)$ in $\overline{Q}$ for every $n \in \mathbb{N}$. Hence, (4.4) and (A3) imply that

$$f'(\varphi_n^\alpha) \rightarrow f'(\varphi) \quad \text{strongly in } C^0(\overline{Q}). \quad (4.5)$$

Moreover, $\mathcal{U}_{\text{ad}}$ is a convex, bounded, and closed subset of $L^\infty(Q)$. Therefore, we may assume that there is some $u^\alpha \in \mathcal{U}_{\text{ad}}$ such that

$$u_n \rightarrow u^\alpha \quad \text{weakly star in } L^\infty(Q). \quad (4.6)$$

Passage to the limit as $n \rightarrow \infty$ in the equations (1.2)–(1.6), written for $((\mu_n^\alpha, \varphi_n^\alpha, w_n^\alpha), u_n)$, taking (4.1)–(4.6) into account, then leads to the conclusion that (μ, φ, w) solves the system (1.2)–(1.6) for the control $u = u^\alpha$. In other words, we have $(\mu, \varphi, w) = (\mu^\alpha, \varphi^\alpha, w^\alpha) = \mathcal{S}^\alpha(u^\alpha)$. It then follows from the semicontinuity properties of the functionals J and G (notice that G is convex and continuous and thus weakly sequentially lower semicontinuous on $L^2(Q)$) that

$$\mathcal{J}(\varphi^\alpha, u^\alpha) = J(\varphi^\alpha, u^\alpha) + \kappa G(u^\alpha) \leq \liminf_{n \rightarrow \infty} (J(\varphi_n^\alpha, u_n) + \kappa G(u_n)) = \inf_{u \in \mathcal{U}_{\text{ad}}} \mathcal{J}(\mathcal{S}_2^\alpha(u), u),$$

which means that $u = u^\alpha$ is an optimal control. $\square$

4.2. Necessary optimality conditions

We begin by deriving a first condition for a control to be locally optimal. To this end, recall that $u^\alpha \in \mathcal{U}_{\text{ad}}$ is called locally optimal for **(CP)** in the sense of $L^p(Q)$ for some $p \in [1, +\infty]$ if and only if there is some $\varepsilon > 0$ such that $\mathcal{J}(\mathcal{S}_2^\alpha(u^\alpha), u^\alpha) \leq \mathcal{J}(\mathcal{S}_2^\alpha(u), u)$ for all $u \in \mathcal{U}_{\text{ad}}$ with $\|u - u^\alpha\|_{L^p(Q)} \leq \varepsilon$. Observe that every control that is locally optimal in the sense of $L^p(Q)$ for some $p \in [1, +\infty)$ is also locally optimal in the sense of $L^\infty(Q)$. Therefore, all necessary conditions derived for locally optimal controls in the sense of $L^\infty(Q)$ are automatically valid also for locally optimal controls in the sense of $L^p(Q)$ for any $p \in [1, +\infty)$ and thus, in particular, for (globally) optimal controls.

We have the following result.

Lemma 4.2. *Suppose that (A1)–(A8) are fulfilled and $\alpha \in (0, 1]$. Moreover, assume that $u^\alpha \in \mathcal{U}_{\text{ad}}$ is a locally optimal control for **(CP)** in the sense of $L^\infty(Q)$ with associated state $(\mu^\alpha, \varphi^\alpha, w^\alpha) = \mathcal{S}^\alpha(u^\alpha)$. Then there is some $\lambda^\alpha \in \partial G(u^\alpha)$ (where $\partial G(u^\alpha) \subset L^2(Q)$ denotes the subdifferential of G at u^α) such that it holds*

$$\begin{aligned} & b_1 \iint_Q \psi_h(\varphi^\alpha - \varphi_Q) + b_2 \int_\Omega \psi_h(T)(\varphi^\alpha(T) - \varphi_\Omega) + b_3 \iint_Q u^\alpha(u - u^\alpha) \\ & + \kappa \iint_Q \lambda^\alpha(u - u^\alpha) \geq 0 \quad \text{for all } u \in \mathcal{U}_{\text{ad}}, \end{aligned} \quad (4.7)$$

where (η_h, ψ_h, v_h) is the unique solution to the linearized system (3.1)–(3.5) associated with $h = u - u^\alpha$.

Proof. For the following, we introduce the reduced functionals $\widehat{J}^\alpha$ and $\widehat{\mathcal{J}}^\alpha$ by setting

$$\widehat{J}^\alpha(u) := J(\mathcal{S}_2^\alpha(u), u), \quad \widehat{\mathcal{J}}^\alpha(u) := \widehat{J}^\alpha(u) + \kappa G(u), \quad \text{for } u \in L^\infty(Q). \quad (4.8)$$

Then, by Theorem 3.2 and the quadratic form of the differentiable part J of the cost functional, it follows from the chain rule that the mapping $\widehat{J}^\alpha : L^\infty(Q) \rightarrow \mathbb{R}$ has a Fréchet derivative $D\widehat{J}^\alpha(u)$ for every $u \in L^\infty(Q)$. From the convexity of $\mathcal{U}_{\text{ad}}$ and of G , we then infer that

$$D\widehat{J}^\alpha(u^\alpha)[u - u^\alpha] + \kappa(G(u) - G(u^\alpha)) \geq 0 \quad \text{for all } u \in \mathcal{U}_{\text{ad}}.$$

A standard argument from convex analysis (for the details, see, e.g., [18, Sect. 4.1]) yields that there is some $\lambda^\alpha \in \partial G(u^\alpha)$ such that

$$D\widehat{J}^\alpha(u^\alpha)[u - u^\alpha] + \kappa \iint_Q \lambda^\alpha(u - u^\alpha) \geq 0 \quad \text{for all } u \in \mathcal{U}_{\text{ad}}. \quad (4.9)$$

Finally, we infer from the chain rule, using Theorem 3.2 and the special form of J (which, in particular, does not involve the first and third solution components), that

$$\begin{aligned} D\widehat{J}^\alpha(u^\alpha)[u - u^\alpha] \\ = b_1 \iint_Q \psi_h(\varphi^\alpha - \varphi_Q) + b_2 \int_\Omega \psi_h(T)(\varphi^\alpha(T) - \varphi_\Omega) + b_3 \iint_Q u^\alpha(u - u^\alpha), \end{aligned}$$

where $h = u - u^\alpha$. This concludes the proof of the assertion. $\square$

Note that the optimality condition (4.7) is still impractical, since it requires solving the linearized system infinitely many times. As usual, we improve it by means of the adjoint variables.

We write the associated adjoint system in the form

$$\alpha \partial_u p - \Delta p - q = 0 \quad \text{a.e. in } Q, \quad (4.10)$$

$$-\partial_t(p + \tau q) - \Delta q + f''(\varphi^\alpha)q = b_1(\varphi^\alpha - \varphi_Q) \quad \text{a.e. in } Q, \quad (4.11)$$

$$-\gamma \partial_t r + r - q = 0 \quad \text{a.e. in } Q, \quad (4.12)$$

$$\partial_n p = \partial_n q = 0 \quad \text{a.e. on } \Sigma, \quad (4.13)$$

$$p(T) = \partial_t p(T) = r(T) = 0, \quad q(T) = \frac{b_2}{\tau}(\varphi^\alpha(T) - \varphi_\Omega) \quad \text{a.e. in } \Omega. \quad (4.14)$$

We have the following well-posedness result.

Theorem 4.3. *Suppose that (A1)–(A8) are fulfilled and that $\alpha \in (0, 1]$. In addition, let $u^\alpha \in \mathcal{U}_R$ be given and $(\mu^\alpha, \varphi^\alpha, w^\alpha) = \mathcal{S}^\alpha(u^\alpha)$. Then the system (4.10)–(4.14) has a unique solution $(p^\alpha, q^\alpha, r^\alpha)$ with the regularity*

$$p^\alpha \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; V) \cap L^\infty(0, T; W), \quad (4.15)$$

$$q^\alpha \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W), \quad (4.16)$$

$$r^\alpha \in H^2(0, T; H) \cap C^1([0, T]; V) \cap H^1(0, T; W). \quad (4.17)$$

Moreover, there is a constant $K_5(\alpha) > 0$, which depends only on R and the data of the state system, such that

$$\begin{aligned} \alpha \|p^\alpha\|_{W^{2,\infty}(0,T;H)} + \alpha^{1/2} \|p^\alpha\|_{W^{1,\infty}(0,T;V)} + \|p^\alpha\|_{L^\infty(0,T;W)} \\ + \|q^\alpha\|_{L^\infty(0,T;V) \cap L^2(0,T;W)} + \|r^\alpha\|_{W^{1,\infty}(0,T;V) \cap H^1(0,T;W)} \\ + \|p^\alpha + \tau q^\alpha\|_{H^1(0,T;H)} + \alpha^{1/2} \|\partial_t q^\alpha\|_{L^2(0,T;H)} + \alpha^{1/2} \|\partial_{tt} r^\alpha\|_{L^2(0,T;H)} \leq K_5(\alpha). \end{aligned} \quad (4.18)$$

Proof. The existence proof follows that of Theorem 3.1: we again employ a Faedo–Galerkin approximation using the same basis functions as in the proof of Theorem 3.1. To keep the paper at a reasonable length, we avoid writing the approximating system explicitly here and only give formal estimates for

(p, q, r) that would be rigorous on the level of the finite-dimensional approximations. In the following, we use the notation

$$Q^t := \Omega \times (t, T), \quad \text{for } t \in [0, T).$$

First estimate. To begin with, we first test (4.10) by $-\partial_t p$, (4.11) by q , and (4.12) by $-\partial_t r$, and add the resulting equations, noting the cancellation of two terms. Then we integrate over (t, T) , where $t \in [0, T)$ is arbitrary. Using the terminal conditions (4.14), we arrive at the identity

$$\begin{aligned} & \frac{\alpha}{2} \|\partial_t p(t)\|_H^2 + \frac{1}{2} \int_{\Omega} |\nabla p(t)|^2 + \frac{\tau}{2} \|q(t)\|_H^2 + \iint_{Q^t} |\nabla q|^2 + \gamma \iint_{Q^t} |\partial_t r|^2 + \frac{1}{2} \|r(t)\|_H^2 \\ &= \frac{b_2^2}{2\tau} \int_{\Omega} |\varphi^\alpha(T) - \varphi_\Omega|^2 - \iint_{Q^t} f''(\varphi^\alpha) q^2 + \iint_{Q^t} (b_1(\varphi^\alpha - \varphi_Q) - \partial_t r) q. \end{aligned} \quad (4.19)$$

The first term on the right-hand side of (4.19) is under control due to (2.6) and (A7). Next, we note that

$$- \iint_{Q^t} f''(\varphi^\alpha) q^2 = - \iint_{Q^t} f_1''(\varphi^\alpha) q^2 - \iint_{Q^t} f_2''(\varphi^\alpha) q^2,$$

with the first contribution being nonpositive and the second one being bounded by $C \iint_{Q^t} q^2$, due to (A2), (2.6), and (A3). About the third term on the right-hand side of (4.19), by (2.6), (A7), and Young's inequality, we have that

$$\iint_{Q^t} (b_1(\varphi^\alpha - \varphi_Q) - \partial_t r) q \leq \frac{\gamma}{2} \iint_{Q^t} |\partial_t r|^2 + C \iint_{Q^t} q^2.$$

Then, using Gronwall's lemma, we immediately see that

$$\alpha^{1/2} \|\partial_t p\|_{L^\infty(0,T;H)} + \|\nabla p\|_{L^\infty(0,T;H)^3} + \|q\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \|r\|_{H^1(0,T;H)} \leq C, \quad (4.20)$$

where the constant $C > 0$ depends on the data of the system, but is independent of $\alpha \in (0, 1]$.

Second estimate. Next, we test (4.11) by $p + \tau q$, obtaining the identity

$$\begin{aligned} & \frac{1}{2} \|(p + \tau q)(t)\|_H^2 + \iint_{Q^t} \nabla q \cdot \nabla (p + \tau q) \\ &= \frac{\tau^2}{2} \|q(T)\|_H^2 + \iint_{Q^t} (b_1(\varphi^\alpha - \varphi_Q) - f''(\varphi^\alpha) q) (p + \tau q). \end{aligned}$$

Hence, by virtue of Young's inequality, (4.20) and (2.9), we find from Gronwall's lemma that $\|p + \tau q\|_{L^\infty(0,T;H)} \leq C_\alpha$. Therefore, using once more (4.20), it turns out that $\|p\|_{L^\infty(0,T;H)} \leq C_\alpha$, which, in turn, implies that

$$\|p\|_{L^\infty(0,T;V)} \leq C_\alpha. \quad (4.21)$$

Third estimate. In this estimate, which is entirely formal (but completely justified on the level of the Faedo–Galerkin approximations), we first test (4.10) by $\Delta \partial_t p$. Taking the boundary conditions (4.13) and the terminal conditions (4.14) into account, we obtain the identity

$$\frac{\alpha}{2} \|\nabla \partial_t p(t)\|_H^2 + \frac{1}{2} \|\Delta p(t)\|_H^2 = \iint_{Q^t} q \Delta \partial_t p = \iint_{Q^t} \Delta q \partial_t p. \quad (4.22)$$

Secondly, we test (4.11) by $-\Delta q$ to see that

$$\begin{aligned} & \iint_{Q'} \Delta q \partial_t p + \frac{\tau}{2} \int_{\Omega} |\nabla q(t)|^2 + \iint_{Q'} |\Delta q|^2 \\ &= \frac{\tau}{2} \int_{\Omega} |\nabla q(T)|^2 + \iint_{Q'} \Delta q (f''(\varphi^\alpha) q - b_1(\varphi^\alpha - \varphi_Q)). \end{aligned} \quad (4.23)$$

In view of (4.14), we recall that $\varphi_\Omega \in V$, by assumption **(A7)**, and consequently, by (2.6), deduce that

$$\int_{\Omega} |\nabla q(T)|^2 \leq C(\|\varphi^\alpha(T)\|_V^2 + \|\varphi_\Omega\|_V^2) \leq C.$$

Thus, adding (4.22) and (4.23) leads to a cancellation of two terms. Using (2.9) and Young's inequality, by standard elliptic estimates we can infer that

$$\alpha^{1/2} \|p\|_{W^{1,\infty}(0,T;V)} + \|p\|_{L^\infty(0,T;W)} + \|q\|_{L^\infty(0,T;V) \cap L^2(0,T;W)} \leq C_\alpha. \quad (4.24)$$

It then directly follows from (4.10) and (4.24) that $\|\alpha \partial_{tt} p\|_{L^\infty(0,T;H)} \leq C_\alpha$, and thus, by virtue of (4.20),

$$\alpha \|p\|_{W^{2,\infty}(0,T;H)} \leq C_\alpha. \quad (4.25)$$

In addition, a comparison of terms in (4.11) yields

$$\|p + \tau q\|_{H^1(0,T;H)} \leq C_\alpha. \quad (4.26)$$

Moreover, due to the estimate for q in (4.24), arguing on (4.12) and using the terminal condition for r in (4.13) enable us to conclude that

$$\|r\|_{W^{1,\infty}(0,T;V) \cap H^1(0,T;W)} \leq C_\alpha. \quad (4.27)$$

The last part regards a similar bound for

$$\alpha^{1/2} \|\partial_t q\|_{L^2(0,T;H)} + \alpha^{1/2} \|\partial_{tt} r\|_{L^2(0,T;H)},$$

which follows then from (4.20), (4.26), and comparison in (4.11) and (4.12). At this point, we have completely shown the relevant estimate (4.18) that suffices to prove the existence of a solution to the adjoint system (4.10)–(4.14) that has the asserted properties. Owing to the linearity of the system, the proof of uniqueness is simple and can be skipped here. $\square$

Remark 4.4. Recalling the contents of Remark 2.2 and the assumption **(A3)**, we aim to point out that in the case when $D(\partial f_1) = \mathbb{R}$ and $f_1 \in C^3(\mathbb{R})$ in **(A3)**, then the constant $K_5(\alpha)$ in (4.18) may be chosen independent of $\alpha \in (0, 1]$, since in this framework the coefficient $f''(\varphi^\alpha)$ appearing in (4.11) is bounded in $L^\infty(Q)$ independently of α . In fact, we can repeat all the argumentation leading to (4.21)–(4.27) with constants C in the right-hand sides that do not depend on α .

Having shown the well-posedness of the adjoint system, it is now a standard procedure to improve the necessary condition for locally optimal controls stated in Lemma 4.2. Indeed, testing the linearized system (3.1)–(3.5) with the adjoint state variables, integrating by parts with respect to time and space, and using the adjoint system (4.10)–(4.14), we arrive at the following result.

Theorem 4.5. Suppose that (A1)–(A8) are fulfilled and $\alpha \in (0, 1]$. Moreover, assume that $u^\alpha \in \mathcal{U}_{\text{ad}}$ is a locally optimal control for $(\mathbf{CP}_\alpha)$ in the sense of $L^\infty(Q)$ with associated state $(\mu^\alpha, \varphi^\alpha, w^\alpha) = \mathcal{S}^\alpha(u^\alpha)$ and adjoint state $(p^\alpha, q^\alpha, r^\alpha)$. Then there is some $\lambda^\alpha \in \partial G(u^\alpha)$ such that

$$\iint_Q (r^\alpha + b_3 u^\alpha + \kappa \lambda^\alpha)(u - u^\alpha) \geq 0 \quad \text{for all } u \in \mathcal{U}_{\text{ad}}. \quad (4.28)$$

4.3. Sparsity of controls

The convex functional G in the cost functional accounts for the sparsity of optimal controls, i.e., the possibility that every locally optimal control may vanish in some subset of the space-time cylinder Q . The form of this region depends on the choice of G , where the sparsity properties can be deduced from the variational inequality (4.28) and the form of the subdifferential ∂G . In the following, we restrict ourselves to the case of *full sparsity*, which is obtained for the case of the $L^1(Q)$ -norm (1.9) that recently has been investigated in [1]. Arguing exactly as there, one obtains the following result (see [1, Thm. 4.7]).

Theorem 4.6. Suppose that (A1)–(A8) are fulfilled, $\alpha \in (0, 1]$, and $\kappa > 0$. Assume that $\underline{u}$ and $\bar{u}$ are constants such that $\underline{u} < 0 < \bar{u}$. If $u^\alpha \in \mathcal{U}_{\text{ad}}$ is locally optimal in the sense of $L^\infty(Q)$ for $(\mathbf{CP}_\alpha)$ with associated state $(\mu^\alpha, \varphi^\alpha, w^\alpha)$ and adjoint state $(p^\alpha, q^\alpha, r^\alpha)$, then there exists a function $\lambda^\alpha \in \partial G(u^\alpha)$ satisfying (4.28), and it holds

$$u^\alpha(x, t) = 0 \quad \text{if and only if} \quad |r^\alpha(x, t)| \leq \kappa, \quad \text{for a.e. } (x, t) \in Q. \quad (4.29)$$

Moreover, if r^α and λ^α are given, then u^α is obtained from the projection formula

$$u^\alpha(x, t) = \max \left\{ \underline{u}, \min \left\{ \bar{u}, -b_3^{-1}(r^\alpha + \kappa \lambda^\alpha)(x, t) \right\} \right\} \quad \text{for a.e. } (x, t) \in Q.$$

Remark 4.7. Owing to the global estimate (4.18), we have

$$\|r^\alpha\|_{H^1(0, T; W)} \leq K_5(\alpha),$$

and thus, thanks to the continuity of the embedding $H^1(0, T; W) \subset C^0(\bar{Q})$,

$$\|r^\alpha\|_{C^0(\bar{Q})} \leq \widehat{\kappa}(\alpha),$$

with a constant $\widehat{\kappa}(\alpha) > 0$. Hence, for each $\alpha \in (0, 1]$, if $\kappa \geq \widehat{\kappa}(\alpha)$, then every locally optimal control in the sense of $L^\infty(Q)$ for $(\mathbf{CP}_\alpha)$ must vanish.

5. Asymptotic analysis

This section is devoted to the study of the asymptotic behavior of the problem $(\mathbf{CP}_\alpha)$ as $\alpha \searrow 0$. The case $\alpha = 0$, which has been thoroughly investigated in [1] for a potential as in (1.13), with $(r_-, r_+) = (-1, 1)$, is denoted by $(\mathbf{CP}_0)$.

5.1. Convergence of the state variables

The asymptotic results for the state system have already been studied in [13] where also the case of the nondifferentiable double obstacle potential (1.14) was included in the analysis (see, [13, Thm. 5.1]). We have the following result.

Theorem 5.1. *Assume that (A1)–(A5) are satisfied, and let $\{\alpha_n\}_{n \in \mathbb{N}} \subset (0, 1]$ and $\{u^{\alpha_n}\}_{n \in \mathbb{N}} \subset \mathcal{U}_{\text{ad}}$ be sequences such that $\alpha_n \searrow 0$ and $u^{\alpha_n} \rightarrow u^0$ weakly star in $L^\infty(Q)$. If $(\mu^{\alpha_n}, \varphi^{\alpha_n}, w^{\alpha_n}) := \mathcal{S}^{\alpha_n}(u^{\alpha_n})$ denotes for $n \in \mathbb{N}$ the unique solution to the state system (1.2)–(1.6) for $u = u^{\alpha_n}$ established in Theorem 2.1, then it holds*

$$\mu^{\alpha_n} \rightarrow \mu^0 \quad \text{weakly star in } L^\infty(0, T; V), \quad (5.1)$$

$$\alpha_n \mu^{\alpha_n} \rightarrow 0 \quad \text{weakly star in } W^{2,\infty}(0, T; V^*) \text{ and strongly in } W^{1,\infty}(0, T; H), \quad (5.2)$$

$$\begin{aligned} \varphi^{\alpha_n} \rightarrow \varphi^0 \quad \text{weakly star in } W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \\ \text{and strongly in } C^0([0, T]; V) \cap C^0(\overline{Q}), \end{aligned} \quad (5.3)$$

$$f'(\varphi^{\alpha_n}) \rightarrow f'(\varphi^0) \quad \text{weakly star in } L^\infty(0, T; H) \text{ and a.e. in } Q, \quad (5.4)$$

$$w^{\alpha_n} \rightarrow w^0 \quad \text{weakly star in } W^{1,\infty}(0, T; L^\infty(\Omega)), \quad (5.5)$$

where (μ^0, φ^0, w^0) is the unique strong solution to the viscous Cahn–Hilliard system

$$\partial_t \varphi - \Delta \mu = 0 \quad \text{a.e. in } Q, \quad (5.6)$$

$$\tau \partial_t \varphi - \Delta \varphi + f'(\varphi) = \mu + w \quad \text{a.e. in } Q, \quad (5.7)$$

$$\gamma \partial_t w + w = u^0 \quad \text{a.e. in } Q, \quad (5.8)$$

$$\partial_n \mu = \partial_n \varphi = 0 \quad \text{a.e. on } \Sigma, \quad (5.9)$$

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0, \quad w(0) = w_0 \quad \text{a.e. in } \Omega. \quad (5.10)$$

Proof. Arguing as in the proof of Theorem 4.1, by (2.6) it is not difficult to check that $(\mu^{\alpha_n}, \varphi^{\alpha_n}, w^{\alpha_n})$ converges to a triple (μ, φ, w) in the sense of (5.1)–(5.3), (5.5), at least on a subsequence that we still index by $n \in \mathbb{N}$. In view of (5.3) and the Lipschitz continuity of f'_2 , we have that $f'_2(\varphi^{\alpha_n}) \rightarrow f'_2(\varphi)$ strongly in $C^0(\overline{Q})$. Moreover, recalling (2.6), (A3), and the estimate (2.9), we also have that $f'_1(\varphi^{\alpha_n})$ weakly star converges to a limit in $L^\infty(0, T; H)$ on one hand, and

$$f'_1(\varphi^{\alpha_n}) \rightarrow f'_1(\varphi) \quad \text{a.e. in } Q$$

on the other. Then, the two limits must coincide and $f'_1(\varphi)$ makes sense in $L^\infty(0, T; H)$, although it is no longer expected that $f'_1(\varphi)$ is uniformly bounded in $L^\infty(Q)$ (cf. (2.9)).

Therefore, passing to the limit as $n \rightarrow \infty$ in the state system (1.2)–(1.6) written for $(\mu^{\alpha_n}, \varphi^{\alpha_n}, w^{\alpha_n})$ and u^{α_n} , reveals that (μ, φ, w) solves the system (5.6)–(5.10) at least in a variational sense. However, as $\mu \in L^\infty(0, T; V)$ and $\partial_t \varphi \in L^\infty(0, T; H)$, from (5.7), (5.9), and standard elliptic regularity results it follows that $\mu \in L^\infty(0, T; W)$, whence (μ, φ, w) is actually a strong solution of (5.6)–(5.10). Using exactly the same proof as for the continuous dependence part of [1, Thm. 2.2], we can conclude that this solution is uniquely determined, and thus we have $(\mu, \varphi, w) = (\mu^0, \varphi^0, w^0)$. Moreover, the unicity of the limit point entails that the convergence properties are in fact valid for the entire sequence and not just for a subsequence. This concludes the proof of the assertion. $\square$

Remark 5.2. Notice that we cannot expect to have an initial condition for $\partial_t \mu$ in the limiting state problem. We also note that it makes sense to introduce the control-to-state operator for the viscous Cahn–Hilliard system (5.6)–(5.10),

$$\mathcal{S}^0 = (\mathcal{S}_1^0, \mathcal{S}_2^0, \mathcal{S}_3^0) : u^0 \in L^\infty(Q) \mapsto (\mathcal{S}_1^0(u^0), \mathcal{S}_2^0(u^0), \mathcal{S}_3^0(u^0)) := (\mu^0, \varphi^0, w^0). \quad (5.11)$$

5.2. Convergence of optimal controls

As we have seen in the previous subsection, the state variables associated with $\alpha \in (0, 1]$ converge, as $\alpha \searrow 0$ along a subsequence, in a well-defined sense to their counterparts for the case $\alpha = 0$. We are now going to show that the same holds true for the optimal controls.

Theorem 5.3. Suppose that (A1)–(A8) are fulfilled, and let $\{\alpha_n\}_{n \in \mathbb{N}} \subset (0, 1]$ be given with $\alpha_n \searrow 0$. For every $n \in \mathbb{N}$, let $u^{\alpha_n} \in \mathcal{U}_{\text{ad}}$ be an optimal control for the problem $(\mathbf{CP}_{\alpha_n})$ with associated state $(\mu^{\alpha_n}, \varphi^{\alpha_n}, w^{\alpha_n}) = \mathcal{S}^{\alpha_n}(u^{\alpha_n})$. If, in addition, $u^{\alpha_n} \rightarrow u^0$ weakly star in $L^\infty(Q)$ as $n \rightarrow \infty$, then u^0 is an optimal control of the problem $(\mathbf{CP}_0)$.

Proof. First, we observe that, according to Theorem 5.1, we have the convergence properties pier(5.1)–(5.5), where $(\mu^0, \varphi^0, w^0) = \mathcal{S}^0(u^0)$, which means, in particular, that the pair $((\mu^0, \varphi^0, w^0), u^0)$ is admissible for the limit problem $(\mathbf{CP}_0)$. Moreover, it obviously holds

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{b_1}{2} \iint_Q |\varphi^{\alpha_n} - \varphi_Q|^2 + \frac{b_2}{2} \int_\Omega |\varphi^{\alpha_n}(T) - \varphi_\Omega|^2 \right) \\ &= \frac{b_1}{2} \iint_Q |\varphi^0 - \varphi_Q|^2 + \frac{b_2}{2} \int_\Omega |\varphi^0(T) - \varphi_\Omega|^2. \end{aligned} \quad (5.12)$$

Now let $u \in \mathcal{U}_{\text{ad}}$ be arbitrary. We recall the weak sequential lower semicontinuity of the $L^2(Q)$ -norm and of the functional G , which is convex and continuous on $L^2(Q)$. Then, from (5.12) and the optimality of the pair $((\mu^{\alpha_n}, \varphi^{\alpha_n}, w^{\alpha_n}), u^{\alpha_n})$ for the problem $(\mathbf{CP}_{\alpha_n})$, it follows that

$$\begin{aligned} \mathcal{J}(\mathcal{S}_2^0(u^0), u^0) &= \frac{b_1}{2} \iint_Q |\varphi^0 - \varphi_Q|^2 + \frac{b_2}{2} \int_\Omega |\varphi^0(T) - \varphi_\Omega|^2 + \frac{b_3}{2} \iint_Q |u^0|^2 + \kappa G(u^0) \\ &\leq \liminf_{n \rightarrow \infty} \left(\frac{b_1}{2} \iint_Q |\varphi^{\alpha_n} - \varphi_Q|^2 + \frac{b_2}{2} \int_\Omega |\varphi^{\alpha_n}(T) - \varphi_\Omega|^2 + \frac{b_3}{2} \iint_Q |u^{\alpha_n}|^2 + \kappa G(u^{\alpha_n}) \right) \\ &= \liminf_{n \rightarrow \infty} \mathcal{J}(\mathcal{S}_2^{\alpha_n}(u^{\alpha_n}), u^{\alpha_n}) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(\mathcal{S}_2^{\alpha_n}(u), u) = \mathcal{J}(\mathcal{S}_2^0(u), u). \end{aligned} \quad (5.13)$$

Since $u \in \mathcal{U}_{\text{ad}}$ was arbitrary, the assertion is proved. $\square$

Remark 5.4. 1. Notice that any sequence $\{u^{\alpha_n}\} \subset \mathcal{U}_{\text{ad}}$ with $\alpha_n \searrow 0$ contains at least one subsequence that converges in the weak star topology of $L^\infty(Q)$ to some limit point $u \in \mathcal{U}_{\text{ad}}$. From this it also follows that the problem $(\mathbf{CP}_0)$ admits at least one solution.

2. Observe that we cannot expect a similar result for controls that are only locally optimal in the sense of $L^\infty(Q)$. Indeed, if u^{α_n} is locally optimal for $(\mathbf{CP}_{\alpha_n})$ in the sense of $L^\infty(Q)$, then we can only guarantee the existence of some $\varepsilon_n > 0$ such that $\widehat{\mathcal{J}}(\mathcal{S}_2^{\alpha_n}(u^{\alpha_n}), u^{\alpha_n}) \leq \widehat{\mathcal{J}}(\mathcal{S}_2^{\alpha_n}(u), u)$ for all $u \in \mathcal{U}_{\text{ad}}$ with $\|u - u^{\alpha_n}\|_{L^\infty(Q)} \leq \varepsilon_n$. Since it is very possible that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, we cannot conclude that limits of subsequences of $\{u^{\alpha_n}\}$ are locally optimal for $(\mathbf{CP}_0)$.

5.3. Convergence of the adjoint variables

Finally, we present a convergence result for the adjoint state variables under the following additional restriction.

(A9) In the framework of **(A3)**, we further suppose that $r_- = -\infty$ and $r_+ = +\infty$ so that both f_1 and f_2 belong to $C^3(\mathbb{R})$.

Theorem 5.5. Suppose that **(A1)**–**(A9)** are fulfilled, and let $\{\alpha_n\}_{n \in \mathbb{N}}$ and $\{u^{\alpha_n}\}_{n \in \mathbb{N}} \subset \mathcal{U}_{\text{ad}}$ be sequences with $\alpha_n \searrow 0$ and $u^{\alpha_n} \rightarrow u^0$ weakly star in $L^\infty(Q)$. If $(\mu^{\alpha_n}, \varphi^{\alpha_n}, w^{\alpha_n}) = \mathcal{S}^{\alpha_n}(u^{\alpha_n})$ and $(p^{\alpha_n}, q^{\alpha_n}, r^{\alpha_n})$ denote the respective state and adjoint state variables associated with u^{α_n} for $n \in \mathbb{N}$, then it holds

$$p^{\alpha_n} \rightarrow p^0 \quad \text{weakly star in } L^\infty(0, T; W), \quad (5.14)$$

$$\alpha_n p^{\alpha_n} \rightarrow 0 \quad \text{weakly star in } W^{2,\infty}(0, T; H) \text{ and strongly in } W^{1,\infty}(0, T; V), \quad (5.15)$$

$$q^{\alpha_n} \rightarrow q^0 \quad \text{weakly star in } L^\infty(0, T; V) \cap L^2(0, T; W), \quad (5.16)$$

$$p^{\alpha_n} + \tau q^{\alpha_n} \rightarrow p^0 + \tau q^0 \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \\ \text{and strongly in } C^0([0, T]; H) \cap L^2(0, T; V), \quad (5.17)$$

$$r^{\alpha_n} \rightarrow r^0 \quad \text{weakly star in } W^{1,\infty}(0, T; V) \cap H^1(0, T; W), \quad (5.18)$$

where the triple (p^0, q^0, r^0) is the unique solution to the adjoint system associated with the system (5.6)–(5.10) and the control u^0 , which is with $(\mu^0, \varphi^0, w^0) := \mathcal{S}^0(u^0)$ given by

$$-\Delta p - q = 0 \quad \text{a.e. in } Q, \quad (5.19)$$

$$-\partial_t(p + \tau q) - \Delta q + f''(\varphi^0)q = b_1(\varphi^0 - \varphi_Q) \quad \text{a.e. in } Q, \quad (5.20)$$

$$-\gamma \partial_t r + r - q = 0 \quad \text{a.e. in } Q, \quad (5.21)$$

$$\partial_n p = \partial_n q = 0 \quad \text{a.e. on } \Sigma, \quad (5.22)$$

$$(p + \tau q)(T) = b_2(\varphi^0(T) - \varphi_\Omega), \quad r(T) = 0 \quad \text{a.e. in } \Omega. \quad (5.23)$$

Proof. From Theorem 5.1 we infer that $(\mu^{\alpha_n}, \varphi^{\alpha_n}, w^{\alpha_n})$ converges to (μ^0, φ^0, w^0) in the sense of (5.1)–(5.5). Moreover, recalling Remark 2.2, it turns out that, under the validity of **(A9)**, the values r_* and r^* in (2.7) and the constant K_2 in (2.9) are independent of α . Then, we can also infer that

$$f''(\varphi^{\alpha_n}) \rightarrow f''(\varphi^0) \quad \text{strongly in } C^0(\overline{Q}).$$

Moreover, it follows from the uniform estimate (4.18) (see Remark 4.4 as well) that $(p^{\alpha_n}, q^{\alpha_n}, r^{\alpha_n})$ converges to a triple (p, q, r) in the sense of (5.14)–(5.18) at least on a subsequence. It is then readily seen that (p, q, r) is a strong solution to the system (5.19)–(5.23). In view of the linearity of this system, the solution is easily shown to be uniquely determined (anyway, the reader may consult and refer to [1, Theorem 4.3]), and thus $(p, q, r) = (p^0, q^0, r^0)$.

We point out the correctness of the first terminal condition in (5.23) since the variable under time derivative in (5.20) is just $(p + \tau q)$, so that the terminal condition follows from the convergence (5.17) and the conditions in (4.14).

Hence, owing to the uniqueness of the limit point (p^0, q^0, r^0) , the convergence properties are in fact valid for the entire sequence $\alpha_n \searrow 0$ and not just for a subsequence. This concludes the proof of the assertion. $\square$

Remark 5.6. It is worth noting that the additional assumption (A9) is not necessary for the conditions (5.19)–(5.23) to be valid. We demonstrate this for the case when f is the logarithmic potential (1.13). Indeed, also in this case it follows from Theorem 5.1 that $(\mu^{\alpha_n}, \varphi^{\alpha_n}, w^{\alpha_n})$ converges to (μ^0, φ^0, w^0) in the sense of (5.1)–(5.5). In particular, we have that φ^{α_n} converges uniformly to φ^0 in $\overline{Q}$ by (5.3). At this point, we recall [1, Thm. 2.2], which states that the solution component φ^0 of the limit problem (5.6)–(5.10) has the separation property. That is, the range of φ^0 is contained in a compact subset of $(-1, 1)$. But this implies that also the ranges of φ^{α_n} are contained in a common compact subset of $(-1, 1)$ for all sufficiently large n . Therefore, we can infer that also in this case

$$f''(\varphi^{\alpha_n}) \rightarrow f''(\varphi^0) \quad \text{strongly in } C^0(\overline{Q})$$

holds true. Having established this strong convergence property, we now can conclude as in the remainder of the proof of Theorem 5.5. In fact, the same argument applies whenever the limit problem enjoys a separation property.

Author contributions

Both authors contributed equally to all aspects of the work, including Conceptualization, Methodology, Formal analysis, Investigation, Writing original draft, and Writing review and editing, in accordance with the CRediT (Contributor Roles Taxonomy) system.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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