



Research article

Grand weighted variable Herz-Morrey spaces estimate for some operators

Ming Liu^{1,2}, Binhua Feng^{1,2,*}

¹ College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

² Shool of Mathematics and Statistics, Qinghai Minzu University, Xining 810007, China

* **Correspondence:** Email: binhuaf@163.com; binhuaf@nwnu.edu.cn.

Abstract: In this paper, we established the boundedness of higher-order commutators $I_{\beta,b}^m$ generated by the fractional integral operator with BMO functions on grand weighted variable-exponent Herz-Morrey spaces $\dot{MK}_{\lambda,p(\cdot)}^{\alpha,r,\theta}(\omega)$. We also obtained the boundedness of the m -order multilinear fractional Hardy operator $\mathcal{H}_{\beta,m}$ and its adjoint operator $\mathcal{H}_{\beta,m}^*$ on weighted variable-exponent Herz-Morrey spaces $\dot{MK}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)$.

Keywords: grand weighted; Herz-Morrey spaces; fractional integrals operator; fractional Hardy operator; variable exponent

Mathematics Subject Classification: 46E30, 47G10

1. Introduction

Since Kováčik and Rákosník established the theory of variable-exponent function spaces in [1], the subject has attracted extensive attention by many scholars. The theory of the variable-exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ has been extensively investigated, see [2–5]. Izuki first introduced the variable-exponent Herz spaces $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ [6] and considered the boundedness of commutators of fractional integrals in these spaces; for more research about the boundedness of operators in the above spaces, see [7, 8]. Subsequently, Izuki generalized the Herz-Morrey spaces $\dot{MK}_{q,p}^{\alpha,\lambda}(\mathbb{R}^n)$ in [9] into the variable-exponent Herz-Morrey spaces $\dot{MK}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ [10], for more research about $\dot{MK}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$, see [11–13]. On the other hand, the Muckenhoupt weight theory is a powerful tool in harmonic analysis, [14–17]. By using the basics on Banach function spaces and the variable-exponent Muckenhoupt theory, Izuki and Noi developed the theory of weighted variable-exponent Herz spaces $\dot{K}_{p(\cdot)}^{\alpha,q}(\omega)$ [18–20]. After that, the research for the boundedness of some operators, such as the commutator of bilinear Hardy operators, commutators of fractional integral operators, and fractional Hardy operators achieved good results on weighted variable Herz-Morrey spaces $\dot{MK}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)$; for

more details, see [21–26]. Consequently, many scholars have contributed to the study of function spaces and related differential equations, [27, 28].

Motivated by the mentioned works, the main goal of this paper is to establish the boundedness of higher-order commutators $I_{\beta,b}^m$ generated by the fractional integral operator with BMO functions on grand weighted variable-exponent Herz-Morrey spaces $\dot{MK}_{\lambda,p(\cdot)}^{\alpha,r,\theta}(\omega)$ and to establish boundedness of the m -order multilinear fractional Hardy operator $\mathcal{H}_{\beta,m}$ and its adjoint operator $\mathcal{H}_{\beta,m}^*$ on weighted variable-exponent Herz-Morrey spaces $\dot{MK}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)$. The paper is organized as follows: In Section 2, we collect some preliminary definitions and lemmas. Our main results and their proof will be given in Section 3 and Section 4.

Now, let us recall some notations that will be used in this paper.

In [29], Hardy defined the classical Hardy operator as follows:

$$\mathcal{P}(f)(x) := \frac{1}{x} \int_0^x f(t) dt, \quad x > 0. \quad (1.1)$$

In [30], Christ and Grafakos defined the n -dimensional Hardy operator as follows:

$$\mathcal{H}(f)(x) := \frac{1}{|x|^n} \int_{|t| < |x|} f(t) dt, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad (1.2)$$

and established the boundedness of $\mathcal{H}(f)(x)$ in $L^p(\mathbb{R}^n)$, obtaining the best constants.

In [31], under the condition of $0 \leq \beta < n$ and $|x| = \sqrt{\sum_{i=1}^n x_i^2}$, Fu et al. defined the n -dimensional fractional Hardy operator and its adjoint operator as follows:

$$\mathcal{H}_\beta f(x) := \frac{1}{|x|^{n-\beta}} \int_{|t| < |x|} f(t) dt, \quad \mathcal{H}_\beta^* f(x) := \int_{|t| \geq |x|} \frac{f(t)}{|t|^{n-\beta}} dt, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad (1.3)$$

and established the boundedness of their commutators in Lebesgue spaces and homogeneous Herz spaces.

Let m and n be positive integers with $m \geq 1$, $n \geq 2$, and $0 \leq \beta < mn$, and let $L_{\text{loc}}^1(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Wu and Zhang in [12] defined the m -order multilinear fractional Hardy operator and its adjoint operator as follows:

$$\mathcal{H}_{\beta,m}(\vec{f})(x) := \frac{1}{|x|^{mn-\beta}} \int_{|t_1| < |x|} \int_{|t_2| < |x|} \cdots \int_{|t_m| < |x|} f_1(t_1) f_2(t_2) \cdots f_m(t_m) dt_1 dt_2 \cdots dt_m, \quad (1.4)$$

$$\mathcal{H}_{\beta,m}^*(\vec{f})(x) := \int_{|t_1| \geq |x|} \int_{|t_2| \geq |x|} \cdots \int_{|t_m| \geq |x|} \frac{f_1(t_1) f_2(t_2) \cdots f_m(t_m)}{|(t_1, t_2, \dots, t_m)|^{mn-\beta}} dt_1 dt_2 \cdots dt_m, \quad (1.5)$$

where $x \in \mathbb{R}^n \setminus \{0\}$, $|(t_1, t_2, \dots, t_m)| = \sqrt{t_1^2 + t_2^2 + \cdots + t_m^2}$. $\vec{f} = (f_1, f_2, \dots, f_m)$ is a vector-valued function, where $f_i (i = 1, 2, \dots, m) \in L_{\text{loc}}^1(\mathbb{R}^n)$.

Obviously, when $m = 1$, $\mathcal{H}_{\beta,m} = \mathcal{H}_\beta$, $\mathcal{H}_{\beta,m}^* = \mathcal{H}_\beta^*$. When $\beta = 0$, $\mathcal{H}_m$ indicates a multilinear operator $\mathcal{H}_{0,m}$ corresponding to the Hardy operator $\mathcal{H}$, and $\mathcal{H}_m^*$ indicates a multilinear operator $\mathcal{H}_{0,m}^*$ corresponding to the adjoint operator $\mathcal{H}^* := \mathcal{H}_0^*$.

Let $L^1_{\text{loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$, and given a function $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, the bounded mean oscillation (BMO) space and the BMO norm are defined, respectively, as follows:

$$\text{BMO}(\mathbb{R}^n) := \{b \in L^1_{\text{loc}}(\mathbb{R}^n) : \|b\|_{\text{BMO}(\mathbb{R}^n)} < \infty\}, \quad (1.6)$$

$$\|b\|_{\text{BMO}(\mathbb{R}^n)} := \sup_{B: \text{ball}} \frac{1}{|B|} \int_B |b(x) - b_B| dx. \quad (1.7)$$

where the supremum is taken over all the balls $B \in \mathbb{R}^n$ and $b_B = |B|^{-1} \int_B b(y) dy$.

Let $b \in \text{BMO}(\mathbb{R}^n)$, $0 < \beta < n$, and the fractional integral operator I_β and the commutator of fractional integral operator $[b, I_\beta]f(x)$ are defined, respectively, as follows:

$$I_\beta(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\beta}} dy, \quad x \in \mathbb{R}^n. \quad (1.8)$$

$$[b, I_\beta]f(x) := b(x)I_\beta(f)(x) - I_\beta(bf)(x), \quad x \in \mathbb{R}^n. \quad (1.9)$$

Let $b \in \text{BMO}(\mathbb{R}^n)$, $0 < \beta < n$, and $m \in \mathbb{N}$. The higher-order commutator of fractional integrals operator $I^m_{\beta,b}$ is defined as follows:

$$I^m_{\beta,b}f(x) := \int_{\mathbb{R}^n} \frac{[b(x) - b(y)]^m}{|x - y|^{n-\beta}} f(y) dy, \quad x \in \mathbb{R}^n. \quad (1.10)$$

Obviously, when $m = 1$, $I^1_{\beta,b}(f)(x) = [b, I_\beta]f(x)$; and when $m = 0$, $I^0_{\beta,b}(f)(x) = I_\beta(f)(x)$.

For $0 \leq \beta < n$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the fractional maximal operator M_β is defined as follows:

$$M_\beta f(x) := \sup_{x \in B} \frac{1}{|B|^{1-\frac{\beta}{n}}} \int_B |f(y)| dy, \quad x \in \mathbb{R}^n. \quad (1.11)$$

Where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x . When $\beta = 0$, we simply write M instead of M_0 , which is exactly the Hardy-Littlewood maximal function.

Throughout this paper, we use the following symbols and notations:

1. For a constant $R > 0$ and a point $x \in \mathbb{R}^n$, we write $B(x, R) := \{y \in \mathbb{R}^n : |x - y| < R\}$.
2. For any measurable set $E \subset \mathbb{R}^n$, $|E|$ denotes the Lebesgue measure and χ_E means the characteristic function.
3. Given $k \in \mathbb{Z}$, we write $B_k := \overline{B(0, 2^k)} = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$.
4. We define a family $\{C_k\}_{k=-\infty}^\infty$ by $C_k := B_k \setminus B_{k-1} = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$. Moreover χ_k denotes the characteristic function of C_k , namely, $\chi_k := \chi_{C_k}$.
5. For any index $1 < p(x) < \infty$, $p'(x)$ is denoted by its conjugate index, namely, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.
6. If there exists a positive constant C independent of the main parameters such that $A \leq CB$, then we write $A \lesssim B$. Additionally $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2. Preliminaries

In this section, we first recall some definitions related to the variable Lebesgue space and variable Muckenhoupt weight theory. On this basis, we review some definitions of weighted variable-exponent Lebesgue spaces, weighted variable-exponent Herz-Morrey spaces, and grand weighted variable-exponent Herz-Morrey spaces. In addition, we recall some definitions of Banach function space and weighted Banach function space. Then, we present several relevant lemmas that will aid in the proof of our main boundedness result.

2.1. Some definitions that will be used in this paper

Definition 2.1 (see [2]) Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be a real-valued measurable function.

(i) The Lebesgue space with variable-exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is a measurable function} : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some constant } \lambda > 0 \right\}$$

(ii) The spaces with variable-exponent $L_{\text{loc}}^{p(\cdot)}(E)$ are defined by

$$L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n) := \{f \text{ is a measurable function} : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset \mathbb{R}^n\}$$

The variable-exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

Definition 2.2 (see [2]) (i) The set $\mathcal{P}_0(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ satisfying

$$0 < p^- \leq p(x) \leq p^+ < \infty, \quad (2.1)$$

where

$$p^- := \text{essinf}\{p(x) : x \in \mathbb{R}^n\} > 0, \quad p^+ := \text{esssup}\{p(x) : x \in \mathbb{R}^n\} < \infty. \quad (2.2)$$

(ii) The set $\mathcal{P}(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ satisfying

$$1 < p^- \leq p(x) \leq p^+ < \infty, \quad (2.3)$$

where

$$p^- := \text{essinf}\{p(x) : x \in \mathbb{R}^n\} > 1, \quad p^+ := \text{esssup}\{p(x) : x \in \mathbb{R}^n\} < \infty. \quad (2.4)$$

(iii) The set $\mathcal{B}(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 2.3 (see [2]) Suppose that $p(\cdot)$ is a real-valued function on $\mathbb{R}^n$. We say that

(i) $\mathcal{C}_{\text{loc}}^{\log}(\mathbb{R}^n)$ is the set of all local log-Hölder continuous functions $p(\cdot)$ satisfying

$$|p(x) - p(y)| \leq \frac{C}{\log(|x - y|)}, \quad |x - y| < \frac{1}{2}, \quad x, y \in \mathbb{R}^n. \quad (2.5)$$

(ii) $C_0^{\log}(\mathbb{R}^n)$ is the set of all local log-Hölder continuous functions $p(\cdot)$ satisfying at origin

$$|p(x) - p_0| \leq \frac{C}{\log(e + \frac{1}{|x|})}, \quad x \in \mathbb{R}^n. \quad (2.6)$$

(iii) $C_\infty^{\log}(\mathbb{R}^n)$ is the set of all local log-Hölder continuous functions satisfying at infinity

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n. \quad (2.7)$$

(iv) $C^{\log}(\mathbb{R}^n) = C_\infty^{\log}(\mathbb{R}^n) \cap C_{\text{loc}}^{\log}(\mathbb{R}^n)$ denotes the set of all global log-Hölder continuous functions $p(\cdot)$.

In [2], the author proved that if $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, then $p'(\cdot) \in C^{\log}(\mathbb{R}^n)$, and also proved that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 2.4 (see [17]) (i) Given a non-negative, measure function ω , for $1 < p < \infty$, $\omega \in A_p$ if

$$[\omega]_{A_p} := \sup_B \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{1-p'} dx \right)^{p-1} < \infty, \quad (2.8)$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

(ii) A weight ω is called a Muckenhoupt weight A_1 if

$$[\omega]_{A_1} := \sup_B \frac{\frac{1}{|B|} \int_B \omega(x) dx}{\text{essinf}\{\omega(x) : x \in B\}} < \infty. \quad (2.9)$$

(iii) A weight ω is called a Muckenhoupt weight A_∞ if

$$A_\infty := \bigcup_{1 < r < \infty} A_r. \quad (2.10)$$

Note that these weights characterize the weighted norm inequalities for the Hardy-Littlewood maximal operator, that is, $\omega \in A_p$, $1 < p < \infty$, if and only if $M : L^p(\omega) \rightarrow L^p(\omega)$.

Definition 2.5 (see [18]) Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a weight ω is in the class $A_{p(\cdot)}$ if

$$\sup_{B: \text{ball}} |B|^{-1} \|\omega^{\frac{1}{p(\cdot)}} \chi_B\|_{L^{p(\cdot)}} \|\omega^{-\frac{1}{p'(\cdot)}} \chi_B\|_{L^{p'(\cdot)}} < \infty. \quad (2.11)$$

Obviously, if $p(\cdot) = p$, $1 < p < \infty$, then the above definition reduces to the classical Muckenhoupt A_p class. In [18], suppose $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $p(\cdot) \leq q(\cdot)$, then $A_1 \subset A_{p(\cdot)} \subset A_{q(\cdot)}$.

Definition 2.6 (see [18]) Let $0 < \beta < n$ and $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}$. A weighted ω is said to be an $A(p_1(\cdot), p_2(\cdot))$ weight, then for all balls $B \subset \mathbb{R}^n$ satisfying

$$\|\omega \chi_B\|_{L^{p_2(\cdot)}} \|\omega^{-1} \chi_B\|_{L^{p_1'(\cdot)}} \leq C |B|^{1-\frac{\beta}{n}}. \quad (2.12)$$

In [14], suppose $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\beta \in (0, n)$ such that $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}$. Then $\omega \in A(p_1(\cdot), p_2(\cdot))$ if and only if $\omega^{p_2(\cdot)} \in A_{1+\frac{p_2(\cdot)}{p_1'(\cdot)}}$.

Definition 2.7 (see [25]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\omega \in A_{p(\cdot)}$, the weighted variable-exponent Lebesgue space $L^{p(\cdot)}(\omega)$ denotes the set of all complex-valued measurable functions f satisfying

$$L^{p(\cdot)}(\omega) = \{f : f \omega^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(\mathbb{R}^n)\}.$$

This is a Banach space equipped with the norm:

$$\|f\|_{L^{p(\cdot)}(\omega)} = \|f\omega^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Definition 2.8 (see [25]) Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 \leq \lambda < \infty$. The homogeneous weighted variable-exponent Herz-Morrey spaces $\dot{\mathbf{MK}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)$ are defined by

$$\dot{\mathbf{MK}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega) = \{f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}, \omega) : \|f\|_{\dot{\mathbf{MK}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)} < \infty\},$$

where

$$\|f\|_{\dot{\mathbf{MK}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)} = \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q} \|f\chi_k\|_{L^{p(\cdot)}(\omega)}^q \right\}^{\frac{1}{q}}.$$

Nonhomogeneous weighted variable-exponent Herz-Morrey spaces can be defined in a similar way. For more details, see [25]. When $\lambda = 0$, the weighted variable-exponent Herz-Morrey spaces become weighted variable-exponent Herz spaces, see [18].

Definition 2.9 (see [32]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $\theta > 0$, $0 < r < \infty$, $0 \leq \lambda < \infty$. The homogeneous grand weighted variable-exponent Herz-Morrey spaces $\dot{\mathbf{MK}}_{\lambda,p(\cdot)}^{\alpha,r,\theta}(\omega)$ are the collection of $L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}, \omega)$ such that

$$\dot{\mathbf{MK}}_{\lambda,p(\cdot)}^{\alpha,r,\theta}(\omega) = \{f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}, \omega) : \|f\|_{\dot{\mathbf{MK}}_{\lambda,p(\cdot)}^{\alpha,r,\theta}(\omega)} < \infty\},$$

where

$$\|f\|_{\dot{\mathbf{MK}}_{\lambda,p(\cdot)}^{\alpha,r,\theta}(\omega)} = \sup_{\delta > 0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k \in \mathbb{Z}} 2^{k\alpha r(1+\delta)} \|f\chi_k\|_{L^{p(\cdot)}(\omega)}^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}}.$$

Nonhomogeneous grand weighted variable-exponent Herz-Morrey spaces can be defined in a similar way. For more details, see [32]. When $\lambda = 0$, the grand weighted variable-exponent Herz-Morrey spaces become grand weighted variable-exponent Herz spaces, see [33].

Definition 2.10 (see [18]) Let $\mathcal{M}$ be the set of all complex-valued measurable functions defined on $\mathbb{R}^n$, and X a linear subspace of $\mathcal{M}$.

1. The space X is said to be a Banach function space if there exists a function

$\|\cdot\|_X : \mathcal{M} \rightarrow [0, \infty]$ satisfying the following properties: Let $f, g, f_j \in \mathcal{M}$ ($j = 1, 2, \dots$), then

(a) $f \in X$ holds if and only if $\|f\|_X < \infty$.

(b) Norm property:

i. Positivity: $\|f\|_X \geq 0$.

ii. Strict positivity: $\|f\|_X = 0$ holds if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

iii. Homogeneity: $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$ holds for all $\lambda \in \mathbb{C}$.

iv. Triangle inequality: $\|f + g\|_X \leq \|f\|_X + \|g\|_X$.

(c) Symmetry: $\|f\|_X = \|\|f\|\|_X$.

(d) Lattice property: If $0 \leq g(x) \leq f(x)$ for almost every $x \in \mathbb{R}^n$, then $\|g\|_X \leq \|f\|_X$.

(e) Fatou property: If $0 \leq f_j(x) \leq f_{j+1}(x)$ for all j and $f_j(x) \rightarrow f(x)$ as $j \rightarrow \infty$ for almost every $x \in \mathbb{R}^n$, then $\lim_{j \rightarrow \infty} \|f_j\|_X = \|f\|_X$.

(f) For every measurable set $F \subset \mathbb{R}^n$ such that $|F| < \infty$, $\|\chi_F\|_X$ is finite. Additionally, there exists a constant $C_F > 0$ depending only on F so that $\int_F |h(x)| dx \leq C_F \|h\|_X$ holds for all $h \in X$.

2. Suppose that X is a Banach function space equipped with a norm $\|\cdot\|_X$. The associated space X' is defined by

$$X' = \{f \in \mathcal{M} : \|f\|_{X'} < \infty\},$$

where

$$\|f\|_{X'} = \sup_g \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| : \|g\|_X \leq 1 \right\}.$$

Definition 2.11 Let (see [18]) Let X be a Banach function spaces. The set $X_{loc}(\mathbb{R}^n)$ consists of all measurable functions f such that $f\chi_E \in X$ for any compact set E with $|E| < \infty$. Given a function $\mathcal{W}$ such that $0 < \mathcal{W}(x) < \infty$ for almost every $x \in (\mathbb{R}^n)$, $\mathcal{W} \in X_{loc}(\mathbb{R}^n)$ and $\mathcal{W}^{-1} \in (X')_{loc}(\mathbb{R}^n)$, the weighted Banach function space is defined by

$$X(\mathbb{R}^n, \mathcal{W}) := \{f \in \mathcal{M} : f\mathcal{W} \in X\}.$$

2.2. Some lemmas that will be used in this paper

Lemma 2.1 (see [34]) Let X be a Banach function space, then we have

- (i) The associated space X' is also a Banach function spaces.
- (ii) $\|\cdot\|_{(X')'}$ and $\|\cdot\|_X$ are equivalent.
- (iii) If $g \in X$ and $f \in X'$, then

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \|f\|_X \|g\|_{X'}, \quad (2.13)$$

is the generalized Hölder inequality.

Lemma 2.2 (see [34]) If X is a Banach function space, then we have, for all balls B ,

$$1 \leq |B|^{-1} \|\chi_B\|_X \|\chi_B\|_{X'}. \quad (2.14)$$

Lemma 2.3 (see [16]) Let X be a Banach function space. Suppose that the Hardy-Littlewood maximal operator M is weakly bounded on X , that is,

$$\|\chi_{\{Mf > \lambda\}}\|_X \lesssim \lambda^{-1} \|f\|_X$$

is true for all $f \in X$ and all $\lambda > 0$. Then, we have

$$\sup_{B: \text{ball}} \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'} < \infty. \quad (2.15)$$

Lemma 2.4 (see [18]) (i) The weighted Banach function space $X(\mathbb{R}^n, \mathcal{W})$ is a Banach function space equipped by the norm

$$\|f\|_{X(\mathbb{R}^n, \mathcal{W})} := \|f\mathcal{W}\|_X.$$

(ii) The associate space of $X(\mathbb{R}^n, \mathcal{W})$ is a Banach function space and equals $X'(\mathbb{R}^n, \mathcal{W}^{-1})$.

Remark 2.5 (see [21]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and by comparing the $L^{p(\cdot)}(\omega^{p(\cdot)})$ and $L^{p'(\cdot)}(\omega^{-p'(\cdot)})$ with the definition of $X(\mathbb{R}^n, \mathcal{W})$, we have

1. If we take $\mathcal{W} = \omega$ and $X = L^{p(\cdot)}(\mathbb{R}^n)$, then we get $L^{p(\cdot)}(\mathbb{R}^n, \omega) = L^{p(\cdot)}(\omega^{p(\cdot)})$.

2. If we consider $\mathcal{W} = \omega^{-1}$ and $X = L^{p'(\cdot)}(\mathbb{R}^n)$, then we get $L^{p'(\cdot)}(\mathbb{R}^n, \omega^{-1}) = L^{p'(\cdot)}(\omega^{-p'(\cdot)})$. By virtue of Lemma 2.4, we get

$$(L^{p(\cdot)}(\mathbb{R}^n, \omega))' = (L^{p(\cdot)}(\omega^{p(\cdot)}))' = L^{p'(\cdot)}(\omega^{-p'(\cdot)}) = L^{p'(\cdot)}(\mathbb{R}^n, \omega^{-1}).$$

Lemma 2.6 (see [18]) Let X be a Banach function space. Suppose that M is bounded on the associate space X' . Then there exists a constant $0 < \delta < 1$ such that for all balls $B \subset \mathbb{R}^n$ and all measurable sets $E \subset B$,

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \lesssim \left(\frac{|E|}{|B|}\right)^\delta. \quad (2.16)$$

The paper [1] shows that $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space and the associated space $L^{p'(\cdot)}(\mathbb{R}^n)$ has equivalent norm.

Lemma 2.7 (see [20]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and $\omega \in A_{p(\cdot)}$, then there are constants $\delta_1, \delta_2 \in (0, 1)$ and $C > 0$ such that for all $k, l \in \mathbb{Z}$ with $k \leq l$,

$$\frac{\|\chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})}}{\|\chi_l\|_{L^{p(\cdot)}(\omega^{p(\cdot)})}} = \frac{\|\chi_k\|_{(L^{p'(\cdot)}(\omega^{-p'(\cdot)}))'}}{\|\chi_l\|_{(L^{p'(\cdot)}(\omega^{-p'(\cdot)}))'}} \leq C \left(\frac{|C_k|}{|C_l|}\right)^{\delta_1}, \quad (2.17)$$

and

$$\frac{\|\chi_k\|_{(L^{p(\cdot)}(\omega^{p(\cdot)}))'}}{\|\chi_l\|_{(L^{p(\cdot)}(\omega^{p(\cdot)}))'}} \leq C \left(\frac{|C_k|}{|C_l|}\right)^{\delta_2}. \quad (2.18)$$

Lemma 2.8 (see [35] Theorem 3.12) Let $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap \text{LH}(\mathbb{R}^n)$ and $0 < \beta < \frac{n}{p_1^+}$. Define $p_2(\cdot)$ by $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta}{n}$. If $\omega \in A_{p_1(\cdot), p_2(\cdot)}$, then I_β is bounded from $L^{p_1(\cdot)}(\omega^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(\omega^{p_2(\cdot)})$.

Lemma 2.9 (see [35] Theorem 3.14) Suppose that $b \in \text{BMO}(\mathbb{R}^n)$ and $m \in \mathbb{N}$. Let $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and $0 < \beta < \frac{n}{p_1^+}$. Define $p_2(\cdot)$ by $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta}{n}$. If $\omega \in A_{p_1(\cdot), p_2(\cdot)}$, then

$$\|I_{\beta, b}^m(f)\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}.$$

Lemma 2.10 (see [36] Theorem 2.3) Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ such that $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$ for $x \in \mathbb{R}^n$. Then, there exists a constant C_{p, p_1} independent of functions f and g such that

$$\|fg\|_{L^{p(\cdot)}} \leq C_{p, p_1} \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}}, \quad (2.19)$$

holds for every $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$.

Lemma 2.11 (see [23] Corollary 3.11) Let $b \in \text{BMO}(\mathbb{R}^n)$, $m \in \mathbb{N}$, and $k, j \in \mathbb{Z}$ with $k > j$. Then we have

$$C^{-1} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \leq \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\omega)}} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\omega)} \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^m, \quad (2.20)$$

and

$$\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\omega)} \leq C(k - j)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_{B_k}\|_{L^{p(\cdot)}(\omega)}. \quad (2.21)$$

3. Higher-order commutators of fractional integrals operator

In this section, under certain hypothetical conditions, we first establish the boundedness of higher-order commutators $I_{\beta,b}^m$ generated by the fractional integrals operator with BMO functions on weighted variable-exponent Herz-Morrey spaces $\dot{MK}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)$. Then, we establish the boundedness of $I_{\beta,b}^m$ on grand weighted variable-exponent Herz-Morrey spaces $\dot{MK}_{\lambda,p(\cdot)}^{\alpha,r,\theta}(\omega)$.

Theorem 3.1 Suppose that $b \in \text{BMO}(\mathbb{R}^n)$ and $m \in \mathbb{N}$. Let $0 < \lambda < \infty$, $0 < q_1 \leq q_2 < \infty$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, $\omega^{p_2(\cdot)} \in A_1$, $\delta_1, \delta_2 \in (0, 1)$ are the constants appearing in (2.17) and (2.18) respectively. α and β are such that

$$(i) -n\delta_1 + \lambda < \alpha < n\delta_2 - \beta + \lambda$$

$$(ii) 0 < \beta < n(\delta_1 + \delta_2).$$

Define $p_1(\cdot)$ by $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} - \frac{\beta}{n}$, then $I_{\beta,b}^m$ are bounded from $\dot{MK}_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\omega^{p_2(\cdot)})$ to $\dot{MK}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\omega^{p_1(\cdot)})$.

Proof We prove the homogeneous case while the nonhomogeneous case is similar. For all $f \in \dot{MK}_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\omega^{p_2(\cdot)})(\mathbb{R}^n)$ and $\forall b \in \text{BMO}(\mathbb{R}^n)$, if we denote $f_j := f\chi_j = f\chi_{C_j}$ for each $j \in \mathbb{Z}$, then $f = \sum_{j=-\infty}^{\infty} f_j$. So we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Because of $0 < \frac{q_1}{q_2} \leq 1$, then the Jensen inequality follows that

$$\left(\sum_{j=-\infty}^{\infty} |a_j| \right)^{\frac{q_1}{q_2}} \leq \sum_{j=-\infty}^{\infty} |a_j|^{\frac{q_1}{q_2}}, \quad (3.1)$$

By virtue of (3.1), we obtain

$$\begin{aligned} \|I_{\beta,b}^m(f)\|_{\dot{MK}_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\omega^{p_2(\cdot)})}^{q_1} &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left(\sum_{k=-\infty}^L 2^{k\alpha q_2} \|I_{\beta,b}^m(f)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{q_2} \right)^{\frac{q_1}{q_2}} \\ &\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \|I_{\beta,b}^m(f)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{q_1} \\ &\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=-\infty}^{k-2} \|I_{\beta,b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right)^{q_1} \right\} \\ &\quad + \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{k+1} \|I_{\beta,b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right)^{q_1} \right\} \\ &\quad + \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k+2}^{\infty} \|I_{\beta,b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right)^{q_1} \right\} \\ &=: (\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3). \end{aligned}$$

First we estimate $\mathcal{J}_1$. Note that if $x \in C_k$, $y \in C_j$, and $j \leq k-2$, then $|x-y| \approx |x| \approx 2^k$. By C_p inequality and generalized Hölder inequality, for every $j, k \in \mathbb{Z}$, we get

$$|I_{\beta,b}^m(f_j)(x)\chi_k| \leq C \int_{C_j} \frac{|b(x) - b(y)|^m}{|x-y|^{n-\beta}} |f_j(y)| dy \chi_k(x)$$

$$\begin{aligned}
&\lesssim 2^{k(\beta-n)} \int_{C_j} |f_j(y)| |b(x) - b(y)|^m dy \chi_k(x) \\
&\lesssim 2^{k(\beta-n)} \left\{ |b(x) - b_{C_j}|^m \int_{C_j} |f_j(y)| dy + \int_{C_j} |f_j(y)| |b(y) - b_{C_j}|^m dy \right\} \chi_k(x) \\
&\lesssim 2^{k(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \left\{ |b(x) - b_{C_j}|^m \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \right. \\
&\quad \left. + \|b(y) - b_{C_j}\|^m \chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \right\} \chi_k(x). \tag{3.2}
\end{aligned}$$

By taking the $L^{p_2(\cdot)}(\omega^{p_2(\cdot)})$ -norm for (3.2), by Lemma 2.11, we have

$$\begin{aligned}
&\|I_{\beta,b}^m(f_j)(x) \chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&\lesssim 2^{k(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \left\{ \|b(x) - b_{C_j}\|^m \chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \right. \\
&\quad \left. + \|b(y) - b_{C_j}\|^m \chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right\} \\
&\lesssim 2^{k(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \left\{ (k-j)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \right. \\
&\quad \left. + \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right\} \\
&\lesssim 2^{k(\beta-n)} (k-j)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}. \tag{3.3}
\end{aligned}$$

By virtue of Lemma 2.6, we have

$$\frac{\|\chi_j\|_X}{\|\chi_{B_j}\|_X} \leq C \left(\frac{|C_k|}{|B_k|} \right)^\delta = C \implies \|\chi_j\|_X \leq C \|\chi_{B_j}\|_X. \tag{3.4}$$

Note that $\chi_{B_j}(x) \lesssim 2^{-j\beta} I_\beta(\chi_{B_j})$ (see [11] p.350), by applying (2.15), (3.4), and Lemma 2.8, we obtain

$$\begin{aligned}
\|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} &\leq \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&\lesssim 2^{-j\beta} \|I_\beta(\chi_{B_j})\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&\lesssim 2^{-j\beta} \|\chi_{B_j}\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \\
&\lesssim 2^{j(n-\beta)} \|\chi_{B_j}\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}^{-1} \\
&\lesssim 2^{j(n-\beta)} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}^{-1}. \tag{3.5}
\end{aligned}$$

By virtue of (2.14) and (2.15), combining (2.18) and (3.5), we have

$$\begin{aligned}
&2^{k(\beta-n)} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&= 2^{k\beta} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} 2^{-kn} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&\lesssim 2^{k\beta} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{-1} \\
&= 2^{k\beta} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{-1} \frac{\|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}}{\|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}} \\
&\lesssim 2^{k\beta} 2^{n\delta_2(j-k)} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{-1} \\
&\lesssim 2^{k\beta} 2^{n\delta_2(j-k)} 2^{j(n-\beta)} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{-1} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{-1} \\
&= 2^{k\beta} 2^{n\delta_2(j-k)} 2^{-j\beta} \left(2^{-jn} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right)^{-1}
\end{aligned}$$

$$\lesssim 2^{(\beta-n\delta_2)(k-j)}. \quad (3.6)$$

Hence by virtue of (3.3) and (3.6), we have

$$\|I_{\beta,b}^m(f_j)(x)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \lesssim 2^{(\beta-n\delta_2)(k-j)}(k-j)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}. \quad (3.7)$$

On the other hand, note the following fact:

$$\begin{aligned} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} &= 2^{-j\alpha} \left(2^{j\alpha q_1} \|f\chi_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{q_1} \right)^{\frac{1}{q_1}} \\ &\leq 2^{-j\alpha} \left(\sum_{i=-\infty}^j 2^{i\alpha q_1} \|f\chi_i\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{q_1} \right)^{\frac{1}{q_1}} \\ &= 2^{j(\lambda-\alpha)} \left\{ 2^{-j\lambda} \left(\sum_{i=-\infty}^j 2^{i\alpha q_1} \|f\chi_i\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{q_1} \right)^{\frac{1}{q_1}} \right\} \\ &\lesssim 2^{j(\lambda-\alpha)} \|f\|_{\dot{\text{MK}}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\omega^{p_1(\cdot)})}. \end{aligned} \quad (3.8)$$

Thus, by virtue of (3.7) and (3.8), remark that $\alpha < n\delta_2 - \beta + \lambda$,

$$\begin{aligned} \mathcal{J}_1 &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=-\infty}^{k-2} \|I_{\beta,b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right)^{q_1} \right\} \\ &\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=-\infty}^{k-2} (k-j)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} 2^{(\beta-n\delta_2)(k-j)} \right)^{q_1} \right\} \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mq_1} \|f\|_{\dot{\text{MK}}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\omega^{p_1(\cdot)})}^{q_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\lambda q_1} \left(\sum_{j=-\infty}^{k-2} (k-j)^m 2^{(k-j)(\alpha+\beta-n\delta_2-\lambda)} \right)^{q_1} \right\} \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mq_1} \|f\|_{\dot{\text{MK}}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\omega^{p_1(\cdot)})}^{q_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left(\sum_{k=-\infty}^L 2^{k\lambda q_1} \right) \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mq_1} \|f\|_{\dot{\text{MK}}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\omega^{p_1(\cdot)})}^{q_1}. \end{aligned}$$

Next, we estimate $\mathcal{J}_2$. Using Lemma 2.9, we get

$$\begin{aligned} \mathcal{J}_2 &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{k+1} \|I_{\beta,b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right)^{q_1} \right\} \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{k+1} \|f_j\chi_k\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \right)^{q_1} \right\} \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \|f_j\chi_k\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{q_1} \right\} \\ &= \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mq_1} \|f\|_{\dot{\text{MK}}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\omega^{p_1(\cdot)})}^{q_1}. \end{aligned}$$

Finally, we estimate $\mathcal{J}_3$. Note that if $x \in C_k$, $y \in C_j$, and $j \geq k + 2$, then $|x - y| \approx |x| \approx 2^j$. By the C_p inequality and generalized Hölder inequality, for every $j, k \in \mathbb{Z}$, we get

$$\begin{aligned}
 |I_{\beta,b}^m(f_j)(x)\chi_k| &\leq C \int_{C_j} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\beta}} |f_j(y)| dy \chi_k(x) \\
 &\lesssim 2^{j(\beta-n)} \int_{C_j} |f_j(y)| |b(x) - b(y)|^m dy \chi_k(x) \\
 &\lesssim 2^{j(\beta-n)} \left\{ |b(x) - b_{C_j}|^m \int_{C_j} |f_j(y)| dy + \int_{C_j} |f_j(y)| |b(y) - b_{C_j}|^m dy \right\} \chi_k(x) \\
 &\lesssim 2^{j(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \left\{ |b(x) - b_{C_j}|^m \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \right. \\
 &\quad \left. + \|b(y) - b_{C_j}\|^m \chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \right\} \chi_k(x). \tag{3.9}
 \end{aligned}$$

Thus, by taking the $L^{p_2(\cdot)}(\omega^{p_2(\cdot)})$ -norm for (3.9), by virtue of Lemma 2.11, we have

$$\begin{aligned}
 \|I_{\beta,b}^m(f_j)(x)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} &\lesssim 2^{j(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \left\{ \|b(x) - b_{C_j}\|^m \chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \right. \\
 &\quad \left. + \|b(y) - b_{C_j}\|^m \chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right\} \\
 &\lesssim 2^{j(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \left\{ (j-k)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \right. \\
 &\quad \left. + \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right\} \\
 &\lesssim 2^{j(\beta-n)} (j-k)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}. \tag{3.10}
 \end{aligned}$$

On the other hand, by (2.14) and (2.15), combining (2.17) and (3.5), we have

$$\begin{aligned}
 &2^{j(\beta-n)} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \\
 &= 2^{j\beta} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} 2^{-jn} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \\
 &\lesssim 2^{j\beta} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{-1} \\
 &= 2^{j\beta} \|\chi_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{-1} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \frac{\|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}}{\|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}} \\
 &\lesssim 2^{j\beta} 2^{n\delta_1(k-j)} \|\chi_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{-1} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
 &\lesssim 2^{j\beta} 2^{n\delta_1(k-j)} 2^{j(n-\beta)} \|\chi_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{-1} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}^{-1} \\
 &= 2^{j\beta} 2^{n\delta_1(k-j)} 2^{-j\beta} \left(2^{-jn} \|\chi_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \right)^{-1} \\
 &\lesssim 2^{n\delta_1(k-j)}. \tag{3.11}
 \end{aligned}$$

Hence, combining (3.10) and (3.11), we obtain

$$\|I_{\beta,b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \lesssim 2^{n\delta_1(k-j)} (j-k)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}. \tag{3.12}$$

Thus, by virtue of (3.9) and (3.12), remark that $\lambda - n\delta_1 < \alpha$, and we conclude that

$$\mathcal{J}_3 = \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k+2}^{\infty} \|I_{\beta,b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right)^{q_1} \right\}$$

$$\begin{aligned}
&\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k+2}^{\infty} (j-k)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} 2^{n\delta_1(k-j)q_1} \right)^{q_1} \right\} \\
&\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mq_1} \|f\|_{\dot{\text{MK}}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\omega^{p_1(\cdot)})}^{q_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{k=-\infty}^L 2^{k\lambda q_1} \left(\sum_{j=k+2}^{\infty} (j-k)^m 2^{(j-k)(\lambda - n\delta_1 - \alpha)q_1} \right)^{q_1} \right\} \\
&\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mq_1} \|f\|_{\dot{\text{MK}}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\omega^{p_1(\cdot)})}^{q_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left(\sum_{k=-\infty}^L 2^{k\lambda q_1} \right) \\
&\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mq_1} \|f\|_{\dot{\text{MK}}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\omega^{p_1(\cdot)})}^{q_1}.
\end{aligned}$$

Combining the estimates of $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$, we complete the proof of Theorem 3.1.

Theorem 3.2

Suppose

that $b \in \text{BMO}(\mathbb{R}^n)$ and $m \in \mathbb{N}$. Let $0 \leq \lambda < \infty$, $1 < r < \infty$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, $\omega^{p_2(\cdot)} \in A_1$, $\delta_1, \delta_2 \in (0, 1)$ be the constants appearing in (2.17) and (2.18) respectively. α and β are such that

- (i) $-n\delta_1 < \alpha < n\delta_2 - \beta$
- (ii) $0 < \beta < n(\delta_1 + \delta_2)$.

Define $p_1(\cdot)$ by $\frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} - \frac{\beta}{n}$, then $I_{\beta, b}^m$ is bounded from $\dot{\text{MK}}_{\lambda, p_2(\cdot)}^{\alpha, r, \theta}(\omega^{p_2(\cdot)})$ to $\dot{\text{MK}}_{\lambda, p_1(\cdot)}^{\alpha, r, \theta}(\omega^{p_1(\cdot)})$.

Proof We prove the homogeneous case, as the nonhomogeneous case is similar. For all $f \in \dot{\text{MK}}_{\lambda, p_2(\cdot)}^{\alpha, r, \theta}(\omega^{p_2(\cdot)})$ and $\forall b \in \text{BMO}(\mathbb{R}^n)$, if we denote $f_j := f\chi_j = f\chi_{C_j}$ for each $j \in \mathbb{Z}$, then $f = \sum_{j=-\infty}^{\infty} f_j$. So we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Then we have

$$\begin{aligned}
&\|I_{\beta, b}^m(f)\|_{\dot{\text{MK}}_{\lambda, p_2(\cdot)}^{\alpha, r, \theta}(\omega^{p_2(\cdot)})} \\
&= \sup_{\delta > 0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{kar(1+\delta)} \|I_{\beta, b}^m(f)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&= \sup_{\delta > 0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{kar(1+\delta)} \sum_{j=-\infty}^{\infty} \|I_{\beta, b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta > 0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{kar(1+\delta)} \sum_{j \leq k-2} \|I_{\beta, b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\quad + \sup_{\delta > 0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{kar(1+\delta)} \sum_{j=k-1}^{k+1} \|I_{\beta, b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\quad + \sup_{\delta > 0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{kar(1+\delta)} \sum_{j \geq k+2} \|I_{\beta, b}^m(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&=: (\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3).
\end{aligned}$$

First, we estimate $\mathcal{J}_1$. Remark that $\alpha < n\delta_2 - \beta$, thus we consider two cases: $1 < r(1 + \delta) < \infty$ and $0 < r(1 + \delta) \leq 1$. For the case $1 < r(1 + \delta) < \infty$, by applying (3.7) and Hölder inequality, we

have

$$\begin{aligned}
\mathcal{J}_1 &= \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k=-\infty}^{\infty} 2^{kar(1+\delta)} \sum_{j=-\infty}^{k-2} \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} 2^{kar(1+\delta)} \left(\sum_{j=-\infty}^{k-2} \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right)^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
&\lesssim \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} 2^{kar(1+\delta)} \left(\sum_{j=-\infty}^{k-2} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \right. \right. \\
&\quad \left. \left. \times (k-j)^m 2^{(\beta-n\delta_2)(k-j)} \right)^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
&\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
&\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{\alpha j} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} (k-j)^m 2^{(\beta-n\delta_2+\alpha)(k-j)} \right)^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
&\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
&\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} (k-j)^{mr(1+\delta)} 2^{(\beta-n\delta_2+\alpha)(k-j)\frac{r(1+\delta)}{2}} \right) \right. \\
&\quad \left. \times \left(\sum_{j=-\infty}^{k-2} 2^{(\beta-n\delta_2+\alpha)(k-j)\frac{(r(1+\delta))'}{2}} \right)^{\frac{r(1+\delta)}{(r(1+\delta))'}} \right\}^{\frac{1}{r(1+\delta)}} \\
&\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
&\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} (k-j)^{mr(1+\delta)} 2^{(\beta-n\delta_2+\alpha)(k-j)\frac{r(1+\delta)}{2}} \right\}^{\frac{1}{r(1+\delta)}} \\
&\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
&\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \sum_{k \geq j+2} (k-j)^{mr(1+\delta)} 2^{(\beta-n\delta_2+\alpha)(k-j)\frac{r(1+\delta)}{2}} \right\}^{\frac{1}{r(1+\delta)}} \\
&\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
&\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{\dot{\text{MK}}_{\lambda,p_1(\cdot)}^{\alpha,r,\theta}(\omega^{p_1(\cdot)})}.
\end{aligned}$$

For $0 < r(1 + \delta) \leq 1$, by virtue of (3.7), we have

$$\begin{aligned}
\mathcal{J}_1 &= \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k=-\infty}^{\infty} 2^{kar(1+\delta)} \sum_{j=-\infty}^{k-2} \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} 2^{kar(1+\delta)} \left(\sum_{j=-\infty}^{k-2} \|I_{\beta,b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \right)^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
&\lesssim \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} 2^{kar(1+\delta)} \left(\sum_{j=-\infty}^{k-2} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& \times (k-j)^m 2^{(\beta-n\delta_2)(k-j)} \Big\}^{\frac{1}{r(1+\delta)}} \\
& \leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
& \quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \Big\{ \delta^\theta \sum_{k=-\infty}^{\infty} \left(2^{\alpha j} \sum_{j=-\infty}^{k-2} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} (k-j)^m 2^{(\beta-n\delta_2+\alpha)(k-j)} \right)^{r(1+\delta)} \Big\}^{\frac{1}{r(1+\delta)}} \\
& \leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
& \quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \Big\{ \delta^\theta \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} (k-j)^{mr(1+\delta)} 2^{(\beta-n\delta_2+\alpha)(k-j)r(1+\delta)} \Big\}^{\frac{1}{r(1+\delta)}} \\
& \leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
& \quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \Big\{ \delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \sum_{k \geq j+2} (k-j)^{mr(1+\delta)} 2^{(\beta-n\delta_2+\alpha)(k-j)r(1+\delta)} \Big\}^{\frac{1}{r(1+\delta)}} \\
& \leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
& \quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \Big\{ \delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \Big\}^{\frac{1}{r(1+\delta)}} \\
& \leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{\text{MK}_{\lambda, p_1(\cdot)}^{\alpha, r, \theta}(\omega^{p_1(\cdot)})}.
\end{aligned}$$

Next, we estimate $\mathcal{J}_2$. Using Lemma 2.9, we get

$$\begin{aligned}
\mathcal{J}_2 &= \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j=k-1}^{k+1} \|I_{\beta, b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j=k-1}^{k+1} \|(f \chi_j)\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{k\alpha r(1+\delta)} \|(f \chi_k)\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{\text{MK}_{\lambda, p_1(\cdot)}^{\alpha, r, \theta}(\omega^{p_1(\cdot)})}.
\end{aligned}$$

Finally, we estimate $\mathcal{J}_3$. By virtue of (3.12), we have

$$\begin{aligned}
\mathcal{J}_3 &= \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j \geq k+2} \|I_{\beta, b}^m(f_j) \chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
&\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\delta^\theta \sum_{k \in \mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j \geq k+2} 2^{n\delta_1(k-j)r(1+\delta)} (j-k)^{mr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
&\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k \in \mathbb{Z}} \left(\sum_{j \geq k+2} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} (j-k)^m 2^{\alpha j} 2^{(\alpha+n\delta_1)(k-j)} \right)^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}}.
\end{aligned}$$

Remark that $\alpha + n\delta_1 > 0$, thus we consider two cases $1 < r(1 + \delta) < \infty$ and $0 < r(1 + \delta) \leq 1$. For the case $1 < r(1 + \delta) < \infty$, by applying Hölder inequality, we have

$$\begin{aligned}
 \mathcal{J}_3 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
 &\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k \in \mathbb{Z}} \left(\sum_{j \geq k+2} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} (j-k)^m 2^{\alpha j} 2^{(\alpha+n\delta_1)(k-j)} \right)^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
 &\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} \left(\sum_{j \geq k+2} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} (j-k)^{mr(1+\delta)} 2^{(n\delta_1+\alpha)(k-j) \frac{r(1+\delta)}{2}} \right) \right. \\
 &\quad \times \left. \left(\sum_{j \geq k+2} 2^{(n\delta_1+\alpha)(k-j) \frac{(r(1+\delta))'}{2}} \right)^{\frac{r(1+\delta)}{(r(1+\delta))'}} \right\}^{\frac{1}{r(1+\delta)}} \\
 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
 &\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} \sum_{j \geq k+2} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} (j-k)^{mr(1+\delta)} 2^{(n\delta_1+\alpha)(k-j) \frac{r(1+\delta)}{2}} \right\}^{\frac{1}{r(1+\delta)}} \\
 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
 &\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \sum_{k \leq j-2} (j-k)^{mr(1+\delta)} 2^{(n\delta_1+\alpha)(k-j) \frac{r(1+\delta)}{2}} \right\}^{\frac{1}{r(1+\delta)}} \\
 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{\dot{\text{MK}}_{\lambda, p_1(\cdot)}^{\alpha, r, \theta}(\omega^{p_1(\cdot)})}.
 \end{aligned}$$

For $0 < r(1 + \delta) \leq 1$, we have

$$\begin{aligned}
 \mathcal{J}_3 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
 &\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k \in \mathbb{Z}} \left(\sum_{j \geq k+2} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} (j-k)^m 2^{\alpha j} 2^{(\alpha+n\delta_1)(k-j)} \right)^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
 &\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{k=-\infty}^{\infty} \sum_{j \geq k+2} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} (j-k)^{mr(1+\delta)} 2^{(n\delta_1+\alpha)(k-j)r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\
 &\quad \times \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \sum_{k \leq j-2} (j-k)^{mr(1+\delta)} 2^{(n\delta_1+\alpha)(k-j)r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \sup_{\delta>0} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}^{r(1+\delta)} \right\}^{\frac{1}{r(1+\delta)}} \\
 &\leq \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{\dot{\text{MK}}_{\lambda, p_1(\cdot)}^{\alpha, r, \theta}(\omega^{p_1(\cdot)})}.
 \end{aligned}$$

Combining the estimates of $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$, we complete the proof of Theorem 3.2.

Remark 3.3 When $\lambda = 0$ and $m = 0$, Theorem 3.1 holds on weighted variable-exponent Herz spaces and generalizes the result of Izuki in [18] (see Theorem 4). When $0 < \lambda < n$ and $m = 1$, Theorem 3.1 has been proved by Zhao in [26] (see Theorem 2.2). When $m = 0$, Theorem 3.2 holds on grand weighted variable-exponent Herz-Morrey spaces, and generalizes the result of Sultan in [32] (see Theorem 2).

4. Multilinear fractional Hardy-type operators

In this section, under some assumed conditions, we first establish the boundedness of the m -order multilinear fractional Hardy operator $\mathcal{H}_{\beta,m}$ on weighted variable exponent Herz-Morrey spaces $\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)$. Then, we establish the boundedness of the adjoint operator of the m -order multilinear fractional Hardy operator $\mathcal{H}_{\beta,m}^*$ on weighted variable-exponent Herz-Morrey spaces $\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)$. As a corollary of the above two results, we also obtain the corresponding result for multilinear Hardy operator $\mathcal{H}_m$ and its adjoint operator $\mathcal{H}_m^*$.

Theorem 4.1 Let $p_i(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ ($i = 1, 2, \dots, m, m \in \mathbb{Z}^+$), $p(\cdot)$ is defined as follows:

$$\sum_{i=1}^m \frac{1}{p_i(x)} - \frac{1}{p(x)} = \frac{\beta}{n}.$$

Let $0 < \beta < \frac{mn}{\max_{1 \leq i \leq m} (p_i)^+}$, $0 < q_i < \infty$, $\lambda_i > 0$, $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} - \frac{\beta}{n}$, $\lambda = \sum_{i=1}^m \lambda_i$, $\alpha = \sum_{i=1}^m \alpha_i$, $\omega \in A_{p(\cdot)}$, $\omega_i \in A_{p_i(\cdot)}$, $\omega = \prod_{i=1}^m \omega_i$, $\alpha_i < \lambda_i + n\delta_{i2}$, where $\delta_{i2} \in (0, 1)$ are the constants in (2.18) for exponents $p_i(\cdot)$ and weights $\omega_i^{p_i(\cdot)}$, then

$$\|\mathcal{H}_{\beta,m}(\vec{f})\|_{\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega^{p(\cdot)})} \leq C \prod_{i=1}^m \|f_i\|_{\dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})}.$$

Proof We prove the homogeneous case, since the nonhomogeneous case is similar. Without loss of generality, we only consider the case $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{Z}^+$ ($m \geq 1$). When $m = 2$, then we have

$$\mathcal{H}_{\beta,2}(\vec{f})(x) = \frac{1}{|x|^{2n-\beta}} \int_{|t_1| < |x|} \int_{|t_2| < |x|} f_1(t_1) f_2(t_2) dt_1 dt_2.$$

For arbitrary $f_i \in \dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})$ ($i = 1, 2$), let $f_{k_i} := f_i \cdot \chi_{k_i} = f_i \cdot \chi_{C_{k_i}}$, then

$$f_i(x) = \sum_{k_i=-\infty}^{\infty} f_i(x) \cdot \chi_{k_i}(x) = \sum_{k_i=-\infty}^{\infty} f_{k_i}(x).$$

By virtue of the definition of $\mathcal{H}_{\beta,2}$ and generalized Hölder inequality (2.13), we have

$$\begin{aligned} |\mathcal{H}_{\beta,2}(\vec{f})(x) \cdot \chi_k(x)| &\leq \frac{1}{|x|^{2n-\beta}} \int_{|t_1| < |x|} \int_{|t_2| < |x|} |f_1(t_1) f_2(t_2)| dt_1 dt_2 \cdot \chi_k(x) \\ &\lesssim 2^{k(\beta-2n)} \int_{B_k} \int_{B_k} |f_1(t_1) f_2(t_2)| dt_1 dt_2 \cdot \chi_k(x) \end{aligned}$$

$$\begin{aligned}
&\lesssim 2^{k(\beta-2n)} \sum_{k_1=-\infty}^k \sum_{k_2=-\infty}^k \int_{C_{k_1}} \int_{C_{k_2}} |f_1(t_1)f_2(t_2)| dt_1 dt_2 \cdot \chi_k(x) \\
&\lesssim 2^{k(\beta-2n)} \sum_{k_1=-\infty}^k \sum_{k_2=-\infty}^k \left(\int_{C_{k_1}} |f_1(t_1)| dt_1 \right) \left(\int_{C_{k_2}} |f_2(t_2)| dt_2 \right) \cdot \chi_k(x) \\
&\lesssim 2^{k(\beta-2n)} \sum_{k_1=-\infty}^k \sum_{k_2=-\infty}^k \left(\|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right. \\
&\quad \left. \cdot \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \right) \chi_k(x). \tag{4.1}
\end{aligned}$$

Note that if $u(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{u(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$ for $x \in \mathbb{R}^n$, and $\omega \in A_{u(\cdot)}$ with $\omega_i \in A_{p_i(\cdot)}$, $\omega = \prod_{i=1}^2 \omega_i$, by (2.19) of Lemma 2.10, we have

$$\begin{aligned}
\|fg\|_{L^{u(\cdot)}(\omega^{u(\cdot)})} &= \|fg\omega\|_{L^{u(\cdot)}(\mathbb{R}^n)} = \|f\omega_1 g\omega_2\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \|f\omega_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|g\omega_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} = \|f\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|g\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})}. \tag{4.2}
\end{aligned}$$

By virtue of (2.16) of Lemma 2.6, we have

$$\frac{\|\chi_k\|_X}{\|\chi_{B_k}\|_X} \leq \left(\frac{|C_k|}{|B_k|} \right)^\delta = C \implies \|\chi_k\|_X \leq C \|\chi_{B_k}\|_X. \tag{4.3}$$

Let $\frac{1}{u(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$, then by the condition of Theorem 4.1, it implies that $\frac{\beta}{n} = \frac{1}{u(x)} - \frac{1}{p(x)}$. Note that $\chi_{B_k} \leq C 2^{-k\beta} I_\beta(\chi_{B_k})(x)$ (see [11] p.350), by virtue of (2.15), (4.2), (4.3), and Lemma 2.8, we have

$$\begin{aligned}
\|\chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} &\leq \|\chi_{B_k}\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} \\
&\lesssim 2^{-k\beta} \|I_\beta(\chi_{B_k})\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} \\
&\lesssim 2^{-k\beta} \|\chi_{B_k}\|_{L^{u(\cdot)}(\omega^{u(\cdot)})} \\
&\lesssim 2^{-k\beta} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|\chi_{B_k}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \\
&\lesssim 2^{k(2n-\beta)} \|\chi_{B_k}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'}^{-1} \|\chi_{B_k}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'}^{-1} \\
&\lesssim 2^{k(2n-\beta)} \|\chi_k\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'}^{-1} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'}^{-1}. \tag{4.4}
\end{aligned}$$

Remark that $k_1 \leq k, k_2 \leq k$. By applying (2.18) and (4.4), we have

$$\begin{aligned}
&2^{k(\beta-2n)} \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \|\chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} \\
&\lesssim \frac{\|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'}}{\|\chi_k\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'}} \\
&\lesssim 2^{(k_1-k)n\delta_{12}} 2^{(k_2-k)n\delta_{22}}. \tag{4.5}
\end{aligned}$$

Thus, by taking the $L^{p(\cdot)}(\omega^{p(\cdot)})$ -norm for (4.1), and by virtue of (4.5), we have

$$\|\mathcal{H}_{\beta,2}(\vec{f}) \cdot \chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} \lesssim 2^{k(\beta-2n)} \sum_{k_1=-\infty}^k \sum_{k_2=-\infty}^k \left(\|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right)$$

$$\begin{aligned}
& \cdot \| \chi_{k_1} \|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \| \chi_{k_2} \|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \| \chi_k \|_{L^{p(\cdot)}(\omega^{p(\cdot)})} \\
& \lesssim \left(\sum_{k_1=-\infty}^k 2^{(k_1-k)n\delta_{12}} \| f_{k_1} \|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \right) \left(\sum_{k_2=-\infty}^k 2^{(k_2-k)n\delta_{22}} \| f_{k_2} \|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right) \\
& \lesssim \prod_{i=1}^2 \left(\sum_{k_i=-\infty}^k 2^{(k_i-k)n\delta_{i2}} \| f_{k_i} \|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right). \tag{4.6}
\end{aligned}$$

Let $0 < \gamma \leq 1$, then by the Jensen inequality it follows that

$$\left(\sum_{j=-\infty}^{\infty} |a_j| \right)^\gamma \leq \sum_{j=-\infty}^{\infty} |a_j|^\gamma, \tag{4.7}$$

Let $\frac{1}{v} = \frac{1}{q_1} + \frac{1}{q_2}$, then $\frac{1}{q} = \frac{1}{v} - \frac{\beta}{n}$, therefore $q > v$. By applying (4.6), (4.7), and Hölder inequality in sequential form, we have

$$\begin{aligned}
\| \mathcal{H}_{\beta,2}(\vec{f}) \|_{\mathbf{MK}_{q,p(\cdot)}^{\alpha,\lambda}(\omega^{p(\cdot)})} &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q} \| \mathcal{H}_{\beta,2}(\vec{f}) \chi_k \|_{L^{p(\cdot)}(\omega^{p(\cdot)})}^q \right\}^{\frac{1}{q}} \\
&\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q} \prod_{i=1}^2 \left(\sum_{k_i=-\infty}^k 2^{(k_i-k)n\delta_{i2}} \| f_{k_i} \|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^q \right\}^{\frac{1}{q}} \\
&\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L \prod_{i=1}^2 \left(\sum_{k_i=-\infty}^k 2^{k\alpha_i + (k_i-k)n\delta_{i2}} \| f_{k_i} \|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^q \right\}^{\frac{1}{q}} \\
&\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L \prod_{i=1}^2 \left(\sum_{k_i=-\infty}^k 2^{k\alpha_i + (k_i-k)n\delta_{i2}} \| f_{k_i} \|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^v \right\}^{\frac{1}{v}} \\
&\lesssim \prod_{i=1}^2 \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^L \left(\sum_{k_i=-\infty}^k 2^{k\alpha_i + (k_i-k)n\delta_{i2}} \| f_{k_i} \|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^{q_i} \right\}^{\frac{1}{q_i}}. \tag{4.8}
\end{aligned}$$

on the other hand, note the following fact:

$$\begin{aligned}
\| f_{k_i} \|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} &= 2^{-k_i\alpha_i} \left(2^{k_i\alpha_i q_i} \| f_i \chi_{k_i} \|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})}^{q_i} \right)^{\frac{1}{q_i}} \\
&\leq 2^{-k_i\alpha_i} \left(\sum_{j_i=-\infty}^{k_i} 2^{j_i\alpha_i q_i} \| f_i \chi_{j_i} \|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})}^{q_i} \right)^{\frac{1}{q_i}} \\
&= 2^{k_i(\lambda_i - \alpha_i)} \left\{ 2^{-k_i\lambda_i} \left(\sum_{j_i=-\infty}^{k_i} 2^{j_i\alpha_i q_i} \| f_i \chi_{j_i} \|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})}^{q_i} \right)^{\frac{1}{q_i}} \right\} \\
&\lesssim 2^{k_i(\lambda_i - \alpha_i)} \| f_i \|_{\mathbf{MK}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})}. \tag{4.9}
\end{aligned}$$

Remark that $\alpha_i < \lambda_i + n\delta_{i2}$. By applying (4.7), (4.8), and (4.9), we have

$$\| \mathcal{H}_{\beta,2}(\vec{f}) \|_{\mathbf{MK}_{q,p(\cdot)}^{\alpha,\lambda}(\omega^{p(\cdot)})}$$

$$\begin{aligned}
&\lesssim \prod_{i=1}^2 \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^L \left(\sum_{k_i=-\infty}^k 2^{k\alpha_i + (k_i-k)n\delta_{i2}} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^{q_i} \right\}^{\frac{1}{q_i}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{\dot{M}_{q_i, p_i(\cdot)}^{\alpha_i, \lambda_i}(\omega_i^{p_i(\cdot)})} \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^L 2^{k\lambda_i q_i} \left(\sum_{k_i=-\infty}^k 2^{(k_i-k)(\lambda_i - \alpha_i + n\delta_{i2})} \right)^{q_i} \right\}^{\frac{1}{q_i}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{\dot{M}_{q_i, p_i(\cdot)}^{\alpha_i, \lambda_i}(\omega_i^{p_i(\cdot)})} \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left(\sum_{k=-\infty}^L 2^{k\lambda_i q_i} \right)^{\frac{1}{q_i}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{\dot{M}_{q_i, p_i(\cdot)}^{\alpha_i, \lambda_i}(\omega_i^{p_i(\cdot)})}.
\end{aligned}$$

This finishes the proof of Theorem 4.1.

Theorem 4.2 Let $p_i(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ ($i = 1, 2, \dots, m, m \in \mathbb{Z}^+$), $p(\cdot)$ is defined as follows:

$$\sum_{i=1}^m \frac{1}{p_i(x)} - \frac{1}{p(x)} = \frac{\beta}{n}.$$

Let $0 < \beta < \frac{mn}{\max_{1 \leq i \leq m} (p_i)^+}$, $0 < q_i < \infty$, $\lambda_i > 0$, $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} - \frac{\beta}{n}$, $\lambda = \sum_{i=1}^m \lambda_i$, $\alpha = \sum_{i=1}^m \alpha_i$, $\omega \in A_{p(\cdot)}$, $\omega_i \in A_{p_i(\cdot)}$, $\omega = \prod_{i=1}^m \omega_i$, $\alpha_i > \lambda_i + \frac{\beta}{m} - n\delta_{i1}$, where $\delta_{i1} \in (0, 1)$ are the constants in (2.17) for exponents $p_i(\cdot)$ and weights $\omega_i^{p_i(\cdot)}$, then

$$\|\mathcal{H}_{\beta, m}^*(\vec{f})\|_{\dot{M}_{q, p(\cdot)}^{\alpha, \lambda}(\omega^{p(\cdot)})} \leq C \prod_{i=1}^m \|f_i\|_{\dot{M}_{q_i, p_i(\cdot)}^{\alpha_i, \lambda_i}(\omega_i^{p_i(\cdot)})}.$$

Proof We prove the homogeneous case, since the nonhomogeneous case is similar. Without loss of generality, we only consider the case $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{Z}^+$ ($m \geq 1$). When $m = 2$, then we have

$$\mathcal{H}_{\beta, 2}^*(\vec{f})(x) = \int_{|t_1| \geq |x|} \int_{|t_2| \geq |x|} \frac{f_1(t_1)f_2(t_2)}{|(t_1, t_2)|^{2n-\beta}} dt_1 dt_2.$$

For arbitrary $f_i \in \dot{M}_{q_i, p_i(\cdot)}^{\alpha_i, \lambda_i}(\omega_i^{p_i(\cdot)})$ ($i = 1, 2$), let $f_{k_i} := f_i \cdot \chi_{k_i} = f_i \cdot \chi_{C_{k_i}}$, then

$$f_i(x) = \sum_{k_i=-\infty}^{\infty} f_i(x) \cdot \chi_{k_i}(x) = \sum_{k_i=-\infty}^{\infty} f_{k_i}(x).$$

Note that $|t_1|^{n-\frac{\beta}{2}} |t_2|^{n-\frac{\beta}{2}} < |(t_1, t_2)|^{2n-\beta}$ (see [37] p.11). By virtue of the definition of $\mathcal{H}_{\beta, 2}^*$ and generalized Hölder inequality, we have

$$\begin{aligned}
|\mathcal{H}_{\beta, 2}^*(\vec{f})(x) \cdot \chi_k(x)| &\leq \int_{|t_1| \geq |x|} \int_{|t_2| \geq |x|} \frac{|f_1(t_1)f_2(t_2)|}{|(t_1, t_2)|^{2n-\beta}} dt_1 dt_2 \cdot \chi_k(x) \\
&\leq \sum_{k_1=k}^{\infty} \sum_{k_2=k}^{\infty} \int_{C_{k_1}} \int_{C_{k_2}} \frac{|f_1(t_1)f_2(t_2)|}{|(t_1, t_2)|^{2n-\beta}} dt_1 dt_2 \cdot \chi_k(x) \\
&\lesssim \sum_{k_1=k}^{\infty} \sum_{k_2=k}^{\infty} 2^{(k_1+k_2)(\frac{\beta}{2}-n)} \left(\int_{C_{k_1}} |f_1(t_1)| dt_1 \right) \left(\int_{C_{k_2}} |f_2(t_2)| dt_2 \right) \cdot \chi_k(x)
\end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{k_1=k}^{\infty} \sum_{k_2=k}^{\infty} 2^{(k_1+k_2)(\frac{\beta}{2}-n)} \left(\|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right. \\ &\quad \left. \cdot \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \right) \chi_k(x). \end{aligned} \quad (4.10)$$

Remark that $k_1 \geq k, k_2 \geq k$. By applying (2.14), (2.17), and (4.4), we have

$$\begin{aligned} &2^{(k_1+k_2)(\frac{\beta}{2}-n)} \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \|\chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} \\ &\lesssim 2^{(k_1+k_2)(\frac{\beta}{2}-n)} \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} 2^{k(2n-\beta)} \|\chi_k\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'}^{-1} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'}^{-1} \\ &\lesssim 2^{(k_1+k_2)(\frac{\beta}{2}-n)} \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} 2^{-k\beta} \|\chi_k\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|\chi_k\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \\ &\lesssim 2^{(k_1+k_2)(\frac{\beta}{2}-n)} 2^{nk_1} \|\chi_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})}^{-1} 2^{nk_2} \|\chi_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})}^{-1} 2^{-k\beta} \|\chi_k\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|\chi_k\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \\ &= 2^{(k_1+k_2-2k)\frac{\beta}{2}} \|\chi_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})}^{-1} \|\chi_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})}^{-1} \|\chi_k\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|\chi_k\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \\ &= 2^{(k_1+k_2-2k)\frac{\beta}{2}} \frac{\|\chi_k\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})}}{\|\chi_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})}} \frac{\|\chi_k\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})}}{\|\chi_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})}} \\ &\lesssim 2^{(k_1+k_2-2k)\frac{\beta}{2}} 2^{(k-k_1)n\delta_{11}} 2^{(k-k_2)n\delta_{21}} \\ &= 2^{(k_1-k)(\frac{\beta}{2}-n\delta_{11})} 2^{(k_1-k)(\frac{\beta}{2}-n\delta_{21})}. \end{aligned} \quad (4.11)$$

Thus, by taking the $L^{p(\cdot)}(\omega^{p(\cdot)})$ -norm for (4.10), and by virtue of (4.11), we have

$$\begin{aligned} \|\mathcal{H}_{\beta,2}^*(\vec{f}) \cdot \chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} &\lesssim \sum_{k_1=k}^{\infty} \sum_{k_2=k}^{\infty} 2^{(k_1+k_2)(\frac{\beta}{2}-n)} \left(\|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right. \\ &\quad \left. \cdot \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \right) \|\chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} \\ &\lesssim \left(\sum_{k_1=k}^{\infty} 2^{(k_1-k)(\frac{\beta}{2}-n\delta_{11})} \|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \right) \left(\sum_{k_2=k}^{\infty} 2^{(k_2-k)(\frac{\beta}{2}-n\delta_{21})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right) \\ &\lesssim \prod_{i=1}^2 \left(\sum_{k_i=k}^{\infty} 2^{(k_i-k)(\frac{\beta}{2}-n\delta_{i1})} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right). \end{aligned} \quad (4.12)$$

Let $\frac{1}{v} = \frac{1}{q_1} + \frac{1}{q_2}$, then $\frac{1}{q} = \frac{1}{v} - \frac{\beta}{n}$; therefore, $q > v$. By applying (4.7), (4.12), and Hölder inequality in sequential form, we have

$$\begin{aligned} \|\mathcal{H}_{\beta,2}^*(\vec{f})\|_{\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega^{p(\cdot)})} &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q} \|\mathcal{H}_{\beta,2}^*(\vec{f}) \chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})}^q \right\}^{\frac{1}{q}} \\ &\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L 2^{k\alpha q} \prod_{i=1}^2 \left(\sum_{k_i=k}^{\infty} 2^{(k_i-k)(\frac{\beta}{2}-n\delta_{i1})} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^q \right\}^{\frac{1}{q}} \\ &\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L \prod_{i=1}^2 \left(\sum_{k_i=k}^{\infty} 2^{k\alpha_i + (k_i-k)(\frac{\beta}{2}-n\delta_{i1})} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^q \right\}^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L \prod_{i=1}^2 \left(\sum_{k_i=k}^{\infty} 2^{k\alpha_i + (k_i-k)(\frac{\beta}{2} - n\delta_{i1})} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^v \right\}^{\frac{1}{v}} \\
&\lesssim \prod_{i=1}^2 \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^L \left(\sum_{k_i=k}^{\infty} 2^{k\alpha_i + (k_i-k)(\frac{\beta}{2} - n\delta_{i1})} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^{q_i} \right\}^{\frac{1}{q_i}}. \quad (4.13)
\end{aligned}$$

Remark that $\alpha_i > \lambda_i + \frac{\beta}{2} - n\delta_{i1}$. By applying (4.7), (4.9), and (4.13), we have

$$\begin{aligned}
&\|\mathcal{H}_{\beta,2}^*(\vec{f})\|_{\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega^{p(\cdot)})} \\
&\lesssim \prod_{i=1}^2 \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^L \left(\sum_{k_i=k}^{\infty} 2^{k\alpha_i + (k_i-k)(\frac{\beta}{2} - n\delta_{i1})} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^{q_i} \right\}^{\frac{1}{q_i}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{\dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})} \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^L 2^{k\lambda_i q_i} \left(\sum_{k_i=k}^{\infty} 2^{(k_i-k)(\lambda_i + \frac{\beta}{2} - \alpha_i - n\delta_{i1})} \right)^{q_i} \right\}^{\frac{1}{q_i}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{\dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})} \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left(\sum_{k=-\infty}^L 2^{k\lambda_i q_i} \right)^{\frac{1}{q_i}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{\dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})}.
\end{aligned}$$

This finishes the proof of Theorem 4.2.

Theorem 4.3 Let $p_i(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ ($i = 1, 2, \dots, m, m \in \mathbb{Z}^+$), $p(\cdot)$ is defined as follows:

$$\sum_{i=1}^m \frac{1}{p_i(x)} = \frac{1}{p(x)}.$$

Let $0 < q_i < \infty$, $\lambda_i > 0$, $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$, $\lambda = \sum_{i=1}^m \lambda_i$, $\omega \in A_{p(\cdot)}$, $\omega_i \in A_{p_i(\cdot)}$, $\omega = \prod_{i=1}^m \omega_i$, $\delta_{i1}, \delta_{i2} \in (0, 1)$ are the constants in Lemma 2.7 for exponents $p_i(\cdot)$ and weights $\omega_i^{p_i(\cdot)}$, then

(i) When $\alpha_i < \lambda_i + n\delta_{i2}$, we have

$$\|\mathcal{H}_m(\vec{f})\|_{\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega^{p(\cdot)})} \leq C \prod_{i=1}^m \|f_i\|_{\dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})}.$$

(ii) When $\alpha_i > \lambda_i - n\delta_{i1}$, we have

$$\|\mathcal{H}_m^*(\vec{f})\|_{\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega^{p(\cdot)})} \leq C \prod_{i=1}^m \|f_i\|_{\dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})}.$$

Proof (i) We prove the homogeneous case, since the nonhomogeneous case is similar. Since the proof method is similar to the Theorem 4.1, we only give the proof idea here and omit the detailed proof. Without loss of generality, we only consider the case $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{Z}^+$ ($m \geq 1$). When $m = 2$, similar to the estimation of (4.1), by virtue of the definition of $\mathcal{H}_2$ and generalized Hölder inequality, we have

$$|\mathcal{H}_2(\vec{f})(x) \cdot \chi_k(x)| \lesssim 2^{-2kn} \sum_{k_1=-\infty}^k \sum_{k_2=-\infty}^k \left(\|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right)$$

$$\cdot \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \chi_k(x). \quad (4.14)$$

By taking the $L^{p(\cdot)}(\omega^{p(\cdot)})$ -norm for (4.14) and applying (2.18) and (4.4), we have

$$\begin{aligned} \|\mathcal{H}_2(\vec{f}) \cdot \chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} &\lesssim 2^{-2kn} \sum_{k_1=-\infty}^k \sum_{k_2=-\infty}^k \left\{ \|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right. \\ &\quad \cdot \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \left. \right\} \|\chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} \\ &\lesssim 2^{-2kn} \sum_{k_1=-\infty}^k \sum_{k_2=-\infty}^k \left\{ \|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right. \\ &\quad \cdot \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \left. \right\} 2^{k(2n-\beta)} \|\chi_k\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'}^{-1} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'}^{-1} \\ &\lesssim \prod_{i=1}^2 \left\{ \sum_{k_i=-\infty}^k \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \|\chi_{k_i}\|_{(L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)}))'} \|\chi_k\|_{(L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)}))'}^{-1} \right\} \\ &= \prod_{i=1}^2 \left\{ \sum_{k_i=-\infty}^k \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \frac{\|\chi_{k_i}\|_{(L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)}))'}}{\|\chi_k\|_{(L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)}))'}} \right\} \\ &\lesssim \prod_{i=1}^2 \left\{ \sum_{k_i=-\infty}^k 2^{(k_i-k)n\delta_{i2}} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right\}. \end{aligned} \quad (4.15)$$

Next, the required results are obtained in a way similar to the proof of Theorem 4.1.

(ii) We prove the homogeneous case, since the nonhomogeneous case is similar. Since the proof method is similar to the Theorem 4.2, we only give the proof idea here and omit the detailed proof. Without loss of generality, we only consider the case $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{Z}^+ (m \geq 1)$. When $m = 2$, similar to the estimation of (4.10), by virtue of the definition of $\mathcal{H}_2^*$ and generalized Hölder inequality, we have

$$\begin{aligned} |\mathcal{H}_2^*(\vec{f})(x) \cdot \chi_k(x)| &\lesssim \sum_{k_1=k}^{\infty} \sum_{k_2=k}^{\infty} 2^{(-n)(k_1+k_2)} \left(\|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right. \\ &\quad \cdot \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \left. \right) \chi_k(x). \end{aligned} \quad (4.16)$$

By taking the $L^{p(\cdot)}(\omega^{p(\cdot)})$ -norm for (4.16), applying (2.14), (2.15), (2.17), and (4.4), we have

$$\begin{aligned} \|\mathcal{H}_2^*(\vec{f}) \cdot \chi_k\|_{L^{p(\cdot)}(\omega^{p(\cdot)})} &\lesssim 2^{(-n)(k_1+k_2)} \sum_{k_1=k}^{\infty} \sum_{k_2=k}^{\infty} \left\{ \|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right. \\ &\quad \cdot \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \left. \right\} 2^{k(2n-\beta)} \|\chi_k\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'}^{-1} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'}^{-1} \\ &\lesssim \sum_{k_1=k}^{\infty} \sum_{k_2=k}^{\infty} \left\{ \|f_{k_1}\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|f_{k_2}\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \right. \\ &\quad \cdot 2^{-nk_1} \|\chi_{k_1}\|_{(L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)}))'} 2^{-nk_2} \|\chi_{k_2}\|_{(L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)}))'} \left. \right\} \|\chi_k\|_{L^{p_1(\cdot)}(\omega_1^{p_1(\cdot)})} \|\chi_k\|_{L^{p_2(\cdot)}(\omega_2^{p_2(\cdot)})} \end{aligned}$$

$$\begin{aligned}
&\lesssim \prod_{i=1}^2 \left\{ \sum_{k_i=k}^{\infty} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \|\chi_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})}^{-1} \|\chi_k\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right\} \\
&= \prod_{i=1}^2 \left\{ \sum_{k_i=k}^{\infty} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \frac{\|\chi_k\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})}}{\|\chi_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})}} \right\} \\
&\lesssim \prod_{i=1}^2 \left\{ \sum_{k_i=k}^{\infty} 2^{(k-k_i)n\delta_{i1}} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right\}. \tag{4.17}
\end{aligned}$$

Similar to the estimation of (4.13). By applying (4.7), (4.17), and Hölder inequality in sequential form, we have

$$\|\mathcal{H}_2^*(\vec{f})\|_{\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega^{p(\cdot)})} \lesssim \prod_{i=1}^2 \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^L \left(\sum_{k_i=k}^{\infty} 2^{k\alpha_i+(k-k_i)n\delta_{i1}} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^{q_i} \right\}^{\frac{1}{q_i}}. \tag{4.18}$$

Remark that $\alpha_i > \lambda_i - n\delta_{i1}$. By applying (4.7), (4.9), and (4.18), we have

$$\begin{aligned}
&\|\mathcal{H}_2^*(\vec{f})\|_{\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega^{p(\cdot)})} \\
&\lesssim \prod_{i=1}^2 \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^L \left(\sum_{k_i=k}^{\infty} 2^{k\alpha_i+(k-k_i)n\delta_{i1}} \|f_{k_i}\|_{L^{p_i(\cdot)}(\omega_i^{p_i(\cdot)})} \right)^{q_i} \right\}^{\frac{1}{q_i}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{\dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})} \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^L 2^{k\lambda_i q_i} \left(\sum_{k_i=k}^{\infty} 2^{(k-k_i)(\alpha_i+n\delta_{i1}-\lambda_i)} \right)^{q_i} \right\}^{\frac{1}{q_i}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{\dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})} \sup_{L \in \mathbb{Z}} 2^{-L\lambda_i} \left(\sum_{k=-\infty}^L 2^{k\lambda_i q_i} \right)^{\frac{1}{q_i}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{\dot{\mathbf{M}}_{q_i,p_i(\cdot)}^{\alpha_i,\lambda_i}(\omega_i^{p_i(\cdot)})}.
\end{aligned}$$

This finishes the proof idea of Theorem 4.3.

Remark 4.4 Because of $\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,0}(\omega) = \dot{\mathbf{K}}_{p(\cdot)}^{\alpha,q}(\omega)$, let $\lambda = 0$ from Theorem 4.1 and Theorem 4.2. Then we can obtain the boundedness of the m -order multilinear fractional Hardy operator $\mathcal{H}_{\beta,m}$ and its adjoint operator $\mathcal{H}_{\beta,m}^*$ from the weighted variable-exponent Herz product space

$$K_{p_1(\cdot)}^{\alpha_1,q_1}(\omega_1^{p_1(\cdot)}) \times K_{p_2(\cdot)}^{\alpha_2,q_2}(\omega_2^{p_2(\cdot)}) \times \cdots \times K_{p_m(\cdot)}^{\alpha_m,q_m}(\omega_m^{p_m(\cdot)})$$

to the homogeneous weighted variable-exponent Herz space $K_{p(\cdot)}^{\alpha,q}(\omega^{p(\cdot)})$. Obviously, from Theorem 4.3, the m -order multilinear Hardy operator $\mathcal{H}$ and its adjoint operator $\mathcal{H}^*$ have similar results.

5. Conclusions

This paper first considered the boundedness of higher-order commutators $I_{\beta,b}^m$ generated by the fractional integral operator with BMO functions on weighted variable-exponent Herz-Morrey spaces $\dot{\mathbf{M}}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)$ and grand weighted variable-exponent Herz-Morrey spaces $\dot{\mathbf{M}}_{\lambda,p(\cdot)}^{\alpha,r,\theta}(\omega)$, and

generalized Theorem 4 of Izuki [18] as well as Theorem 2 of Sultan [32]. Then, we considered the boundedness of the m -order multilinear fractional Hardy operator $\mathcal{H}_{\beta,m}$ and its adjoint operator $\mathcal{H}_{\beta,m}^*$ on weighted variable-exponent Herz-Morrey spaces $\dot{M}_{q,p(\cdot)}^{\alpha,\lambda}(\omega)$, and generalized some relevant results of Wu [12].

Author contributions

Ming Liu and Binhua Feng wrote the main manuscript text and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We wish to thank the handing editor and the referees for their valuable comments and suggestions. B. Feng is supported by the National Natural Science Foundation of China (No.12461035,12261079), the Natural Science Foundation of Gansu Province (No.24JRRA242).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.*, **41** (1991), 582–618. <http://dx.doi.org/10.21136/CMJ.1991.102493>
2. D. Cruz-Uribe, A. Fiorenza, C. J. Neugebauer, The maximal function on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.*, **28** (2003), 223–238.
3. L. Diening, Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, *Math. Nachr.*, **268** (2004), 31–43. <https://doi.org/10.1002/mana.200310157>
4. A. Nekvinda, Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$, *Math. Inequal. Appl.*, **7** (2004), 255–265. <https://doi.org/10.7153/mia-07-28>
5. H. Wang, Z. Fu, Z. Liu, Higher order commutators of Marcinkiewicz integrals on variable Lebesgue spaces, *Acta Math. Sci. A*, **32** (2012), 1092–1101.
6. M. Izuki, Commutators of fractional integrals on Lebesgue and Herz spaces with variable exponent, *Rend. Circ. Mat. Palermo*, **59** (2010), 461–472. <https://doi.org/10.1007/s12215-010-0034-y>
7. M. Izuki, Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization, *Anal. Math.*, **36** (2010), 33–50. <https://doi.org/10.1007/s10476-010-0102-8>

8. L. Wang, M. Qu, L. Shu, Higher Order Commutators of Fractional Integral Operator on the Homogeneous Herz Spaces with Variable Exponent, *J. Funct. Space. Appl.*, **2013** (2013), 1–7. <https://doi.org/10.1155/2013/257537>
9. S. Lu, L. Xu, Boundedness of rough singular integral operators on the Homogeneous Morrey-Herz spaces, *Hokkaido Math. J.*, **34** (2005), 299–314. <https://doi.org/10.14492/hokmj/1285766224>
10. M. Izuki, Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponent, *Math. Sci. Res. J.*, **13** (2009), 243–253.
11. M. Izuki, Fractional integrals on Herz-Morrey spaces with variable exponent, *Hiroshima Math. J.*, **40** (2010), 343–355. <https://doi.org/10.32917/hmj/1291818849>
12. J. Wu, P. Zhang, Boundedness of Multilinear Fractional Hardy Operators on the Product of Herz-Morrey Spaces with Variable Exponent, *J. Coll. Univ.*, **2** (2013), 154–164.
13. J. Wu, Boundedness for commutators of fractional integrals on Herz-Morrey spaces with variable exponent, *Kyoto J. Math.*, **54** (2014), 483–495. <https://doi.org/10.1215/21562261-2693397>
14. D. Cruz-Uribe, A. Fiorenza, C. J. Neugebauer, Weighted norm inequalities for the maximal operator on variable Lebesgue spaces, *J. Math. Anal. Appl.*, **394** (2012), 223–238. <https://doi.org/10.1016/j.jmaa.2012.04.044>
15. D. Cruz-Uribe, L. A. Wang, Extrapolation and weighted norm inequalities in the variable Lebesgue spaces, *Trans. Am. Math. Soc.*, **369** (2017), 1205–1235. <https://doi.org/10.1090/tran/6730>
16. M. Izuki, Remarks on Muckenhoupt weights with variable exponent, *J. Anal. Appl.*, **11** (2013), 27–41.
17. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.*, **165** (1972), 207–226. <https://doi.org/10.1090/S0002-9947-1972-0293384-6>
18. M. Izuki, T. Noi, Boundedness of fractional integrals on weighted Herz spaces with variable exponent, *J. Inequal. Appl.*, **2016** (2016), 1–15. <https://doi.org/10.1186/s13660-016-1142-9>
19. M. Izuki, T. Noi, An intrinsic square function on weighted Herz spaces with variable exponent, *J. Math. Inequal.*, **11** (2017), 799–816. <https://doi.org/10.7153/jmi-2017-11-62>
20. M. Izuki, T. Noi, Two weighted Herz spaces with variable exponents, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 169–200. <https://doi.org/10.1007/s40840-018-0671-4>
21. M. Asim, A. Hussain, N. Sarfraz, Weighted variable Morrey-Herz estimates for fractional Hardy operators, *J. Inequal. Appl.*, **2022** (2022), 1–12. <https://doi.org/10.1186/s13660-021-02739-z>
22. A. Hussain, M. Asim, M. Aslam, F. Jarad, Commutators of the Fractional Hardy Operator on Weighted Variable Herz-Morrey Spaces, *J. Funct. Space.*, **2021** (2021), 1–10. <https://doi.org/10.1155/2021/9705250>
23. S. Wang, J. Xu, Commutators of bilinear hardy operators on weighted Herz-Morrey spaces with variable exponent, *Acta Math. Sin.*, **64** (2021), 123–138.
24. S. Wang, J. Xu, Boundedness of vector-valued sublinear operators on weighted Herz-Morrey spaces with variable exponent, *Open Math.*, **19** (2021), 412–426. <https://doi.org/10.1515/math-2021-0024>

25. D. Xiao, L. Shu, Boundedness of Marcinkiewicz integrals in weighted variable exponent Herz-Morrey spaces, *Math. Res. Commun.*, **34** (2018), 371–382.
26. H. Zhao, Z. Liu, Boundedness of commutators of fractional integral operators on variable weighted Herz-Morrey spaces, *Adv. Math.*, **51** (2022), 103–116.
27. H. Ahmad, M. Tariq, S. K. Sahoo, J. Baili, C. Cesarano, New Estimations of Hermite-Hadamard Type Integral Inequalities for Special Functions, *Fractal Fract.*, **5** (2021), 144. <https://doi.org/10.3390/fractalfract5040144>
28. F. Wang, I. Ahmad, H. Ahmad, M. D. Alsulami, K. S. Alimgeer, C. Cesarano, et al., Meshless method based on RBFs for solving three-dimensional multi-term time fractional PDEs arising in engineering phenomenons, *J. King Saud Univ. Sci.*, **33** (2021), 101604. <https://doi.org/10.1016/j.jksus.2021.101604>
29. G. H. Hardy, Note on a theorem of Hilbert, *Math. Z.*, **6** (1920), 314–317. <https://doi.org/10.1007/BF01199965>
30. M. Chirst, L. Grafakos, Best constants for two non-convolution inequalities, *Proc. Amer. Math. Soc.*, **123** (1995), 1687–1693. <https://doi.org/10.1090/S0002-9939-1995-1239796-6>
31. Z. Fu, Z. Liu, S. Lu, H. Wong, Characterization for commutators of n -dimensional fractional Hardy operators, *Sci. China, Ser. A: Math.*, **50** (2007), 1418–1426. <https://doi.org/10.1007/s11425-007-0094-4>
32. B. Sultan, F. Azmi, M. Sultan, T. Mahmood, N. Mlaiki, N. Souayah, Boundedness of Fractional Integrals on Grand Weighted Herz-Morrey Spaces with Variable Exponent, *Fractal Fract.*, **6** (2022), 660. <https://doi.org/10.3390/fractalfract6110660>
33. B. Sultan, M. Sultan, M. Mehmood, M. Azmi, F. Alghafli, N. Mlaiki, Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent, *AIMS Math.*, **8** (2023), 752–764. <https://doi.org/10.3934/math.2023036>
34. C. Bennett, R. C. Sharpley, *Interpolation of Operators*, Springer-Verlag, New York, 1988.
35. A. L. Bernardis, E. D. Dalmasso, G. G. Pradolini, Generalized maximal functions and related operators on weighted Musielak-Orlicz spaces, *Ann. Acad. Sci. Fenn. Math.*, **39** (2014), 23–50. <https://doi.org/10.5186/aasfm.2014.3904>
36. A. W. Huang, J. S. Xu, Multilinear singular integrals and commutators in variable exponent Lebesgue spaces, *Appl. Math. J. Chin. Univ.*, **25** (2010), 69–77. <https://doi.org/10.1007/s11766-010-2167-3>
37. C. Kening, E. M. Stein, Multilinear estimates and fractional integration, *Math. Res. Lett.*, **6** (1991), 1–15. <http://dx.doi.org/10.4310/mrl.1999.v6.n1.a1>



AIMS Press

© 2025 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)