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Research article

Normalized solutions for Kirchhoff equations with Choquard nonlinearity: mass Super-Critical Case

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Abstract: In the present paper, we investigated the existence of normalized solutions for the following Kirchhoff equation with Choquard nonlinearity

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u - \lambda u = \mu|u|^{q-2}u + (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^3$$

with prescribed mass $\int_{\mathbb{R}^3} |u|^2 dx = c^2$, where $a, b, c > 0$, $\mu \in \mathbb{R}$, $\alpha \in (0, 3)$, $\frac{10}{3} \le q < 6$, $3 + \frac{\alpha}{3} \le p < 3 + \alpha$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier. We first considered the case of $\mu > 0$ and obtained mountain pass type solutions. For the defocusing situation $\mu < 0$, we proved the existence result by constructing a minimax characterization for the energy functional. Finally, we discussed the asymptotic behavior of normalized solutions obtained above as $b \to 0^+$ when $\mu > 0$.

Keywords: Choquard nonlinearity; Kirchhoff equation; normalized solutions; defocusing case; asymptotic behavior
Mathematics Subject Classification: 35A15, 35J60, 58E05

1. Introduction and main results

In this paper we are interested in the following Kirchhoff equation with Choquard nonlinearity:

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u - \lambda u = \mu|u|^{q-2}u + (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^3 \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^3} |u|^2 dx = c^2, \tag{1.2}$$

where $a, b, c > 0$, $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $\alpha \in (0, 3)$, $\frac{10}{3} \le q < 6$, $3 + \frac{\alpha}{3} \le p < 3 + \alpha$. $I_\alpha : \mathbb{R}^N \setminus \{0\} \mapsto \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{with} \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})}.$$

The problem (1.1) is closely related to the equation

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u = f(x, u), \tag{1.3}$$

which is the stationary analog of the equation

$$u_{tt} - \Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u = f(x, u), \tag{1.4}$$

where $f(x, u)$ is a general nonlinearity. The problem (1.4) was proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equations for free vibration of elastic strings. In problem (1.4), u denotes the displacement, the nonlinear term f is the external force, and a is the initial tension while b is related to the intrinsic properties of the string. Mathematically, the problem (1.4) is often referred to be nonlocal as the appearance of the term $(\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u$, which depends not only on the pointwise value of Δu, but also on the integral of $|\nabla u|_2$ over the whole space. This phenomenon causes some mathematical difficulties, which make the study of Kirchhoff type equations particularly interesting.

After the pioneering work of Lions [2], (1.3) began to receive much attention and many researchers studied its steady-state model, see [3–6] for more important research progress. Some scholars have also considered generalizations of fixed-frequency solutions for Kirchhoff equations. Gao et al. [7] studied the nonlinear coupled Kirchhoff system with purely Sobolev critical exponent

$$\begin{cases} -\Big(a_1 + b_1 \int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\Delta u = \mu_1 |u|^{2^*-2} u + \frac{\alpha\gamma}{2^*} |u|^{\alpha-2} u |v|^\beta & x \in \mathbb{R}^N, \\ -\Big(a_2 + b_2 \int_{\mathbb{R}^N} |\nabla v|^2 dx\Big)\Delta v = \mu_2 |v|^{2^*-2} v + \frac{\beta\gamma}{2^*} |u|^\alpha |v|^{\beta-2} v & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $N \ge 3$, $a_i, b_i \ge 0$, $i = 1, 2$, $\mu_1, \mu_2, \gamma > 0$, and $\alpha + \beta = 2^*$. They gave a complete classification of positive ground states for (1.5) in any dimension 3 or 4. Sun et al. [8] extended the results to the p-sub-Laplacians and obtained the multiple solutions. As to the case of bounded domains in $\mathbb{R}^N$, Cabanillas [9] studied the global existence theorem and its exponential decay. In addition, Yang and Tang [10] dealt with the nonlinear Kirchhoff problem with a sign potential

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3, \tag{1.6}$$

where $b > 0$ and the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ exhibits subcritical growth. By using a more general global compactness lemma and a sign-changing Nehari manifold, they showed the existence of a least energy sign-changing solution for $b > 0$ that is sufficiently small and established the asymptotic behavior when $b \to 0^+$.

Now there are two substantially different viewpoints in terms of the frequency λ in (1.1). One is to regard the frequency λ as a given constant. In this situation, solutions of (1.1) are critical points of the following functional

$$\begin{aligned} I_\mu(u) : = & \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2-\frac{\lambda}{2}\int_{\mathbb{R}^3}|u|^2dx-\frac{\mu}{q}\int_{\mathbb{R}^3}|u|^qdx \\ & -\frac{1}{2p}\int_{\mathbb{R}^3}(I_\alpha*|u|^p)|u|^pdx. \end{aligned}$$

Although this is not the concern of our present article, we still refer the readers to [11–14]. Naturally, the other one is to regard λ as an unknown parameter, which is exactly what our article is concerned. For this case, standing wave solutions are required to possess a priori prescribed L^2-norm, which also have attracted widespread attention during recent years. These solutions are commonly called as normalized solutions, which provide valuable insights into dynamical properties of stationary solutions, such as the stability or instability of orbits. In addition, it is natural to prescribe the value of the mass so that λ can be interpreted as a Lagrange multiplier. For example, from a physical point of view, the normalized condition may represent the number of particles of each component in Bose-Einstein condensates or the power supply in the nonlinear optics framework. To obtain the solutions of (1.1)-(1.2), it suffices to consider critical points of the functional

$$E_\mu(u) := \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2-\frac{\mu}{q}\int_{\mathbb{R}^3}|u|^qdx-\frac{1}{2p}\int_{\mathbb{R}^3}(I_\alpha*|u|^p)|u|^pdx \tag{1.7}$$

on

$$S_c := \Big\{u\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}|u|^2dx=c^2\Big\}$$

with the parameter $\lambda\in\mathbb{R}$ appearing as a Lagrange multiplier.

In order to narrate the relevant results and state our motivation conveniently, we consider the following Kirchhoff-type equations with convolutional terms:

$$\begin{cases} -\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u=\lambda u+\mu|u|^{q-2}u+\gamma(I_\alpha*|u|^p)|u|^{p-2}u, & x\in\mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2dx=c^2, \end{cases} \tag{1.8}$$

where $a>0, b\geq 0, c>0, N\geq 3, \mu,\gamma\in\mathbb{R}, 2<q<\frac{2N}{N-2}$, and $\frac{N+\alpha}{N}<p<\frac{N+\alpha}{N-2}$.

Taking $a=1$ and $b=0$, (1.8) reduces to the nonlinear Choquard equation with combined nonlinearities. The endpoints of $\frac{N+\alpha}{N}$ and $\frac{N+\alpha}{N-2}$ established in [15] are called lower- and upper-critical exponent. The upper-critical exponent plays a similar role as the Sobolev critical exponent in the local semilinear equations, for instance, Li [16] proved the existence and orbital stability of ground states of (1.8) when $p=\frac{N+\alpha}{N-2}, \gamma=1$. As to the lower critical exponent, it seems to be a new feature for the Choquard equation, which is related to a new phenomenon of bubbling at infinity, see [17, 18].

If $b>0$, $N=3$, then (1.8) is a nonlinear Kirchhoff equation. Zeng and Zhang [19] proved the existence and uniqueness of solutions to (1.8) with $q\in(2,6)$, $\gamma=0$, and $\mu=1$ by using some simple energy estimates rather than the concentration-compactness principles. In addition, Ye [20] considered the existence and mass concentration of solutions of (1.8) under the case of L^2-critical exponent, namely, $q=\frac{14}{3}$. If $\gamma\neq 0$, then (1.8) can be viewed as a Kirchhoff-Choquard type equation. Liu [21] considered the case of $\gamma=1$, $\mu=0$, and $\frac{N+\alpha}{N}<p<\frac{N+\alpha}{N-2}$, and he provided threshold values c_* and c_{**} related to c separating the existence and nonexistence of normalized solutions of it when p belongs to different ranges. In addition, he also deduced that (1.8) has no nontrivial solutions in the cases of $p = \frac{N+\alpha}{N}$ or $p = \frac{N+\alpha}{N-2}$. For the cases of non-autonomous Kirchhoff equations, Qiu et al. [22] considered the following non-autonomous Kirchhoff equation with a perturbation:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + \lambda u = h(x)|u|^{q-2}u + |u|^{p-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c^2, \end{cases} \tag{1.9}$$

where $1 \le N \le 3$, $a, b, c > 0$, $1 \le q < 2$, $2 < p < 2^*$, $h(x) \in \mathbb{R}$. They proved the existence of mountain pass solutions and bound-state solutions. Furthermore, Ni et al. [23] dealt with the following non-autonomous Kirchhoff equations with general nonlinearities:

$$\begin{cases} -\left(a\varepsilon + b\varepsilon\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = \lambda u + f(u), & x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = c^2\varepsilon, \end{cases} \tag{1.10}$$

where $a, b, c > 0$, V is a nonnegative continuous function, and f is a continuous function with L^2-subcritical growth. When $\varepsilon > 0$ is small enough, by using minimization techniques and the Lusternik-Schnirelmann theory, they pointed out that the number of normalized solutions was related to the topological richness of the set where the potential V attained its minimum value.

Motivated by the above analysis, we consider (1.8) with $\gamma = 1$, $\mu \neq 0$, and $N = 3$. Compared to the case of $\mu = 0$, at this time, (1.8) is regarded as a Kirchhoff type mixed equation with convolutional terms, which leads to a more complex geometric structure of the energy functional and makes the compactness analysis and energy estimates more difficult. In addition, we need to accurately determine the range of the parameter p to ensure that the convolution term $(I_\alpha * |u|^p)|u|^{p-2}u$ is the leading term.

In the present paper, we study the existence and asymptotic behavior of solutions to (1.1)-(1.2). We say that u_0 is a ground state to (1.1)-(1.2) if it is a solution to (1.1)-(1.2) having minimal energy among all the solutions belong to S_c

$$dE_\mu|_{S_c}(u_0) = 0 \quad \text{and} \quad E_\mu(u_0) = \inf\{E_\mu(u) : dE_\mu|_{S_c}(u) = 0, u \in S_c\}. \tag{1.11}$$

The following Gagliardo-Nirenberg inequality [24] is also crucial in our argument, that is, there exists a best constant C_q depending on q such that

$$\|u\|_q \le C_q \|\nabla u\|_2^{\delta_q} \cdot \|u\|_2^{1-\delta_q}, \quad q \in (2, 6), \quad \forall u \in H^1(\mathbb{R}^3), \tag{1.12}$$

where $\delta_q := \frac{3(q-2)}{2q}$.

The following inequality introduced in [25] is called Gagliardo-Nirenberg inequality of Hartree type:

$$\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p dx \le \frac{p}{\|Q_p\|_2^{2p-2}} \|\nabla u\|_2^{2p\gamma_p} \cdot \|u\|_2^{3+\alpha-p}, \quad \forall u \in H^1(\mathbb{R}^3), \tag{1.13}$$

where $\gamma_p := \frac{3p-3-\alpha}{2p}$ and equality holds for $u = Q_p$, Q_p is a nontrivial solution of

$$-\frac{3p-3-\alpha}{2}\Delta Q_p + \frac{3+\alpha-p}{2} Q_p = (I_\alpha * |Q_p|^p)|Q_p|^{p-2}Q_p, \quad x \in \mathbb{R}^3.$$

Now, we state our main results:

Theorem 1.1. *Let $a, b, c > 0$, $3 + \frac{\alpha}{3} \leq p < 3 + \alpha$, $\alpha \in (0, 3)$, and $\mu > 0$.*

(1) *If $q = \frac{10}{3}$ with $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$ or $\frac{14}{3} \leq q < 6$, the problem* (1.1)-(1.2) *has a positive radial ground state solution of mountain pass type at a positive level m for some $\lambda < 0$.*

(2) *If $\frac{10}{3} < q < \frac{14}{3}$, the problem* (1.1)-(1.2) *has a positive radial solution of mountain pass type at a positive level m for some $\lambda < 0$.*

Theorem 1.2. *Let $a, b, c > 0$, $\frac{10}{3} \leq q \leq \frac{14}{3} \leq p < 3 + \alpha$, and $\alpha \in (\frac{5}{3}, 3)$. If the following inequality*

$$\Big(1 - \frac{1}{\gamma_p}\Big)\Big(\frac{\gamma_p}{\gamma_p - \delta_q}\Big)\frac{c^{q(\delta_q - 1)}}{C_q^q}(a + b\widetilde{C_0})\widetilde{C_0}^{\frac{2 - q\delta_q}{2}} < \mu < 0 \tag{1.14}$$

holds, where $\widetilde{C_0} = \Big(\frac{ac^{p-3-\alpha}}{p\gamma_p}\|Q_p\|_2^{2p-2}\Big)^{\frac{2}{2p\gamma_p - 2}}$, then the problem (1.1)-(1.2) *has a mountain pass type ground state $\tilde{u}$, with the following properties: $\tilde{u}$ is a radial function, and solves* (1.1)-(1.2) *for some $\lambda < 0$ and $E_\mu(\tilde{u}) > 0$.*

Theorem 1.3. *Let $u_b \in S_{c,r}$ be the solution of* (1.1) *obtained by Theorem 1.1. Then, up to a subsequence, we have $u_b \to u$ in $H_r^1(\mathbb{R}^3)$ as $b \to 0^+$, where $u \in S_{c,r}$ is a solution of*

$$-a\Delta u = \lambda u + \mu |u|^{q-2}u + (I_\alpha * |u|^p)|u|^{p-2}u, \quad in\ \mathbb{R}^3 \tag{1.15}$$

for some $\lambda < 0$.

Remark 1.4. Compared with $b = 0$, the case of $b > 0$ is more delicate because of the presence of the nonlocal term $(\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u$, which causes that the weak limit u of a Palais-Smale sequence $\{u_n\}$ may not solve (1.1)-(1.2) and makes compactness analysis more complex. In addition, dealing with the convergence of the convolutional term $(I_\alpha * |u|^p)|u|^{p-2}u$ is a challenge. Finally, motivated by the defocusing case of Schrödinger equations, which were studied by Soave [26] and Luo et al. [27], we discuss the existence of solutions to (1.1)-(1.2) under the case $\mu < 0$. Compared to [26] and [27], $b > 0$ has a significant impact on the analysis of compactness in the defocusing case.

Remark 1.5. When $p = 3 + \alpha$, the term $(I_\alpha * |u|^p)|u|^{p-2}u$ can be seen a Sobolev critical term and the lack of compactness is a challenge. Soave [26] and Li et al. [28] both used the method introduced by Brezis and Nirenberg [29] in Sobolev critical case; this method ensured that energy level is less than a threshold, which is an essential ingredient in compactness argument. However, in our article, the existence of $(\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u$ and $(I_\alpha * |u|^p)|u|^{p-2}u$ makes it difficult for us to accurately estimate the energy level. So, the convergence of a Palais-Smale sequence is a very delicate problem, which at the moment we could not solve.

2. Preliminaries

In this section, we give some preliminary results that will be used throughout the rest of the paper. To start, we introduce the following notations:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the norm $\|u\| = (\|\nabla u\|_2^2 + \|u\|_2^2)^{\frac{1}{2}}$.
- $H_r^1(\mathbb{R}^3)$ denotes the subspace of functions in $H^1(\mathbb{R}^3)$ which are radially symmetric with respect to 0, $S_{c,r} = H_r^1(\mathbb{R}^3) \cap S_c$.
- $L^p(\mathbb{R}^3)(1 \leq p < \infty)$, denotes the Lebesgue space with the norm $\|u\|_p = (\int_{\mathbb{R}^3} |u|^p dx)^{\frac{1}{p}}$.

- $D^{1,2}(\mathbb{R}^3) = \{u \in L^{2^*}(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$.
- $o_n(1)$ denotes the vanishing quantities as $n \to \infty$.

Next, we give some lemmas that will be used throughout the rest of the paper.

Lemma 2.1. *If $u \in H^1(\mathbb{R}^3)$ is a weak solution of* (1.1)*, then the Nehari-Pohozaev identity*

$$P_\mu(u) := a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \mu\delta_q\|u\|_q^q - \gamma_p \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p dx = 0$$

holds.

Proof. If u is a weak solution of

$$-\Delta u + \lambda u = \mu|u|^{q-2}u + \gamma(I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^3,$$

then we get

$$\frac{1}{2}\|\nabla u\|_2^2 + \frac{3}{2}\|u\|_2^2 = \mu\frac{3}{q}\|u\|_q^q + \gamma\frac{3+\alpha}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx. \tag{2.1}$$

by [30, Corollary 2.5]. Furthermore, we can regard the term $(a + b\int_{\mathbb{R}^3}|\nabla u|^2 dx)$ in (1.1) as a constant coefficient motivated by [28, Lemma 2.3]. Therefore, combining (2.1) with the conclusion of Pohozaev identity of Schrödinger equation [31], we see immediately that

$$\frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 = \frac{3}{2}\lambda\|u\|_2^2 + \mu\frac{3}{q}\|u\|_q^q + \frac{3+\alpha}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx. \tag{2.2}$$

In addition, since $u \in H^1(\mathbb{R}^3)$ is a weak solution of (1.1)-(1.2), we have

$$a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 = \lambda\|u\|_2^2 + \mu\|u\|_q^q + \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx. \tag{2.3}$$

Combining (2.2) with (2.3), we infer that

$$a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \mu\delta_q\|u\|_q^q - \gamma_p\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx = 0.$$

□

When the energy functional E_μ is unbounded from below on S_c, we introduce the Pohozaev set:

$$\mathcal{P}_{c,\mu} := \{u \in S_c : P_\mu(u) = 0\}. \tag{2.4}$$

Lemma 2.1 implies that any critical point of $E_\mu|_{S_c}$ is contained in $\mathcal{P}_{c,\mu}$. For $t \in \mathbb{R}$ and $u \in S_c$, we define

$$(t \star u)(x) := e^{\frac{3t}{2}}u(e^t x).$$

Then, $t \star u \in S_c$. The map

$$(t, u) \in \mathbb{R} \times H^1(\mathbb{R}^3) \mapsto (t \star u) \in H^1(\mathbb{R}^3) \text{ is continuous}, \tag{2.5}$$

see [32, Lemma 3.5]. Similar to [33], we define the fiber map

$$\Psi^u_\mu(t) := E_\mu(t \star u) = \frac{a}{2}e^{2t}\|\nabla u\|_2^2 + \frac{b}{4}e^{4t}\|\nabla u\|_2^4 - \frac{\mu}{q}e^{q\delta_q t}\|u\|_q^q - \frac{1}{2p}e^{2p\gamma_p t}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx. \tag{2.6}$$

An easy computation shows that

$$(\Psi^u_\mu)'(t) = P_\mu(t \star u). \tag{2.7}$$

So we see immediately that for $u \in S_c$, t is a critical point of $\Psi^u_\mu(t)$ if and only if $t \star u \in \mathcal{P}_{c,\mu}$.

We need to recall the Hardy-Littlewood-Sobolev inequality.

Lemma 2.2. (Hardy-Littlewood-Sobolev inequality) *[34] Let $N \geq 1$, $p, r > 1$, and $0 < \alpha < N$ with $\frac{1}{p} + \frac{N-\alpha}{N} + \frac{1}{r} = 2$, $u \in L^p(\mathbb{R}^N)$, $v \in L^r(\mathbb{R}^N)$. Then, there exists a sharp constant $C(N, \alpha, p)$ independent of u and v such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u(x)v(y)}{|x-y|^{N-\alpha}}dxdy\right| \leq C(N,\alpha,p)\|u\|_p\|v\|_r.$$

Lemma 2.3. *[35, Lemma 2.3] Let $N \geq 3$, $\alpha \in (0, N)$, and $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$. Assume that the sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ satisfies $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Then,*

$$(I_\alpha * |u_n|^p)|u_n|^{p-2}u_n \rightharpoonup (I_\alpha * |u|^p)|u|^{p-2}u \text{ in } H^{-1}(\mathbb{R}^N) \quad \text{as} \quad n \to \infty.$$

Lemma 2.4. *[32, Lemma 3.6] For $u \in S_c$ and $t \in \mathbb{R}$, the map $\varphi \mapsto t * \varphi$ from $T_u S_c$ to $T_{t*u} S_c$ is a linear isomorphism with inverse $\psi \mapsto (-t) * \psi$, where $T_u S_c = \{\varphi \in S_c : \int_{\mathbb{R}^3} u\varphi = 0\}$.*

Lemma 2.5. *Assume $\frac{3+\alpha}{3} \leq p \leq 3 + \alpha$. Then the energy functional ϕ is invariant under any orthogonal transformation in $\mathbb{R}^3$, where*

$$\phi(u) = \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx, \quad u \in H^1(\mathbb{R}^3).$$

Proof. We define the following group with orthogonal invariance:

$$O(3) := \{A \in R^{3\times 3} | A^T A = I\}$$

and

$$\tilde{x} := Ax, \quad \tilde{y} := Ay, \quad u_A(x) := u(\tilde{x}), \quad u_A(y) := u(\tilde{y}),$$

with $A \in O(3)$. We obtain

$$\begin{aligned}
\phi(u_A) &= \int_{\mathbb{R}^3}(I_\alpha * |u_A|^p)|u_A|^p dx \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u_A(x)|^p|u_A(y)|^p}{|x-y|^{3-\alpha}}dxdy \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(\tilde{x})|^p|u(\tilde{y})|^p}{|A^{-1}\tilde{x} - A^{-1}\tilde{y}|^{3-\alpha}}d\tilde{x}d\tilde{y} \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(\tilde{x})|^p|u(\tilde{y})|^p}{|\tilde{x}-\tilde{y}|^{3-\alpha}}d\tilde{x}d\tilde{y} \\
&= \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx = \phi(u)
\end{aligned}$$

by $\left|\det\left(\frac{\partial\tilde{x}}{\partial x}\right)\right| = \left|\det\left(\frac{\partial\tilde{y}}{\partial y}\right)\right| = |\det(A)| = 1$. Thus, ϕ is invariant under any orthogonal transformation in $\mathbb{R}^3$. □

3. Supercritical leading term with focusing perturbation

In this section, we prove the existence of mountain pass type critical points for $E_\mu|_{S_{c,r}}$ when $\mu > 0$ and we assume that $3 + \frac{\alpha}{3} \leq p < 3 + \alpha$.

We first investigate the mountain pass geometry of E_μ on $S_{c,r}$.

Lemma 3.1. *Suppose that* $\frac{10}{3} < q < 6$ *or* $q = \frac{10}{3}$ *with* $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$.

(i) *There exist two positive numbers* $k_1 < k_2$ *sufficiently small such that*

$$0 < \sup_{\overline{A_{k_1}}} E_\mu(u) < \inf_{\partial A_{k_2}} E_\mu(u) \quad \text{and} \quad E_\mu(u) > 0,\ P_\mu(u) > 0 \quad \text{for} \quad u \in A_{k_2},$$

where

$$A_k := \{u \in S_{c,r} : \|\nabla u\|_2 < k\}. \tag{3.1}$$

(ii) *There exists* $u_0 \in S_{c,r} \setminus A_{k_2}$ *such that* $E_\mu(u_0) < 0$.

Proof. (i) In view of (1.12), (1.13) and $2 \leq q\delta_q < 2p\gamma_p$, we obtain

$$\begin{aligned} E_\mu(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx \\ &\geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{q}C_q^q c^{q(1-\delta_q)}\|\nabla u\|_2^{q\delta_q} - \frac{c^{3+\alpha-p}}{2\|Q_p\|_2^{2p-2}}\|\nabla u\|_2^{2p\gamma_p} \end{aligned}$$

and

$$\begin{aligned} P_\mu(u) &= a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \mu\delta_q\|u\|_q^q - \gamma_p\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx \\ &\geq a\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \mu\delta_q C_q^q c^{q(1-\delta_q)}\|\nabla u\|_2^{q\delta_q} - \frac{p\gamma_p c^{3+\alpha-p}}{\|Q_p\|_2^{2p-2}}\|\nabla u\|_2^{2p\gamma_p}. \end{aligned}$$

It is also clear that

$$E_\mu(u) \leq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4.$$

Notice that $2 \leq q\delta_q < 6$ and $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$ when $q\delta_q = 2$. Taking two small positive numbers $k_1 < k_2$, we arrive at the desired result.

(ii) For $u \in S_{c,r}$, we have

$$\lim_{t\to+\infty}\|\nabla(t \star u)\|_2 = +\infty, \quad \lim_{t\to+\infty} E_\mu(t \star u) = -\infty.$$

Choosing $u_0 = t \star u$ with $t > 0$ large enough, we deduce that $u_0 \in S_{c,r} \setminus A_{k_2}$ and $E_\mu(u_0) < 0$. □

By Lemma 3.1, we define the mountain pass level of the functional E_μ on $S_{c,r}$ by

$$\sigma(c,\mu) := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} E_\mu(\gamma(t)). \tag{3.2}$$

where

$$\Gamma := \{\gamma \in C([0,1], S_{c,r}) : \gamma(0) \in \overline{A_{k_1}}, E_\mu(\gamma(1)) \leq 0\}. \tag{3.3}$$

Clearly, we have

$$\sigma(c,\mu) \geq \inf_{\partial A_{k_2}} E_\mu(u) > 0. \tag{3.4}$$

Lemma 3.2. *Suppose that* $\frac{10}{3} < q < 6$ *or* $q = \frac{10}{3}$ *with* $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$. *Then, there exists a Palais-Smale sequence* $\{u_n\} \subset S_{c,r}$ *for* $E_\mu|_{S_{c,r}}$ *at the level* $\sigma(c,\mu)$ *with* $P_\mu(u_n) \to 0$ *as* $n \to \infty$.

Proof. Motivated by [33], we define the augmented functional $\tilde{E}_\mu : \mathbb{R} \times H^1(\mathbb{R}^3) \to \mathbb{R}$

$$\tilde{E}_\mu(t, u) := E_\mu(t \star u) = \Psi_\mu^u(t), \tag{3.5}$$

where $\Psi_\mu^u(t)$ is defined in (2.6). Notice that $\tilde{E}_\mu$ is of class C^1. By Lemma 2.5, we know that $\tilde{E}_\mu$ is invariant under rotations applied to u. Therefore, [31, Theorem 1.28] indicates that a critical point for $\tilde{E}_\mu|_{\mathbb{R}\times S_{c,r}}$ is a critical point for $\tilde{E}_\mu|_{\mathbb{R}\times S_c}$.

Now, we denote

$$\tilde{\Gamma} := \{\tilde{\gamma} \in C([0,1], \mathbb{R} \times S_{c,r}) : \tilde{\gamma}(0) \in \{0\} \times \overline{A_{k_1}}, \tilde{\gamma}(1) \in \{0\} \times E_\mu^0\},$$

where $E_\mu^0 := \{u \in S_{c,r} : E_\mu(u) \leq 0\}$. We easily see that if $\gamma \in \Gamma$, then $\tilde{\gamma} := (0, \gamma) \in \tilde{\Gamma}$ and $\tilde{E}_\mu(\tilde{\gamma}(t)) = E_\mu(\gamma(t))$ for $t \in [0,1]$; while if $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) \in \tilde{\Gamma}$, then $\gamma(\cdot) := \tilde{\gamma}_1 \star \tilde{\gamma}_2 \in \Gamma$ and $\tilde{E}_\mu(\tilde{\gamma}(t)) = E_\mu(\gamma(t))$ for $t \in [0,1]$. Therefore, we have

$$\sigma(c,\mu) = \inf_{\tilde{\gamma}\in\tilde{\Gamma}} \max_{t\in[0,1]} \tilde{E}_\mu(\tilde{\gamma}(t)).$$

By the definition of $\sigma(c,\mu)$, for $\varepsilon_n = \frac{1}{n^2}$, there exists $\gamma_n \in \Gamma$ such that

$$\max_{t\in[0,1]} E_\mu(\gamma(t)) \leq \sigma(c,\mu) + \frac{1}{n^2},$$

setting $\tilde{\gamma}_n = (0, \gamma_n)$, we obtain

$$\max_{t\in[0,1]} \tilde{E}_\mu(\tilde{\gamma}(t)) \leq \sigma(c,\mu) + \frac{1}{n^2}.$$

According to Ekeland's variational principle [33, Lemma 2.3], there exists a sequence $\{(t_n, v_n)\} \subset \mathbb{R}\times S_{c,r}$ such that

$$\tilde{E}_\mu(t_n, v_n) \to \sigma(c,\mu) \quad \text{and} \quad (\tilde{E}_\mu|_{\mathbb{R}\times S_{c,r}})'(t_n, v_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.6}$$

with the additional property that

$$|t_n| + dist_{H^1(\mathbb{R}^3)}(v_n, \beta_n([0,1])) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Note that $\tilde{E}_\mu(t_n, v_n) = \tilde{E}_\mu(0, t_n \star v_n)$ and

$$\langle(\tilde{E}_\mu|_{\mathbb{R}\times S_{c,r}})'(t_n, v_n), (t, \psi)\rangle = \langle(\tilde{E}_\mu|_{\mathbb{R}\times S_{c,r}})'(0, t_n \star v_n), (t, t_n \star \psi)\rangle \tag{3.8}$$

for $(t, \psi) \in \mathbb{R} \times H_r^1(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} v_n\psi = 0$. Setting $u_n = t_n \star v_n \in S_{c,r}$, by (3.6), we obtain

$$E_\mu(u_n) = \tilde{E}_\mu(0, t_n \star v_n) = \tilde{E}_\mu(t_n, v_n) \to \sigma(c,\mu), \quad \text{as} \quad n \to \infty.$$

We take $(1, 0)$ as a test function in (3.8), and it follows from (3.6) that

$$P_\mu(u_n) = \partial_t \tilde{E}_\mu(0, u_n) \to 0, \quad \text{as} \quad n \to \infty.$$

For $w \in H_r^1(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3}(t_n \star v_n)w = 0$, we take $(0, (-t_n) \star w)$ as a test function in (3.8). In view of (3.6) and (3.7), we have that $E'_\mu|_{S_{c,r}}(u_n) \to 0$ as $n \to \infty$. □

Lemma 3.3. *If $r_i > 0, i = 1, 2, 3$, then the function*

$$g(t) = r_1 e^{2t} + r_2 e^{4t} - r_3 e^{2p\gamma_p t}, \quad t \in \mathbb{R}$$

has a unique critical point at which g achieves its maximum.

Proof. By direct computation, we have

$$g'(t) = 2r_1 e^{2t} + 4r_2 e^{4t} - 2p\gamma_p r_3 e^{2p\gamma_p t} = e^{4t}(2r_1 e^{-2t} + 4r_2 - 2p\gamma_p r_3 e^{(2p\gamma_p - 4)t}) := e^{4t}\tilde{g}(t).$$

Obviously, $\tilde{g}$ is decreasing, $\lim_{t\to-\infty} \tilde{g}(t) = +\infty$ and $\lim_{t\to+\infty} \tilde{g}(t) = -\infty$, so there exists a unique $t_0 \in \mathbb{R}$ such that $\tilde{g}(t_0) = 0$, $\tilde{g}(t) < 0$ if $t > t_0$, and $\tilde{g}(t) > 0$ if $t < t_0$. Then, t_0 is the unique critical point of the function $g(t)$ and $g(t_0) = \max_{t\in\mathbb{R}} g(t) > 0$ since $g(-\infty) = 0^+$ and $g(+\infty) = -\infty$. □

Similar to Lemma 3.3, we have

Lemma 3.4. *If $r_i > 0, i = 1, 2, 3, 4$ and $\tau \geq 4$, then the function*

$$g(t) = r_1 e^{2t} + r_2 e^{4t} - r_3 e^{\tau t} - r_4 e^{2p\gamma_p t}, \quad t \in \mathbb{R}$$

has a unique critical point at which g achieves its maximum.

Lemma 3.5. *Suppose that $\frac{14}{3} \leq q < 6$ or $q = \frac{10}{3}$ with $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$. For any $u \in S_c$, there exists a unique $t_u \in \mathbb{R}$ such that $t_u \star u \in \mathcal{P}_{c,\mu}$, t_u is the unique critical point of Ψ_μ^u, and it is a strict maximum at a positive level. Moreover, Ψ_μ^u is strictly decreasing on $(t_u, +\infty)$ and $t_u < 0$ implies $P_\mu(u) < 0$. The map $u \in S_c \mapsto t_u \in \mathbb{R}$ is of class C^1.*

Proof. If $q = \frac{10}{3}$, recalling $q\delta_q = 2$, letting $u \in S_c$, by (2.6) and (2.7), we have

$$\Psi_\mu^u(t) = \Big(\frac{a}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}\|u\|_q^q\Big)e^{2t} + \frac{b}{4}e^{4t}\|\nabla u\|_2^4 - \frac{1}{2p}e^{2p\gamma_p t}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx$$

and $(\Psi_\mu^u)'(t) = 0 \Leftrightarrow P_\mu(t \star u) = 0 \Leftrightarrow t \star u \in \mathcal{P}_{c,\mu}$. In view of (1.12), we obtain

$$\Psi_\mu^u(t) \geq \Big(\frac{a}{2} - \frac{\mu}{q}C_q^q c^{q(1-\delta_q)}\Big)e^{2t}\|\nabla u\|_2^2 + \frac{b}{4}e^{4t}\|\nabla u\|_2^4 - \frac{1}{2p}e^{2p\gamma_p t}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx. \tag{3.9}$$

Since $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$ and $2p\gamma_p > 4$, in view of Lemma 3.3 and (3.9), there exists a unique $t_u \in \mathbb{R}$ such that $(\Psi_\mu^u)'(t_u) = 0$ and $\Psi_\mu^u(t_u) = \max_{t\in\mathbb{R}} \Psi_\mu^u(t) > 0$.

Similarly, we deduce that $4 \leq q\delta_q < 2p\gamma_p$ when $\frac{14}{3} \leq q < 6$, in view of Lemma 3.4 and (2.6), there exists a unique $t_u \in \mathbb{R}$ such that $(\Psi_\mu^u)'(t_u) = 0$ and $\Psi_\mu^u(t_u) = \max_{t\in\mathbb{R}} \Psi_\mu^u(t) > 0$. Thus, we infer that

$$\Psi_\mu^u(-\infty) = 0^+, \quad \Psi_\mu^u(+\infty) = -\infty$$

and Ψ_μ^u is strictly decreasing on $(t_u, +\infty)$. Since $(\Psi_\mu^u)'(t) < 0$ if and only if $t > t_u$, we deduce that $t_u < 0$ implies $P_\mu(u) = (\Psi_\mu^u)'(0) < 0$.

Define $\Phi : \mathbb{R} \times H_r^1(\mathbb{R}^3) \to \mathbb{R}$ by $\Phi(t, u) = (\Psi_\mu^u)'(t_u)$. It is clear that Φ is of class C^1, $\Phi(t_u, u) = 0$ and $\partial_t\Phi(t_u, u) = (\Psi_\mu^u)''(t_u) < 0$. Applying the implicit function theorem, we see that the map $u \in S_c \mapsto t_u \in \mathbb{R}$ is of class C^1. □

Lemma 3.6. *Suppose that* $\frac{14}{3} \le q < 6$ *or* $q = \frac{10}{3}$ *with* $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$. *We define*

$$m(c,\mu) := \inf_{u \in \mathcal{P}_{c,\mu}} E_\mu(u), \tag{3.10}$$

then $m(c,\mu) > 0$.

Proof. If $u \in \mathcal{P}_{c,\mu}$, combining $P_\mu(u) = 0$ with (1.12), (1.13), we obtain

$$a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 \le \mu\delta_q C_q^q c^{q(1-\delta_q)}\|\nabla u\|_2^{q\delta_q} + \frac{p\gamma_p c^{3+\alpha-p}}{\|Q_p\|_2^{2p-2}}\|\nabla u\|_2^{2p\gamma_p}.$$

So, in view of $4 \le q\delta_q < 2p\gamma_p$ or $q\delta_q = 2$ when $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$, there exists a positive constant C_0 such that

$$\inf_{u \in \mathcal{P}_{c,\mu}} \|\nabla u\|_2 \ge C_0 > 0. \tag{3.11}$$

In addition, for $u \in \mathcal{P}_{c,\mu}$, we have

$$\begin{aligned} E_\mu(u) = E_\mu(u) - \frac{1}{4}P_\mu(u) &= \frac{a}{4}\|\nabla u\|_2^2 + \mu\frac{q\delta_q - 4}{4}\|u\|_q^q + \left(\frac{\gamma_p}{4} - \frac{1}{2p}\right)\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx \\ &\ge \begin{cases} \frac{1}{4}(a - \frac{2\mu}{q}C_q^q c^{q(1-\delta_q)})\|\nabla u\|_2^2 > 0, & q = \frac{10}{3} \\ \frac{a}{4}\|\nabla u\|_2^2 > 0, & \frac{14}{3} \le q < 6 \end{cases}. \end{aligned} \tag{3.12}$$

Then, we get $m(c,\mu) > 0$. □

Lemma 3.7. *Suppose that* $\frac{14}{3} \le q < 6$ *or* $q = \frac{10}{3}$ *with* $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$. *Then,* $\sigma(c,\mu) = m(c,\mu)$*, where* $\sigma(c,\mu)$ *is defined in* (3.2).

Proof. Denote

$$m_r(c,\mu) := \inf_{u \in \mathcal{P}_{c,\mu} \cap S_{c,r}} E_\mu(u). \tag{3.13}$$

For any $u \in \mathcal{P}_{c,\mu} \cap S_{c,r}$, recalling that

$$\lim_{t \to -\infty} \|\nabla(t \star u)\|_2 = 0, \quad \lim_{t \to +\infty} \|\nabla(t \star u)\|_2 = +\infty, \quad \lim_{t \to +\infty} E_\mu(t \star u) = -\infty.$$

Then, there exist $t_1 < 0$ and $t_2 > 0$ such that $t_1 \star u \in A_{k_1}$, $t_2 \star u \in S_{c,r} \setminus A_{k_2}$ and $E_\mu(t_2 \star u) < 0$ by Lemma 3.1. Setting $\gamma(t) = ((1-t)t_1 + tt_2) \star u$ for $t \in [0,1]$, we easily see that $\gamma \in \Gamma$. In addition, by Lemma 3.5, we have $\max_{t \in [0,1]} E_\mu(\gamma(t)) = E_\mu(u)$ if $(1-t)t_1 + tt_2 = 0$, from which it follows that $m_r(c,\mu) \ge \sigma(c,\mu)$.

On the other hand, we prove $\sigma(c,\mu) \ge m_r(c,\mu)$, it suffices to verify that $\gamma([0,1]) \cap \mathcal{P}_{c,\mu} \ne \emptyset$ for any $\gamma \in \Gamma$. If $u \in S_{c,r}$, by (3.12), we get

$$E_\mu(u) - \frac{1}{4}P_\mu(u) > 0.$$

So, we have $P_\mu(\gamma(1)) < 4E_\mu(\gamma(1)) \le 0$ for any $\gamma \in \Gamma$. Note that $P_\mu(\gamma(0)) > 0$. Let us consider the function

$$P_\gamma : \tau \in [0,1] \mapsto P_\mu(\gamma(\tau)), \quad \text{for } u \in S_{c,r}.$$

Obviously , P_γ is continuous by (2.5). Hence, we infer that there exists $\tau_\gamma \in (0,1)$ such that $P_\gamma(\tau_\gamma) = 0$, namely, $\gamma(\tau_\gamma) \in \mathcal{P}_{c,\mu}$, which implies that $\gamma(\tau_\gamma) \in \gamma([0,1]) \cap \mathcal{P}_{c,\mu}$, namely, $\gamma([0,1]) \cap \mathcal{P}_{c,\mu} \neq \emptyset$. Thus, we obtain

$$m_r(c,\mu) = \sigma(c,\mu). \tag{3.14}$$

Finally, we prove $m_r(c,\mu) = m(c,\mu)$. Since $m_r(c,\mu) \geq m(c,\mu)$, we only prove $m(c,\mu) \geq m_r(c,\mu)$. Suppose by contradiction that there exists $\bar{u} \in \mathcal{P}_{c,\mu} \setminus S_{c,r}$ such that

$$E_\mu(\bar{u}) < m_r(c,\mu). \tag{3.15}$$

Let $\bar{v} = |\bar{u}|^*$ be the symmetric decreasing rearrangement of $|\bar{u}|$, then by the properties of symmetric decreasing rearrangement, we have

$$\begin{gathered} ||\nabla\bar{v}||_2 \leq ||\nabla\bar{u}||_2, \quad ||\bar{v}||_q = ||\bar{u}||_q, \quad ||\bar{v}||_2 = ||\bar{u}||_2, \\ \int_{\mathbb{R}^3} (I_\alpha * |\bar{v}|^p)|\bar{v}|^p dx \geq \int_{\mathbb{R}^3} (I_\alpha * |\bar{u}|^p)|\bar{u}|^p dx. \end{gathered} \tag{3.16}$$

Since $\bar{v} \in S_{c,r}$, by Lemma 3.5, there exists a unique $t_{\bar{v}} \in \mathbb{R}$ such that $t_{\bar{v}} \star \bar{v} \in \mathcal{P}_{c,\mu}$ and

$$E_\mu(t_{\bar{v}} \star \bar{v}) = \max_{t\in\mathbb{R}} E_\mu(t \star \bar{v}). \tag{3.17}$$

Hence, in view of (3.15)-(3.17), we have

$$\begin{aligned} E_\mu(\bar{u}) &< m_r(c,\mu) \leq E_\mu(t_{\bar{v}} \star \bar{v}) \\ &= \frac{a}{2}e^{2t_{\bar{v}}}||\nabla\bar{v}||_2^2 + \frac{b}{4}e^{4t_{\bar{v}}}||\nabla\bar{v}||_2^4 - \frac{\mu}{q}e^{q\delta_q t_{\bar{v}}}||\bar{v}||_q^q - \frac{1}{2p}e^{2p\gamma_p t_{\bar{v}}}\int_{\mathbb{R}^3}(I_\alpha * |\bar{v}|^p)|\bar{v}|^p dx \\ &\leq \frac{a}{2}e^{2t_{\bar{v}}}||\nabla\bar{u}||_2^2 + \frac{b}{4}e^{4t_{\bar{v}}}||\nabla\bar{u}||_2^4 - \frac{\mu}{q}e^{q\delta_q t_{\bar{v}}}||\bar{u}||_q^q - \frac{1}{2p}e^{2p\gamma_p t_{\bar{v}}}\int_{\mathbb{R}^3}(I_\alpha * |\bar{u}|^p)|\bar{u}|^p dx \\ &= E_\mu(t_{\bar{v}} \star \bar{u}) \leq E_\mu(t_{\bar{u}} \star \bar{u}) = E_\mu(\bar{u}), \end{aligned}$$

which is impossible. Therefore, $m_r(c,\mu) = m(c,\mu)$. Combining with (3.14), we have

$$\sigma(c,\mu) = m(c,\mu).$$

□

In what follows, we will discuss the convergence of Palais-Smale sequences satisfying suitable additional conditions, following the ideas introduced by [28, Proposition 3.1].

Proposition 3.8. *Suppose that $\frac{10}{3} < q < 6$ or $q = \frac{10}{3}$ with $\mu c^{q(1-\delta_q)} < \frac{aq}{2C_q^q}$, $\{u_n\} \subset S_{c,r}$ is a Palais-Smale sequence for $E_\mu|_{S_{c,r}}$ at the positive level $\sigma(c,\mu)$ with $P_\mu(u_n) \to 0$ as $n \to \infty$. Then, up to a subsequence, $u_n \to u$ in $H_r^1(\mathbb{R}^3)$, and $u \in S_{c,r}$ is a solution to* (1.1) *for some $\lambda < 0$.*

Proof. **Step 1**. Boundedness of $\{u_n\}$ in $H_r^1(\mathbb{R}^3)$.

If $\frac{10}{3} \le q < \frac{14}{3}$, then $2 \le q\delta_q < 4$. Using (1.12) and $P_\mu(u_n) = o_n(1)$ yields

$$\begin{aligned}\sigma(c,\mu) + 1 &\ge E_\mu(u_n) - \frac{1}{2p\gamma_p} P_\mu(u_n) + o_n(1)\\ &= a\Big(\frac{1}{2} - \frac{1}{2p\gamma_p}\Big)\|\nabla u_n\|_2^2 + b\Big(\frac{1}{4} - \frac{1}{2p\gamma_p}\Big)\|\nabla u_n\|_2^4 - \mu\Big(\frac{2p\gamma_p - q\delta_q}{2pq\delta_q}\Big)\|u_n\|_q^q + o_n(1)\\ &\ge b\Big(\frac{1}{4} - \frac{1}{2p\gamma_p}\Big)\|\nabla u_n\|_2^4 - \mu\Big(\frac{2p\gamma_p - q\delta_q}{2pq\delta_q}\Big)C_q^q c^{q(1-\delta_q)}\|\nabla u_n\|_2^{q\delta_q} + o_n(1),\end{aligned}$$

from which we see that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$.

If $\frac{14}{3} \le q < 6$, then $4 \le q\delta_q < 6$. Using (1.12) and $P_\mu(u_n) = o_n(1)$ again, we obtain

$$\begin{aligned}\sigma(c,\mu) + 1 &\ge E_\mu(u_n) - \frac{1}{4} P_\mu(u_n) + o_n(1)\\ &= \frac{a}{4}\|\nabla u_n\|_2^2 + \mu\Big(\frac{q\delta_q - 4}{4q}\Big)\|u_n\|_q^q + \frac{p\gamma_p - 2}{4p}\int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p dx + o_n(1)\\ &\ge \frac{a}{4}\|\nabla u_n\|_2^2 + o_n(1).\end{aligned}$$

Hence, $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$.

Step 2. We prove that there exist Lagrange multipliers $\lambda_n \to \lambda \in \mathbb{R}$.

Since $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$, by the compactness of $H_r^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)(q \in (2,6))$, up to a subsequence, there exists a function $u \in H_r^1(\mathbb{R}^3)$, such that $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, $u_n \to u$ in $L^q(\mathbb{R}^3)$ and $u_n \to u$ a.e. on $\mathbb{R}^3$. Notice $\{u_n\}$ is a bounded Palais-Smale sequence of $E_\mu|_{S_{c,r}}$, by the Lagrange multipliers rule, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\begin{aligned}(a + b\|\nabla u_n\|_2^2)\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla\varphi dx - \mu\int_{\mathbb{R}^3} |u_n|^{q-2}u_n\varphi dx\\ - \int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n\varphi dx - \lambda_n\int_{\mathbb{R}^3} u_n\varphi dx = o_n(1)\|\varphi\|,\end{aligned} \tag{3.18}$$

for every $\varphi \in H^1(\mathbb{R}^3)$. Choosing $\varphi = u_n$, then

$$\lambda_n c^2 = (a + b\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 - \mu\|u_n\|_q^q - \int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p dx + o_n(1). \tag{3.19}$$

By (1.12) and (1.13), the boundedness of $\{u_n\}$ implies that $\{\|u_n\|_q\}$ and $\{\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^p dx\}$ are bounded. Thus, $\{\lambda_n\}$ is bounded as well, up to a subsequence, $\lambda_n \to \lambda \in \mathbb{R}$.

Step 3. We show that $\lambda < 0$ and $u \not\equiv 0$.

We first prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p dx = \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p dx. \tag{3.20}$$

By Hardy-Littlewood-Sobolev inequality and Minkowski inequality, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^p dx - \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx\right| \\
&= \left|\int_{\mathbb{R}^3}(I_\alpha * (|u_n|^p + |u|^p))(|u_n|^p - |u|^p)dx\right| \\
&\leq C_1 \||u_n|^p + |u|^p\|_{\frac{6}{3+\alpha}} \cdot \||u_n|^p - |u|^p\|_{\frac{6}{3+\alpha}} \\
&\leq C_1 \left(\|u_n\|^p_{\frac{6p}{3+\alpha}} + \|u\|^p_{\frac{6p}{3+\alpha}}\right) \cdot \||u_n|^p - |u|^p\|_{\frac{6}{3+\alpha}}.
\end{aligned}$$

By virtue of (1.12), $2 < \frac{6p}{3+\alpha} < 6$, and the boundedness of $\{u_n\}$ in $H^1_r(\mathbb{R}^3)$, we deduce that $\{\|u_n\|_{\frac{6p}{3+\alpha}}\}$ is bounded. In addition, we have

$$\||u_n|^p - |u|^p\|_{\frac{6}{3+\alpha}} \to 0 \tag{3.21}$$

by $\|u_n - u\|_{\frac{6p}{3+\alpha}} \to 0$. Thus, (3.20) is established. Combining (3.19) with $P_\mu(u_n) = o_n(1)$, we get

$$\lambda_n c^2 = \mu(\delta_q - 1)\|u_n\|_q^q + \frac{p-3-\alpha}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^p dx + o_n(1).$$

Then, we obtain

$$\lambda c^2 = \mu(\delta_q - 1)\|u\|_q^q + \frac{p-3-\alpha}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx \tag{3.22}$$

by virtue of $\lambda_n \to \lambda$, $u_n \to u$ in $L^q(\mathbb{R}^3)$ and (3.20). Since $\mu > 0$, $0 < \delta_q < 1$ and $p < 3 + \alpha$, we deduce that $\lambda \leq 0$, with $\lambda = 0$ if and only if $u \equiv 0$. If $\lambda_n \to 0$, we have

$$\|u_n\|_q \to 0, \quad \int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^p dx \to 0$$

by the compactness of $H^1_r(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ and (3.20). Using again $P_\mu(u_n) = o_n(1)$, we have $E_\mu(u_n) \to 0$, which is a contradiction with $E_\mu(u_n) \to m > 0$, and thus $\lambda_n \to \lambda < 0$ and $u \neq 0$.

Step 4. $u_n \to u$ in $H^1_r(\mathbb{R}^3)$.

Since $u_n \rightharpoonup u \neq 0$ in $H^1_r(\mathbb{R}^3)$ and $H^1_r(\mathbb{R}^3) \hookrightarrow D^{1,2}(\mathbb{R}^3)$, we get

$$\tilde{B} = \lim_{n\to\infty}\|\nabla u_n\|_2^2 \geq \|\nabla u\|_2^2 > 0. \tag{3.23}$$

Then, (3.18) and Lemma 2.3 imply that

$$\begin{aligned}
&(a + b\tilde{B})\int_{\mathbb{R}^3}\nabla u \cdot \nabla\varphi dx - \lambda\int_{\mathbb{R}^3}u\varphi dx \\
&- \mu\int_{\mathbb{R}^3}|u|^{q-2}u\varphi dx - \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^{p-2}u\varphi dx = 0, \quad \forall\varphi \in H^1(\mathbb{R}^3).
\end{aligned} \tag{3.24}$$

Test (3.18)-(3.24) with $\varphi = u_n - u$, we obtain

$$\begin{aligned}
(a + \tilde{B}b)\|\nabla(u_n - u)\|_2^2 - \lambda\|u_n - u\|_2^2 &= \mu\int_{\mathbb{R}^3}(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx \\
&+ \int_{\mathbb{R}^3}\left[(I_\alpha * |u_n|^p)|u_n|^{p-2}u_n - (I_\alpha * |u|^p)|u|^{p-2}u\right](u_n - u)dx + o_n(1).
\end{aligned} \tag{3.25}$$

By using the Hölder inequality and the strong convergence of u_n to u in $L^q(\mathbb{R}^3)$, we obtain

$$\left|\int_{\mathbb{R}^3}(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx\right| \to 0. \tag{3.26}$$

In addition, by the Hardy-Littlewood-Sobolev inequality and generalized Hölder inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^{p-2}u_n(u_n - u)dx\right| \\
&\leq C\||u_n|^p\|_{\frac{6}{3+\alpha}}\||u_n|^{p-2}u_n(u_n - u)\|_{\frac{6}{3+\alpha}} \\
&\leq C\|u_n\|^p_{\frac{6p}{3+\alpha}}\||u_n|^{p-2}u_n\|_{\frac{p}{p-1}\frac{6}{3+\alpha}}\|u_n - u\|_{\frac{6p}{3+\alpha}} \\
&= C\|u_n\|^{2p-1}_{\frac{6p}{3+\alpha}}\|u_n - u\|_{\frac{6p}{3+\alpha}}.
\end{aligned}$$

So, by the boundedness of $\{\|u_n\|_{\frac{6p}{3+\alpha}}\}$ and $\|u_n - u\|_{\frac{6p}{3+\alpha}} \to 0$, we obtain

$$\left|\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^{p-2}u_n(u_n - u)dx\right| \to 0. \tag{3.27}$$

Similarly, we get

$$\left|\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^{p-2}u(u_n - u)dx\right| \to 0. \tag{3.28}$$

In view of (3.26), (3.27), (3.28), and (3.25), we have

$$(a + \tilde{B}b)\|\nabla(u_n - u)\|_2^2 - \lambda\|u_n - u\|_2^2 \to 0,$$

which implies that $u_n \to u$ in $H^1_r(\mathbb{R}^3)$ by $\lambda < 0$, and $u \in S_{c,r}$ solves (1.1) for some $\lambda < 0$. □

3.1. Proof of Theorem 1.1

According to Lemma 3.2, there exists a sequence $\{u_n\} \subset S_{c,r}$ with the following properties

$$E_\mu(u_n) \to \sigma(c,\mu), \quad E'_\mu|_{S_{c,r}}(u_n) \to 0, \quad P_\mu(u_n) \to 0, \quad \text{as } n \to \infty.$$

Then, by Propostion 3.8, $u_n \to u$ in $H^1_r(\mathbb{R}^3)$, $u \in S_{c,r}$ is a solution to (1.1) for some $\lambda < 0$ with $E_\mu(u) = \sigma(c,\mu)$. In addition, by Lemma 3.7, we get $E_\mu(u) = \sigma(c,\mu) = m(c,\mu)$, namely, u is a ground state solution under the cases of $q = \frac{10}{3}$ and $\frac{14}{3} \leq q < 6$. Recalling that $\beta_n(\tau) \geq 0$ a.e. in $\mathbb{R}^3$ for every τ, (3.7) and the convergence imply that u is non-negative. The strong maximum principle implies that $u > 0$. □

4. Supercritical leading term with defocusing perturbation

In this section, we prove the existence of a mountain pass type solution when $\mu < 0$.

Lemma 4.1. *Suppose that $\frac{10}{3} \leq q \leq \frac{14}{3} \leq p < 3 + \alpha$, $\alpha \in (\frac{5}{3}, 3)$ and $\mu < 0$. For any $u \in S_c$, there exists a unique $t_u \in \mathbb{R}$ such that $t_u \star u \in \mathcal{P}_{c,\mu}$, t_u is the unique critical point of Ψ^u_μ and it is a strict maximum at a positive level. Moreover, Ψ^u_μ is strictly decreasing on $(t_u, +\infty)$ and $t_u < 0$ implies $P_\mu(u) < 0$. The map $u \in S_c \mapsto t_u \in \mathbb{R}$ is of class C^1.*

Proof. In view of $\mu < 0$, $2 \le q\delta_q \le 4 < 2p\gamma_p$ and (2.6), we have

$$\Psi_\mu^u(-\infty) = 0^+ \quad \text{and} \quad \Psi_\mu^u(+\infty) = -\infty. \tag{4.1}$$

Therefore, $\Psi_\mu^u(t)$ has a global maximum point at positive level. The rest of the proof is similar to that of Lemma 3.5. □

Lemma 4.2. *Suppose that* $\frac{10}{3} \le q \le \frac{14}{3} \le p < 3+\alpha$, $\alpha \in (\frac{5}{3}, 3)$ *and* $\mu < 0$. *Then,* $m(c,\mu) > 0$, *where* $m(c,\mu)$ *is defined in* (3.10).

Proof. Combining (1.13) with $P_\mu(u) = 0$, we obtain

$$a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \mu\delta_q\|u\|_q^q = \gamma_p \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p dx \le \frac{p\gamma_p c^{3+\alpha-p}}{\|Q_p\|_2^{2p-2}} \|\nabla u\|_2^{2p\gamma_p}. \tag{4.2}$$

By virtue of $\mu < 0$ and (4.2), we have

$$\|\nabla u\|_2^2 \le \frac{p\gamma_p c^{3+\alpha-p}}{a\|Q_p\|_2^{2p-2}} \|\nabla u\|_2^{2p\gamma_p}. \tag{4.3}$$

Thus,

$$\|\nabla u\|_2^2 \ge \left(\frac{ac^{p-3-\alpha}}{p\gamma_p}\|Q_p\|_2^{2p-2}\right)^{\frac{2}{2p\gamma_p-2}} = \widetilde{C_0} > 0. \tag{4.4}$$

For any $u \in \mathcal{P}_{c,\mu}$, the energy functional can be seen

$$E_\mu(u) = a\Big(\frac{1}{2} - \frac{1}{2p\gamma_p}\Big)\|\nabla u\|_2^2 + b\Big(\frac{1}{4} - \frac{1}{2p\gamma_p}\Big)\|\nabla u\|_2^4 - \mu\frac{2p\gamma_p - 4}{2pq\gamma_p}\|u\|_q^q.$$

In view of $2p\gamma_p > 4$, $\mu < 0$ and (4.4), we obtain

$$E_\mu(u) \ge a\Big(\frac{1}{2} - \frac{1}{2p\gamma_p}\Big)\|\nabla u\|_2^2 \ge a\Big(\frac{1}{2} - \frac{1}{2p\gamma_p}\Big)\widetilde{C_0},$$

and the desired result follows from the inequality above. □

Lemma 4.3. *Suppose that* $\frac{10}{3} \le q \le \frac{14}{3} \le p < 3+\alpha$, $\alpha \in (\frac{5}{3}, 3)$ *and* $\mu < 0$. *Then, there exists* $k > 0$ *sufficiently small such that*

$$E_\mu(u) > 0, \ \ P_\mu(u) > 0 \ \text{ for } \ u \in \overline{A_k} \ \text{ and } \ \sup_{\overline{A_k}} E_\mu(u) < m(c,\mu),$$

where $\overline{A_k} := \{u \in S_c : \|\nabla u\|_2 \le k\}$.

Proof. In view of (1.13), $2p\gamma_p > 4$, and $\mu < 0$, if $u \in \overline{A_k}$ with k small enough, then we obtain

$$\begin{aligned} E_\mu(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx \\ &\ge \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{c^{3+\alpha-p}}{2\|Q_p\|_2^{2p-2}}\|\nabla u\|_2^{2p\gamma_p} > 0 \end{aligned}$$

and

$$P_\mu(u) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \mu\delta_q\|u\|_q^q - \gamma_p \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p dx$$
$$\geq a\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{p\gamma_p c^{3+\alpha-p}}{\|Q_p\|_2^{2p-2}}\|\nabla u\|_2^{2p\gamma_p} > 0.$$

If we could replace k with a smaller positive number, combining (1.12) with Lemma 4.2, then it is obvious that

$$E_\mu(u) \leq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{q}C_q^q c^{q(1-\delta_q)}\|\nabla u\|_2^2 < m(c,\mu).$$

□

Proposition 4.4. *Suppose that* $\frac{10}{3} \leq q \leq \frac{14}{3} \leq p < 3+\alpha$, $\alpha \in (\frac{5}{3}, 3)$, *and* $\mu < 0$. *If* $\{u_n\} \subset S_{c,r}$ *is a Palais-Smale sequence for* $E_\mu|_{S_{c,r}}$ *at non-zero level* $\tilde{c}$ *with* $P_\mu(u_n) \to 0$ *as* $n \to \infty$ *and* (1.14) *holds, then up to a subsequence,* $u_n \to \tilde{u}$ *in* $H_r^1(\mathbb{R}^3)$, *and* $\tilde{u} \in S_{c,r}$ *is a solution to* (1.1) *for some* $\lambda < 0$.

Proof. We divide the proof into four steps.

Step 1. Boundedness of $\{u_n\}$ in $H_r^1(\mathbb{R}^3)$.

Combining $\mu < 0$ with $P_\mu(u_n) = o_n(1)$, we obtain

$$\tilde{c} + 1 \geq E_\mu(u_n) \geq a\Big(\frac{1}{2} - \frac{1}{2p\gamma_p}\Big)\|\nabla u_n\|_2^2 + b\Big(\frac{1}{4} - \frac{1}{2p\gamma_p}\Big)\|\nabla u_n\|_2^4 + o_n(1),$$

then we deduce that $\{\|\nabla u_n\|_2\}$ is bounded by $2p\gamma_p > 4$. Thus, $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$.

Step 2. We prove that there exist Lagrange multipliers $\lambda_n \to \lambda$.

By the compactness of $H_r^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3) (q \in (2,6))$, up to a subsequence, there exists a function $\tilde{u} \in H_r^1(\mathbb{R}^3)$, such that $u_n \rightharpoonup \tilde{u}$ in $H_r^1(\mathbb{R}^3)$, $u_n \to \tilde{u}$ in $L^q(\mathbb{R}^3)$ and $u_n \to \tilde{u}$ a.e. on $\mathbb{R}^3$. Notice $\{u_n\}$ is a bounded Palais-Smale sequence of $E_\mu|_{S_{c,r}}$, by the Lagrange multipliers rule, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$(a + b\|\nabla u_n\|_2^2)\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla\varphi dx - \mu\int_{\mathbb{R}^3} |u_n|^{q-2}u_n\varphi dx$$
$$- \int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n\varphi dx - \lambda_n \int_{\mathbb{R}^3} u_n\varphi dx = o_n(1)\|\varphi\|,$$

for every $\varphi \in H_r^1(\mathbb{R}^3)$. Choosing $\varphi = u_n$, then we obtain

$$\lambda_n = \frac{1}{c^2}\Big((a + b\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 - \mu\|u_n\|_q^q - \int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^p dx\Big) + o_n(1). \tag{4.5}$$

By (1.12) and (1.13), the boundedness of $\{u_n\}$ implies that $\{\|u_n\|_q\}$ and $\{\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^p dx\}$ are bounded. Thus, $\{\lambda_n\}$ is bounded as well, up to a subsequence, $\lambda_n \to \lambda \in \mathbb{R}$.

Step 3. $\lambda < 0$.

Combining $\mu < 0$ with $P_\mu(u_n) = o_n(1)$, we get

$$a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 = \mu\delta_q\|u_n\|_q^q + \gamma_p\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^p dx + o_n(1)$$
$$\leq \gamma_p\int_{\mathbb{R}^3}(I_\alpha * |u_n|^p)|u_n|^p dx + o_n(1),$$

using (1.13) again, we have

$$a\|\nabla u_n\|_2^2 \le \gamma_p \int_{\mathbb{R}^3} (I_\alpha * |u_n|^p)|u_n|^p dx + o_n(1) \le \frac{p\gamma_p c^{3+\alpha-p}}{\|Q_p\|_2^{2p-2}} \|\nabla u_n\|_2^{2p\gamma_p} + o_n(1). \tag{4.6}$$

Now, we may assume that

$$\hat{B} = \lim_{n\to\infty} \|\nabla u_n\|_2^2 \ge \|\nabla \tilde{u}\|_2^2 \ge 0, \tag{4.7}$$

by (4.6) and (4.7), we obtain

$$\hat{B} \ge \left(\frac{ac^{p-3-\alpha}}{p\gamma_p}\|Q_p\|_2^{2p-2}\right)^{\frac{2}{2p\gamma_p-2}} = \widetilde{C_0} > 0. \tag{4.8}$$

Combining (4.5) with $P_\mu(u_n) = o_n(1)$, we obtain

$$\lambda_n = \frac{1}{c^2}\Big[\Big(1 - \frac{1}{\gamma_p}\Big)(a + b\|\nabla u_n\|_2^2)\|\nabla u_n\|_2^2 + \mu\Big(\frac{\delta_q}{\gamma_p} - 1\Big)\|u_n\|_q^q\Big] + o_n(1). \tag{4.9}$$

Since $\frac{10}{3} \le q \le \frac{14}{3} \le p < 3 + \alpha$, we get $1 - \frac{1}{\gamma_p} < 0$ and $\frac{\delta_q}{\gamma_p} < 1$, in view of (4.7), (4.8), (4.9), and (1.12), we infer that

$$\lambda_n \le \frac{1}{c^2}\widetilde{C_0}^{\frac{q\delta_q}{2}}\Big[\Big(1 - \frac{1}{\gamma_p}\Big)(a + b\widetilde{C_0})\widetilde{C_0}^{\frac{2-q\delta_q}{2}} + \mu\Big(\frac{\delta_q}{\gamma_p} - 1\Big)C_q^q c^{q(1-\delta_q)}\Big] + o_n(1). \tag{4.10}$$

Thus, observing assumption (1.14), taking to the limit of (4.10) as $n \to \infty$, we obtain

$$\lambda_n \to \lambda < 0.$$

Step 4. We show that $u_n \to \tilde{u}$ in $H^1(\mathbb{R}^3)$.

Similar to (3.24)-(3.28), we easily obtain

$$(a + \hat{B}b)\|\nabla(u_n - \tilde{u})\|_2^2 - \lambda\|u_n - \tilde{u}\|_2^2 \to 0, \quad \text{as} \quad n \to \infty,$$

which, being $\lambda < 0$, implies that $u_n \to \tilde{u}$ in $H^1(\mathbb{R}^3)$. □

Denoting by E_μ^d the closed sublevel set $\{u \in S_{c,r} : E_\mu(u) \le d\}$, we introduce the minimax class

$$\Gamma_0 := \{\gamma = (\alpha, \beta) \in C([0,1], \mathbb{R} \times S_{c,r}) : \gamma(0) \in (0, \overline{A_k}), \gamma(1) \in (0, E_\mu^0)\},$$

with associated minimax level

$$\hat{\sigma}(c,\mu) := \inf_{\gamma\in\Gamma_0} \max_{(t,u)\in\gamma([0,1])} \tilde{E}_\mu(t,u),$$

where A_k and $\tilde{E}_\mu(t,u)$ are defined in (3.1) and (3.5), respectively.

4.1. Proof of Theorem 1.2

We split the proof into the following steps.

Step 1. We show $\hat{\sigma}(c,\mu) = m_r(c,\mu)$, where $m_r(c,\mu)$ is defined (3.13).

Let $u \in S_{c,r}$, by (4.1), there exist $t_0 \ll -1$ and $t_1 \gg 1$ such that

$$\gamma_u : \tau \in [0,1] \mapsto (0, ((1-\tau)t_0 + \tau t_1)) \star u \in \mathbb{R} \times S_{c,r} \tag{4.11}$$

is a path in Γ_0(the continuity from (2.5)), then $\hat{\sigma}(c,\mu)$ is a real number.

We claim that for every $\gamma \in \Gamma_0$, there exists $\tau_\gamma \in (0,1)$ such that $\alpha(\tau_\gamma) \star \beta(\tau_\gamma) \in \mathcal{P}_{c,\mu}$. Indeed, let us consider the function

$$\tilde{P}_\gamma : \tau \in [0,1] \mapsto P_\mu(\alpha(\tau) \star \beta(\tau)) \in \mathbb{R}.$$

By Lemma 4.3, we get $\tilde{P}_\gamma(0) = P_\mu(\beta(0)) > 0$. On the other hand, by Lemma 4.1, since $(\Psi_\mu^{\beta(1)})'(t) > 0$ for every $t \in (-\infty, t_{\beta(1)}]$ and $\Psi_\mu^{\beta(1)}(0) = E_\mu(\beta(1)) \le 0$, then $t_{\beta(1)} < 0$. Again by Lemma 4.1, we can see that $P_\mu(\beta(1)) < 0$. Therefore, by the continuity of $\tilde{P}_\gamma$(see (2.5)), we can deduce that there exists $\tau_\gamma \in (0,1)$ such that $\tilde{P}_\gamma(\tau_\gamma) = 0$, so $\alpha(\tau_\gamma) \star \beta(\tau_\gamma) \in \mathcal{P}_{c,\mu}$. This implies that

$$\max_{\gamma([0,1])} \tilde{E}_\mu \ge \tilde{E}_\mu(\gamma(\tau_\gamma)) = E_\mu(\alpha(\tau_\gamma) \star \beta(\tau_\gamma)) \ge \inf_{\mathcal{P}_{c,\mu} \cap S_{c,r}} E_\mu.$$

Thus,

$$\hat{\sigma}(c,\mu) \ge \inf_{\mathcal{P}_{c,\mu} \cap S_{c,r}} E_\mu. \tag{4.12}$$

On the other hand, γ_u is the corresponding path defined by (4.11), and if $u \in \mathcal{P}_{c,\mu} \cap S_{c,r}$, then by Lemma 4.1, we get

$$E_\mu(u) = \tilde{E}_\mu(0,u) = \max_{\gamma_u([0,1])} \tilde{E}_\mu \ge \hat{\sigma}(c,\mu).$$

So, we infer that

$$\inf_{\mathcal{P}_{c,\mu} \cap S_{c,r}} E \ge \hat{\sigma}(c,\mu). \tag{4.13}$$

Combining (4.12) with (4.13), we have

$$\inf_{\mathcal{P}_{c,\mu} \cap S_{c,r}} E = \hat{\sigma}(c,\mu).$$

Then, we have

$$m_r(c,\mu) = \hat{\sigma}(c,\mu) \tag{4.14}$$

and $\hat{\sigma}(c,\mu) > 0$ by Lemma 4.2.

Step 2. We prove the existence of a Palais-Smale sequence $\{u_n\}$ of $E_\mu|_{S_{c,r}}$ at the level $\hat{\sigma}(c,\mu)$.

By Lemma 4.3, we infer that

$$\hat{\sigma}(c,\mu) = m_r(c,\mu) > \sup_{\overline{A_k} \cup E_\mu^0} E_\mu = \sup_{(0,\overline{A_k}) \cup (0,E_\mu^0)} \tilde{E}.$$

Taking

$$X = \mathbb{R} \times S_{c,r}, \quad \mathcal{F} = \{\gamma([0,1]) : \gamma \in \Gamma_0\}, \quad B = (0, \overline{A_k}) \cup (0, E_\mu^0),$$

$$F = \{(t,u) \in \mathbb{R} \times S_{c,r} | \tilde{E}_\mu(t,u) \ge \hat{\sigma}(c,\mu)\}, \quad A = \gamma([0,1]), \quad A_n = \gamma_n([0,1]).$$

Then, we consider that $\mathcal{F} = \{\gamma([0,1]) : \gamma \in \Gamma_0\}$ is a homotopy stable family of compact subsets of $\mathbb{R} \times S_{c,r}$ with extended boundary $B = (0, \overline{A_k}) \cup (0, E_\mu^0)$ and the assumptions of [36, Theorem 5.2] hold with the superlevel $\{\tilde{E} \geq \hat{\sigma}(c,\mu)\}$. Thus, taking any minimizing sequence $\{\gamma_n = (\alpha_n, \beta_n)\} \subset \Gamma_0$ for $\hat{\sigma}(c,\mu)$ with the property that $\alpha_n \equiv 0$ and $\beta_n(\tau) \geq 0$ a.e. in $\mathbb{R}^3$ for every $\tau \in [0,1]$, there exists a Palais-Smale sequence $\{(t_n, v_n)\} \subset \mathbb{R} \times S_{c,r}$ for $\tilde{E}_\mu|_{\mathbb{R}\times S_{c,r}}$ at the level $\hat{\sigma}(c,\mu) > 0$ such that

$$\partial_s \tilde{E}_\mu(t_n, v_n) \to 0 \quad \text{and} \quad \|\partial_u \tilde{E}_\mu(t_n, v_n)\|_{(T_{v_n} S_{c,r})^*} \to 0, \tag{4.15}$$

as $n \to \infty$. Moreover,

$$|t_n| + dist_{H^1(\mathbb{R}^3)}(v_n, \beta_n([0,1])) \to 0, \quad \text{as } n \to \infty. \tag{4.16}$$

From (4.15), we have $P_\mu(t_n \star v_n) \to 0$ as $n \to \infty$. In addition, we have

$$\begin{aligned} ae^{2t_n} \int_{\mathbb{R}^3} \nabla v_n \nabla \varphi + be^{4t_n} \|\nabla v_n\|_2^2 \int_{\mathbb{R}^3} \nabla v_n \nabla \varphi - \mu e^{q\delta_q t_n} \int_{\mathbb{R}^3} |v_n|^{q-2} v_n \varphi \\ -e^{(3p-3-\alpha)t_n} \int_{\mathbb{R}^3} (I_\alpha * |v_n|^p)|v_n|^{p-2} v_n \varphi = o_n(1)\|\varphi\|, \end{aligned} \tag{4.17}$$

for every $\varphi \in T_{v_n} S_{c,r}$. Thus, (4.17) leads to

$$\langle E'_\mu(t_n \star v_n), t_n \star \varphi \rangle = o_n(1)\|\varphi\|_{H^1(\mathbb{R}^3)} = o_n(1)\|t_n \star \varphi\|_{H^1(\mathbb{R}^3)}, \tag{4.18}$$

for every $\varphi \in T_{v_n} S_{c,r}$, with $o_n(1) \to 0$ as $n \to \infty$, in the last equality, we used that $\{t_n\}$ is bounded, due to (4.16). From Lemma 2.4 and (4.18), we see that $\{u_n := t_n \star v_n\} \subset S_{c,r}$ is a Pohozaev-Palais-Smale sequence for $E_\mu|_{S_{c,r}}$ at the level $m_r(c,\mu) = \hat{\sigma}(c,\mu)$.

Step 3. We notice that the assumptions of Proposition 4.4 are satisfied for the Palais-Smale sequence $\{u_n\}$ obtained in the previous step. Then, up to a subsequence, we have $u_n \to \tilde{u}$ strongly in $H_r^1(\mathbb{R}^3)$ and $\tilde{u}$ is a radial normalized solution to (1.1)-(1.2) for some $\hat{\lambda} < 0$. We complete the proof. □

5. Asymptotic behavior

In this section, we study the asymptotic behavior of normalized solutions obtained in Theorem 1.1 as $b \to 0^+$ and give the proof of Theorem 1.3.

5.1. Proof of Theorem 1.3

Let $\{u_b : 0 < b < \bar{b}\}$ be the solutions to (1.1)-(1.2) obtained by Theorem 1.1, where $\bar{b}$ is small enough. We split the proof into three steps.

Step 1. We show that the family of radial solutions $\{u_b\}$ is bounded in $H_r^1(\mathbb{R}^3)$.

Note that $P_\mu(u_b) = 0$. If $\frac{10}{3} \leq q < \frac{14}{3}$, namely, $2 \leq q\delta_q < 4 < 2p\gamma_p$, then we obtain

$$\begin{aligned} \sigma(c,\mu) + 1 \geq E_\mu(u_b) &= E_\mu(u_b) - \frac{1}{2p\gamma_p} P_\mu(u_b) \\ &= a\Big(\frac{1}{2} - \frac{1}{2p\gamma_p}\Big)\|\nabla u_b\|_2^2 + b\Big(\frac{1}{4} - \frac{1}{2p\gamma_p}\Big)\|\nabla u_b\|_2^4 - \mu\Big(\frac{2p\gamma_p - q\delta_q}{2pq\delta_q}\Big)\|u_b\|_q^q \\ &\geq b\Big(\frac{1}{4} - \frac{1}{2p\gamma_p}\Big)\|\nabla u_b\|_2^4 - \mu\Big(\frac{2p\gamma_p - q\delta_q}{2pq\delta_q}\Big)C_q^q c^{q(1-\delta_q)}\|\nabla u_b\|_2^{q\delta_q}. \end{aligned}$$

While if $\frac{14}{3} \le q < 6$, namely, $4 \le q\delta_q < 2p\gamma_p$, then

$$\begin{aligned}\sigma(c,\mu) + 1 \ge E_\mu(u_b) &= E_\mu(u_b) - \frac{1}{4}P_\mu(u_b)\\ &= \frac{a}{4}\|\nabla u_b\|_2^2 + \mu\Big(\frac{q\delta_q - 4}{4q}\Big)\|u_b\|_q^q + \frac{p\gamma_p - 2}{4p}\int_{\mathbb{R}^3}(I_\alpha * |u_b|^p)|u_b|^p dx\\ &\ge \frac{a}{4}\|\nabla u_b\|_2^2.\end{aligned}$$

Hence, we deduce from the above two cases that $\{u_b\}$ is bounded in $H_r^1(\mathbb{R}^3)$.

Step 2. We prove that there exist Lagrange multipliers $\lambda_n \to \lambda \in \mathbb{R}$.

Since $\{u_b\}$ is bounded in $H_r^1(\mathbb{R}^3)$, there exists $u \in H_r^1(\mathbb{R}^3)$ such that, up to a subsequence, $u_b \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, and $u_b \to u$ in $L^q(\mathbb{R}^3)$, $u_b \to u$ a.e. on $\mathbb{R}^3$. We also know that

$$(E_\mu|_{S_{c,r}})'(u_b) \to 0 \text{ in } H^{-1}(\mathbb{R}^3) \Leftrightarrow E_\mu'(u_{b_n}) - \frac{1}{c^2}\langle E_\mu'(u_b), u_b\rangle u_b \to 0 \text{ in } H^{-1}(\mathbb{R}^3).$$

Thus, for every $\varphi \in H_r^1(\mathbb{R}^3)$, we have

$$\begin{aligned}(a + b\|\nabla u_b\|_2^2)\int_{\mathbb{R}^3}\nabla u_b \cdot \nabla\varphi dx - \mu\int_{\mathbb{R}^3}|u_b|^{q-2}u_b\varphi dx\\ - \int_{\mathbb{R}^3}(I_\alpha * |u_b|^p)|u_b|^{p-2}u_b\varphi dx - \lambda_n\int_{\mathbb{R}^3}u_b\varphi dx = o_n(1)\|\varphi\|.\end{aligned} \tag{5.1}$$

Choosing $\varphi = u_b$, then

$$\begin{aligned}\lambda_n c^2 &= a\|\nabla u_b\|_2^2 + b\|\nabla u_b\|_2^4 - \mu\|u_b\|_q^q - \int_{\mathbb{R}^3}(I_\alpha * |u_b|^p)|u_b|^p dx + o_n(1)\\ &= \mu(\delta_q - 1)\|u_b\|_q^q + \frac{p - 3 - \alpha}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u_b|^p)|u_b|^p dx + o_n(1).\end{aligned} \tag{5.2}$$

By (1.12) and (1.13), the boundedness of $\{u_b\}$ implies that $\{\|u_b\|_q\}$ and $\{\int_{\mathbb{R}^3}(I_\alpha * |u_b|^p)|u_b|^p dx\}$ are bounded. Thus, $\{\lambda_n\}$ is bounded as well, up to a subsequence, $\lambda_n \to \lambda \in \mathbb{R}$.

Step 3. We prove that u is a weak solution of (1.15) and $u_b \to u \in H_r^1(\mathbb{R}^3)$.

By (3.20) and $u_b \to u$ in $L^q(\mathbb{R}^3)$, we obtain

$$\lambda c^2 = \mu(\delta_q - 1)\|u\|_q^q + \frac{p - 3 - \alpha}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p dx$$

from (5.2). Obviously, we deduce that $\lambda \le 0$, with "=" if and only if $u \equiv 0$. Similar to Step 3 of the proof of Proposition 3.8, we get $\lambda < 0$ and $u \not\equiv 0$. In view of $u_b \rightharpoonup u \ne 0$ in $H_r^1(\mathbb{R}^3)$, $H_r^1(\mathbb{R}^3) \hookrightarrow D^{1,2}(\mathbb{R}^3)$, Lemma 2.3 and (5.1), we get

$$a\int_{\mathbb{R}^3}\nabla u \cdot \nabla\varphi dx - \mu\int_{\mathbb{R}^3}|u|^{q-2}u\varphi dx - \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^{p-2}u\varphi dx - \lambda\int_{\mathbb{R}^3}u\varphi dx = 0 \tag{5.3}$$

for any $\varphi \in H_r^1(\mathbb{R}^3)$ as $b \to 0^+$. That is, u satisfies

$$-a\Delta u = \lambda u + \mu|u|^{q-2}u + (I_\alpha * |u|^p)|u|^{p-2}u \text{ in } \mathbb{R}^3,$$

test (5.1)-(5.3) with $\varphi = u_b - u$, as $b \to 0^+$, we have

$$a\|\nabla(u_b - u)\|_2^2 - \lambda\|u_b - u\|_2^2 \to 0,$$

which implies that $u_b \to u$ in $H_r^1(\mathbb{R}^3)$ by $\lambda < 0$, and $u \in S_{c,r}$ solves (1.1) for some $\lambda < 0$.

By (5.3) and $H_r^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$, u is a weak solution of (1.15) and $\|u\|_2 = c$, namely, u is a normalized solution of (1.15). □

Author contributions

Zhi-Jie Wang: Writing-original draft, Writing-review and editing; Hong-Rui Sun: Supervision, Writing-review and editing, Methodology, Validation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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