Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 17(1): 263–289.
DOI: 10.3934/cam.2025011
Received: 28 August 2024
Revised: 01 March 2025
Accepted: 07 March 2025
Published: 18 March 2025

Theory article

Global existence and blow-up to coupled fourth-order parabolic systems arising from modeling epitaxial thin film growth

Tingfu Feng[1], **Yan Dong**[2,*], **Kelei Zhang**[3] **and Yan Zhu**[1]

[1] Department of Mathematics, Kunming University, Kunming, Yunnan, China
[2] Department of Applied Mathematics, Hubei University of Economics, Wuhan, Hubei, China
[3] School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin, Guangxi, China

* **Correspondence:** Email: dongyan@hbue.edu.cn.

Abstract: This paper focuses on a class of fourth-order parabolic systems involving logarithmic and Rellich nonlinearities arising from modeling epitaxial thin film growth:

$$\begin{cases} u_t + \Delta^2 u = |v|^p|u|^{p-2}u\ln|uv| - \mu\frac{u}{|x|^4}, \\ v_t + \Delta^2 v = |u|^p|v|^{p-2}v\ln|uv| - \gamma\frac{v}{|x|^4}. \end{cases}$$

By using some new techniques to deal with the Rellich nonlinearities $\mu\frac{u}{|x|^4}$ and $\gamma\frac{v}{|x|^4}$, as well as the coupled logarithmic nonlinearities $|v|^p|u|^{p-2}u\ln|uv|$ and $|u|^p|v|^{p-2}v\ln|uv|$, we prove the global existence and finite time blow-up of weak solutions. Furthermore, we not only obtain a new algebraic decay estimate and study the behavior of global weak solutions, but we also derive a new upper bound estimate for the blow-up time in case of the occurrence of blow-up.

Keywords: epitaxial thin film growth; coupled fourth-order parabolic systems; global existence; blow-up; asymptotic behavior
Mathematics Subject Classification: 35A01, 35B44, 35K52

---

1. Introduction

In this paper, we consider the following initial-boundary value problem for a class of coupled fourth-order parabolic systems arising from modeling epitaxial thin film growth

$$\begin{cases} u_t + \Delta^2 u = |v|^p|u|^{p-2}u\ln|uv| - \mu\frac{u}{|x|^4}, & x \in \Omega, \quad t > 0, \\ v_t + \Delta^2 v = |u|^p|v|^{p-2}v\ln|uv| - \gamma\frac{v}{|x|^4}, & x \in \Omega, \quad t > 0, \\ u = \frac{\partial u}{\partial \nu} = 0, v = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0, v(x,0) = v_0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 5$) is a bounded smooth domain, ν denotes the unit outward normal, $|v|^p|u|^{p-2}u \ln |uv|$ and $|u|^p|v|^{p-2}v \ln |uv|$ are the coupled logarithmic nonlinearities, $\mu \frac{u}{|x|^4}$ and $\gamma \frac{v}{|x|^4}$ are the Rellich nonlinearities, the constants $p > 1$, $0 \leq \mu < \frac{N^2(N-4)^2}{16}$, and $0 \leq \gamma < \frac{N^2(N-4)^2}{16}$, where $\frac{N^2(N-4)^2}{16}$ is the best constant of Rellich inequality (see Rellich [1], Rellich and Berkowitz [2], Davies and Hinz [3], Caldiroli and Musina [4]).

In recent years, the epitaxial growth of nanoscale thin film has received increasing attention in materials science. The growth of crystal thin films from molecular or atomic beams is commonly referred to as molecular beam epitaxy, which is a technology used to manufacture computer chips and other semiconductor devices. To qualitatively and quantitatively understand the growth process of thin films in order to formulate better control laws for the film growth process, one can optimize the flatness, electrical conductivity, and other characteristics of the film. This is highly practical and meaningful for the manufacture of computer chips and other semiconductor devices. Consequently, mathematical models arising from epitaxial growth of nanoscale thin film have attracted a lot of attention, such as the evolution of epitaxial growth of nanoscale thin film (Zangwill [5]), a phenomenological continuum model of film growth based on a series expansion of the deposition flux in powers of the profile gradient, consideration of the energetics of the film-substrate interface, and the enforcement of Onsager's reciprocity relations (Ortiz, Repetto, and Si [6]), a geometric model for coarsening during spiral-mode growth of thin film (Schulze and Kohn [7]), and a minimal deposition equation for amorphous thin film growth (Raible, Linz, and Hanggi [8], see also [9, 10]). These can be described by a kind of fourth-order parabolic equations in the following form:

$$u_t + \Delta^2 u - \operatorname{div}(f(\nabla u)) = g(x, t, u), \tag{1.2}$$

where u represents the height from the surface of the thin film, $\Delta^2 u$ denotes the capillarity-driven surface diffusion, $\operatorname{div}(f(\nabla u))$ denotes the upward hopping of atoms effects, and $g(x, t)$ denotes the source term. Stein and Winkler [11] considered a fourth-order nonlinear parabolic equation (1.2) in the one-dimensional case

$$u_t + u_{xxxx} + u_{xx} = -(|u_x|^\alpha)_{xx}, \alpha > 1,$$

which arises in the modeling of epitaxial growth of thin film of certain metallic glasses. Solutions from two different regularity classes are proved based on the value range of α: (i) The unique mild solutions exist locally in time for any $\alpha > 1$ and initial data $u_0 \in W^{1,q}(\Omega)$ $(q > \alpha)$, and they exist globally if $\alpha \leq \frac{5}{3}$ without nonlinear source term, i.e., $g(x, t, u) = 0$; (ii) The global weak solutions are constructed by a semidiscrete approximation scheme for $\alpha \leq \frac{10}{3}$, and by transforms of such solutions, the existence of a bounded absorbing set in $L^1(\Omega)$ for $\alpha \leq \left[2, \frac{10}{3}\right)$. Furthermore, some numerical examples are given in order to illustrate these results about absorbing sets. For the equation $u_t = -u_{xxxx} + \left(u_x{}^2\right)_{xx}$, the uniqueness and smoothness of global solutions were verified rigorously based on numerical data and a posteriori analysis in [12]. Additionally, the conservation of energy for weak solutions of this equation was studied in [13].

Blomker and Gugg [14] (see also [15]) addressed the existence of solutions and statistical quantities for a class of stochastic PDEs arising in amorphous thin film growth,

$$u_t + A_1 \Delta u + \Delta^2 u + \Delta |\nabla u|^2 = \eta, x \in \Omega, t > 0.$$

Kohn and Yan [16] obtained an upper bound on the coarsening rate for an epitaxial growth model

$$u_t + \Delta^2 u + \operatorname{div}\left(2\left(1 - |\nabla u|^2\right)\nabla u\right) = 0, x \in \Omega, t > 0,$$

where $\Omega \subset \mathbb{R}^2$ is a square domain. King, Stein, and Winkler [17] studied the continuum model

$$u_t + \Delta^2 u + \operatorname{div}\left(|\nabla u|^{p-2}\nabla u - \nabla u\right) = g, x \in \Omega, t > 0,$$

and they demonstrated the existence, uniqueness, and regularity of solutions in an appropriate function space under certain assumptions on g. Furthermore, they characterized the existence of nontrivial equilibria in terms of the size of the underlying domain. A fourth-order parabolic equation modeling the evolution of a thin surface when exposed to molecular beam epitaxy is given by

$$u_t = -\Delta^2 u - \mu\Delta u - \lambda\Delta|\nabla u|^2 + f(x), x \in \Omega, t > 0,$$

and was studied by Winkler [18]. He obtained global solutions in higher dimensions by utilizing a Rothe-type approximation scheme under certain appropriate structural conditions.

Liu [19] (see also [20]) studied a fourth-order parabolic equation

$$u_t + \operatorname{div}\left(m(u) k\nabla\Delta u - |\nabla u|^{p-2}\nabla u\right) = 0, k > 0, p > 2$$

with a nonlinear principal part modeling epitaxial thin film growth in one-dimensional space and two-dimensional space, respectively. He proved the global existence of classical solutions based on Schauder-type estimates and Campanato spaces, provided that $m(u)$ satisfied appropriate structural conditions. Li and Melcher [21] studied the well-posedness and stability of a system

$$u_t + \Delta^2 u = \operatorname{div}(f(\nabla u)),$$

under the condition that $f(\nabla u)$ satisfies

$$|f'(\xi_1) - f'(\xi_2)| \le C\left(|\xi_1|^{\alpha-1} + |\xi_2|^{\alpha-1}\right)|\xi_1 - \xi_2|, \forall\xi_1, \xi_2 \in \mathbb{R}^N, \alpha > 1.$$

Zhao, Guo, and Wang [22] dealt with the global existence and blow-up of weak solutions when $f(\nabla u) = |\nabla u|^{p-2}\nabla u$. Additionally, the existence and blow-up of weak solutions under the case $1 < p < 2$ can be found in [23].

Agelas [24] considered the following general equation of surface growth models arising in the context of epitaxial thin film in the presence of the coarsening process, density variations, and the Ehrlich-Schwoebel effects:

$$u_t + A_1\Delta u + A_2\Delta^2 u - A_3\operatorname{div}\left(|\nabla u|^2\nabla u\right) + A_4\Delta|\nabla u|^2 = A_5|\nabla u|^2, x \in \Omega, t > 0;$$

showed the existence and uniqueness of global strong solutions for any initial data $u_0 \in H^s\left(\mathbb{R}^d\right)$, where $d \in \{1, 2\}, s \ge 3$.

Xu, Chen, Liu, and Ding [25] studied a class of fourth-order semilinear parabolic equations

$$u_t - q\Delta u + \Delta^2 u = g(u), x \in \Omega, t > 0,$$

which includes the extended Fisher-Kolmogorov equation that arises in the study of bistable systems (Dee and Van Saarloos [26]). They obtained a global attractor in $H^k(\Omega)$ by using the iteration technique for regularity estimates and derived global existence and nonexistence of solutions with initial data in the potential well when $g(u)$ satisfied appropriate structural conditions. Moreover, Liu and Li [27] added a p-Laplace diffusion term $-div\left(|\nabla u|^{p-2}\nabla u\right)$ the side of the above equation and extended these results from [25].

Zhou [28] considered a thin film equation with a p-Laplace term and nonlocal source term

$$u_t + \Delta^2 u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{q-2}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-2}u dx, x \in \Omega, t > 0,$$

and the global asymptotic behavior and some new blow-up conditions of solutions were obtained by exploiting the boundary condition and the variational structure of the equation. These results generalized the former results in [29].

In recent years, logarithmic nonlinearities have been widely used in partial differential equations describing physical phenomena [30, 31] and biological phenomena [32–34] due to their particular structures. In mathematics, the logarithmic nonlinearity has more profound effects on the properties of solutions than polynomial nonlinearity. For the semilinear heat equation $u_t - \Delta u = g(u)$, the results of [35] and [36, 37] indicated, respectively, that the polynomial nonlinearity $g(u) = |u|^{p-1}u$ caused solutions to blow up in finite time, whereas the logarithmic nonlinearity $g(u) = u\ln|u|$ caused solutions to blow up in infinite time at high energy levels. It is difficult to study the fourth-order parabolic equation with logarithmic nonlinearity $|u|^{p-1}u\ln|u|$; because the logarithmic nonlinearity $|u|^{p-1}u\ln|u|$ satisfies neither the monotonicity condition nor the Ambrosetti-Rabinowitz condition, which does not ensure the boundedness of the Palais-Smale sequence of the Euler-Lagrange functional associated with the equation. Hence, it brings some difficulties to the application of the potential well method. Recently, many scholars [38–44] have shown that these difficulties can be overcome by a modified logarithmic Sobolev inequality that deals with the logarithmic nonlinearity $|u|^{p-1}u\ln|u|$, and have obtained the existence, asymptotic behavior, and finite time blow-up of weak solutions.

Han, Gao, and Shi [38] studied an initial-boundary value problem for a thin film equation with logarithmic nonlinearity,

$$\begin{cases} u_t + \Delta^2 u = u\ln|u|, & x \in \Omega, \quad t > 0, \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0, & x \in \Omega. \end{cases} \tag{1.3}$$

Since the presence of the logarithmic nonlinear term $u\ln|u|$ brings some difficulties to the application of the potential well method, in order to deal with this logarithmic nonlinear term in problem (1.3), they established a modified logarithmic Sobolev inequality. Then, they obtained the existence and decay estimates of global solutions by using the Galerkin method in conjunction with the modified logarithmic Sobolev inequality, the Gronwall inequality, and the potential well method. Furthermore, the blow-up of solutions at infinite time under some suitable conditions was also derived.

Liao and Li [40] studied the initial-boundary value problem to a fourth-order parabolic equation with logarithmic nonlinearity

$$\begin{cases} u_t + \Delta^2 u - div\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u\ln|u|, & x \in \Omega, \quad t > 0, \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0, & x \in \Omega, \end{cases}$$

and they gave some sufficient conditions for the global existence and blow-up of weak solutions for the supercritical initial energy by using the modified potential well method and the logarithmic Sobolev inequality. These results extend and improve upon many of the findings presented in Zhou [41] for a fourth-order nonlinear parabolic equation with logarithmic nonlinearity

$$\begin{cases} u_t + \Delta^2 u + c\Delta u = u \ln |u|, & x \in \Omega, \quad t > 0, \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0 \in H_0^2(\Omega), & x \in \Omega. \end{cases}$$

Liu, Ma, and Tang [42] considered a fourth-order equation modeling epitaxial thin film growth with logarithmic nonlinearity

$$\begin{cases} u_t + \Delta^2 u = -div\left(|\nabla u|^{q-2}\nabla u \ln |\nabla u|\right), & x \in \Omega, \quad t > 0, \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0, & x \in \Omega, \end{cases}$$

and they obtained a blow-up result. Furthermore, the lower bound of the blow-up time and the blow-up rate were derived.

Xu, Lian, and Niu [45] considered a coupled parabolic systems

$$\begin{cases} u_t - \Delta u = \left(|u|^{2p} + |v|^{p+1}|u|^{p-1}\right)u, & x \in \Omega, \quad t > 0, \\ v_t - \Delta v = \left(|v|^{2p} + |u|^{p+1}|v|^{p-1}\right)v, & x \in \Omega, \quad t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

They studied the global existence, finite-time blow-up, and long-time decay of the solutions through considerations of low initial energy scenarios, critical initial energy scenarios, and high initial energy scenarios. Some sufficient initial conditions for finite-time blow-up and global existence were obtained.

Motivated and inspired by the above research work, in this paper; we consider a class of coupled fourth-order parabolic systems arising from modeling epitaxial thin film growth. The results of this paper are established in the framework of potential well theory, which was proposed by Payne and Sattinger in [46, 47] to study well-posedness of the solution to the equations without positive definite energy. This method has gradually developed into an important tool for investigating the classification of the initial data to various evolution equations involving hyperbolic equations [48–51] and parabolic equations [52, 53]. On account of the singularity of Rellich nonlinearities $\mu\frac{u}{|x|^4}$ and $\gamma\frac{v}{|x|^4}$; and the coupled logarithmic nonlinearities $|v|^p|u|^{p-2}u \ln |uv|$ and $|u|^p|v|^{p-2}v \ln |uv|$, which satisfy neither the monotonicity condition nor the Ambrosetti-Rabinowitz condition, this poses some difficulties in applying potential well theory. Moreover, compared to the general logarithmic nonlinearity $|u|^{p-2}u \ln |u|$, the coupled logarithmic nonlinearities $|v|^p|u|^{p-2}u \ln |uv|$ and $|u|^p|v|^{p-2}v \ln |uv|$ are more complex and informative. Noting that the classical logarithmic Sobolev inequality (see Gross [54], Lieb and Loss [55], Pino and Dolbeault [56])

$$p \int_\Omega |u|^p \ln \frac{|u|}{\|u\|_{L^p(\Omega)}} dx + \frac{n}{p} \ln\left(\frac{p\mu e}{nl_p}\right) \int_\Omega |u|^p dx \le \mu \int_\Omega |\nabla u|^p dx$$

is no longer applicable with the coupled logarithmic nonlinearities $|v|^p|u|^{p-2}u \ln |uv|$ and $|u|^p|v|^{p-2}v \ln |uv|$, brings some difficulties for ensuring the compactness of the Euler-Lagrange functional associated with problem (1.1). By using some new techniques to deal with the Rellich nonlinearities and the coupled logarithmic nonlinearities, we prove the global existence and finite time blow-up of weak solutions. Furthermore, we not only obtain a new algebraic decay estimate and study the large time behavior of global weak solutions, but also derive a new upper bound estimate for the blow-up time in the case of blow-up occurrence.

The organization of this paper is as follows. In Section 2, we present some preliminaries. In Section 3, we prove the global existence of a weak solution to problem (1.1) using the Galerkin method. Additionally, we provide a new algebraic decay estimate for this solution and discuss its behavior. In Section 4, we demonstrate the blow-up of a weak solution to problem (1.1) in finite time using a contradiction argument. Furthermore, we obtain a new upper bound estimate for the blow-up time by solving a minimization problem.

2. Preliminaries

Throughout this paper, $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$, and $(\cdot,\cdot)_2$ denotes the inner product in $L^2(\Omega)$. We are equipped with the norm $\|u\|_{H_0^2(\Omega)} = \left(\|u\|_2^2 + \|\nabla u\|_2^2 + \|\Delta u\|_2^2\right)^{\frac{1}{2}}$, which is equivalent to $\|u\|_{H_0^2(\Omega)} = \|\Delta u\|_2$ due to the Poincaré inequality, Cauchy inequality with ε, and Green's formulas under the Dirichlet boundary condition $u = \frac{\partial u}{\partial v} = 0$ on $\partial\Omega$.

Both the logarithmic inequality introduced in Lemma 2.1 and the Rellich inequality presented in Lemma 2.2 are crucial to the development of this paper.

Lemma 2.1 ([57], logarithmic inequality). *Assume that σ is a suitable small positive constant. Then, for the continuous function Ψ, we have*

$$\Psi^p \ln \Psi \leq \frac{1}{e\sigma}\Psi^{p+\sigma}, \Psi \geq 1,$$

and

$$|\Psi^p \ln \Psi| \leq (ep)^{-1}, 0 < \Psi < 1.$$

Lemma 2.2 ([3], Rellich Inequality). *Assume that $\Phi \in C_c^\infty\left(\mathbb{R}^N\backslash\{0\}\right)$. Then,*

$$\frac{N^2(N-4)^2}{16}\int_{\mathbb{R}^N}\frac{\Phi^2}{|x|^4}\,dx \leq \int_{\mathbb{R}^N}|\Delta\Phi|^2\,dx,$$

where $\frac{N^2(N-4)^2}{16}$ is the best constant, and the dimension $N \geq 5$. For $\Phi \in H_0^2(\Omega)$, we can define $\Phi = 0$ for $x \in \mathbb{R}^N\backslash\Omega$, hence

$$\frac{N^2(N-4)^2}{16}\int_{\Omega}\frac{\Phi^2}{|x|^4}\,dx \leq \int_{\Omega}|\Delta\Phi|^2\,dx.$$

Since the stationary problem of (1.1) is given by

$$\begin{cases} \Delta^2 u = |v|^p|u|^{p-2}u\ln|uv| - \mu\frac{u}{|x|^4}, & x \in \Omega, \\ \Delta^2 v = |u|^p|v|^{p-2}v\ln|uv| - \gamma\frac{v}{|x|^4}, & x \in \Omega, \\ u = \frac{\partial u}{\partial v} = 0, v = \frac{\partial v}{\partial v} = 0, & x \in \partial\Omega. \end{cases}$$

Hence, we can define the energy functional

$$J(u,v)=\frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\Delta v\|_2^2+\frac{1}{p^2}\|uv\|_p^p-\frac{1}{p}\int_\Omega |uv|^p\ln|uv|\,dx \\ +\frac{\mu}{2}\int_\Omega\frac{u^2}{|x|^4}dx+\frac{\gamma}{2}\int_\Omega\frac{v^2}{|x|^4}dx \tag{2.1}$$

and the Nehari functional

$$I(u,v)=\|\Delta u\|_2^2+\|\Delta v\|_2^2-2\int_\Omega |uv|^p\ln|uv|\,dx+\mu\int_\Omega\frac{u^2}{|x|^4}dx+\gamma\int_\Omega\frac{v^2}{|x|^4}dx. \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$J(u,v)=\frac{1}{2p}I(u,v)+\frac{p-1}{2p}\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2\right)+\frac{1}{p^2}\|uv\|_p^p \\ +\frac{\mu(p-1)}{2p}\int_\Omega\frac{u^2}{|x|^4}dx+\frac{\gamma(p-1)}{2p}\int_\Omega\frac{v^2}{|x|^4}dx. \tag{2.3}$$

By virtue of the Nehari functional (2.2), we can define a Nehari manifold

$$\mathbb{N}:=\left\{(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\setminus\{0,0\}\,\middle|\,I(u,v)=0\right\}.$$

Furthermore, the potential well W and its corresponding set V are by

$$\mathrm{W}:=\left\{(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\,\middle|\,0<J(u,v)<d,I(u,v)>0\right\}\cup\{0,0\},$$

$$\mathrm{V}:=\left\{(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\,\middle|\,0<J(u,v)<d,I(u,v)<0\right\},$$

where

$$d=\inf_{(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\setminus\{0,0\}}\sup_{\lambda>0}J(\lambda u,\lambda v)=\inf_{(u,v)\in\mathbb{N}}J(u,v)$$

is the depth of the potential well W.

Lemmas 2.3 and 2.4 show, respectively, that the Nehari manifold $\mathbb{N}$ is not empty, that the depth d of potential well W on $\mathbb{N}$ can be attained, and that d is positive.

Lemma 2.3. *For any $(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\setminus\{0,0\}$, define $g(\lambda)=J(\lambda u,\lambda v)$ for $\lambda>0$. Then,*

$$I(\lambda u,\lambda v)=\lambda g'(\lambda)\begin{cases}>0, & 0<\lambda<\lambda_*,\\ =0, & \lambda=\lambda_*,\\ <0, & \lambda>\lambda_*,\end{cases} \tag{2.4}$$

where

$$\lambda_*=\exp\left(\frac{\|uv\|_p^p+(p-1)\int_\Omega|uv|^p\ln|uv|\,dx}{-2(p-1)\|uv\|_p^p}\right). \tag{2.5}$$

Proof. Inspired by Drabek and Pohozaev [58], who first introduced the concept of fibering maps, we consider a fibering map

$$g : \lambda \mapsto J(\lambda u, \lambda v), \lambda > 0,$$

defined by

$$\begin{aligned} g(\lambda) &= J(\lambda u, \lambda v) \\ &= \frac{\lambda^2}{2}\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) + \left(\frac{\lambda^{2p}}{p^2} - \frac{2\lambda^{2p}}{p}\ln\lambda\right)\|uv\|_p^p - \frac{\lambda^{2p}}{p}\int_\Omega |uv|^p \ln|uv|\,dx \\ &\quad + \frac{\lambda^2\mu}{2}\int_\Omega \frac{u^2}{|x|^4}dx + \frac{\lambda^2\gamma}{2}\int_\Omega \frac{v^2}{|x|^4}dx. \end{aligned}$$

By a straightforward calculation, we obtain

$$\begin{aligned} g'(\lambda) &= \lambda\|\Delta u\|_2^2 + \lambda\|\Delta v\|_2^2 - 4\lambda^{2p-1}\ln\lambda\|uv\|_p^p - 2\lambda^{2p-1}\int_\Omega |uv|^p\ln|uv|\,dx \\ &\quad + \lambda\mu\int_\Omega \frac{u^2}{|x|^4}dx + \lambda\gamma\int_\Omega \frac{v^2}{|x|^4}dx \end{aligned}$$

and

$$\begin{aligned} I(\lambda u, \lambda v) &= \lambda^2\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) - 4\lambda^{2p}\ln\lambda\|uv\|_p^p - 2\lambda^{2p}\int_\Omega |uv|^p\ln|uv|\,dx \\ &\quad + \lambda^2\mu\int_\Omega \frac{u^2}{|x|^4}dx + \lambda^2\gamma\int_\Omega \frac{v^2}{|x|^4}dx \\ &= \lambda g'(\lambda). \end{aligned}$$

For any $\lambda > 0$, setting $g'(\lambda) = \lambda k(\lambda)$, namely

$$\begin{aligned} k(\lambda) &= \lambda^{-1}g'(\lambda) \\ &= \|\Delta u\|_2^2 + \|\Delta v\|_2^2 - 4\lambda^{2p-2}\ln\lambda\|uv\|_p^p - 2\lambda^{2p-2}\int_\Omega |uv|^p\ln|uv|\,dx \\ &\quad + \mu\int_\Omega \frac{u^2}{|x|^4}dx + \gamma\int_\Omega \frac{v^2}{|x|^4}dx, \end{aligned}$$

by a straightforward calculation, we have

$$k'(\lambda) = -4(2p-2)\lambda^{2p-3}\ln\lambda\|uv\|_p^p - 4\lambda^{2p-3}\|uv\|_p^p - 2(2p-2)\lambda^{2p-3}\int_\Omega |uv|^p\ln|uv|\,dx,$$

and setting $k'(\lambda) = 0$, there exists a $\lambda_* = \exp\left(\frac{\|uv\|_p^p + (p-1)\int_\Omega |uv|^p \ln|uv|dx}{-2(p-1)\|uv\|_p^p}\right)$, which implies (2.4) holds. The proof of Lemma 2.3 is complete. □

Lemma 2.4. *The depth d of potential well* W *on* $\mathbb{N}$ *is positive.*

Proof. For any $(u,v)\in\mathbb{N}$, we can get

$$\begin{aligned}&\|\Delta u\|_2^2+\|\Delta v\|_2^2+\mu\int_\Omega\frac{u^2}{|x|^4}dx+\gamma\int_\Omega\frac{v^2}{|x|^4}dx\\&=2\int_\Omega|uv|^p\ln|uv|\,dx\\&=2\int_{\Omega_1}|uv|^p\ln|uv|\,dx+2\int_{\Omega_2}|uv|^p\ln|uv|\,dx,\end{aligned}\tag{2.6}$$

where $\Omega_1=\{x\in\Omega|\,|uv|\le 1\}$, $\Omega_2=\{x\in\Omega|\,|uv|>1\}$. By virtue of (2.6) and Lemma 2.1, it follows from the logarithmic inequality and Young's inequality that

$$\begin{aligned}&2\int_{\Omega_1}|uv|^p\ln|uv|\,dx+2\int_{\Omega_2}|uv|^p\ln|uv|\,dx\\&\le 2\int_{\Omega_2}|uv|^p\ln|uv|\,dx\\&\le\frac{2}{e\sigma}\int_{\Omega_2}|uv|^{p+\sigma}dx\\&\le\frac{2}{e\sigma}\|uv\|_{p+\sigma}^{p+\sigma}\\&\le\frac{1}{e\sigma}\|u\|_{2(p+\sigma)}^{2(p+\sigma)}+\frac{1}{e\sigma}\|v\|_{2(p+\sigma)}^{2(p+\sigma)}.\end{aligned}$$

If $1<p<\frac{N}{N-4}$, then there exists a suitable small σ satisfying $2(p+\sigma)<\frac{2N}{N-4}$ such that $H_0^2(\Omega)\hookrightarrow L^{2(p+\sigma)}(\Omega)$, and from the above inequality we have

$$\begin{aligned}&2\int_{\Omega_1}|uv|^p\ln|uv|\,dx+2\int_{\Omega_2}|uv|^p\ln|uv|\,dx\\&\le\frac{1}{e\sigma}S^{2(p+\sigma)}\|\Delta u\|_2^{2(p+\sigma)}+\frac{1}{e\sigma}S^{2(p+\sigma)}\|\Delta v\|_2^{2(p+\sigma)},\end{aligned}\tag{2.7}$$

where S is the best constant for the embedding $H_0^2(\Omega)\hookrightarrow L^{2(p+\sigma)}(\Omega)$. We can deduce from (2.6) and (2.7) that

$$\|\Delta u\|_2^2+\|\Delta v\|_2^2\ge\left(\frac{e\sigma}{S^{2(p+\sigma)}}\right)^{\frac{1}{p+\sigma-1}}>0.\tag{2.8}$$

Since $(u,v)\in\mathbb{N}$, $0\le\mu<\frac{N^2(N-4)^2}{16}$ and $0\le\gamma<\frac{N^2(N-4)^2}{16}$, it follows from (2.3) and (2.8) that

$$J(u,v)\ge\frac{p-1}{2p}\left(\frac{e\sigma}{S^{2(p+\sigma)}}\right)^{\frac{1}{p+\sigma-1}},$$

which implies $d=\inf\limits_{(u,v)\in\mathbb{N}}J(u,v)>0$. The proof of Lemma 2.4 is complete. □

Lemma 2.5 and Remark 2.1 show that $I(u,v)<0$ and $I(u,v)>0$ determine the range of λ_*, respectively.

Lemma 2.5. *Assume that* $(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\setminus\{0,0\}$*, satisfying* $I(u,v)<0$*. Then,*

$$I(u,v)<2p\,(J(u,v)-d)\,.\tag{2.9}$$

Proof. According to $I(u,v)<0$, it follows from (2.2) that

$$0<\|\Delta u\|_2^2+\|\Delta v\|_2^2+\mu\int_\Omega \frac{u^2}{|x|^4}dx+\gamma\int_\Omega \frac{v^2}{|x|^4}dx<2\int_\Omega |uv|^p\ln|uv|\,dx,$$

in $\Omega_2=\{x\in\Omega|\,|uv|>1\}$, such that $\frac{\|uv\|_p^p+(p-1)\int_\Omega |uv|^p\ln|uv|dx}{-2(p-1)\|uv\|_p^p}<0$. Combining this with (2.5) in Lemma 2.3, there exists a constant $\lambda_*\in(0,1)$ such that $I(\lambda_* u,\lambda_* v)=0$. Setting

$$h(\lambda)=2pJ(\lambda u,\lambda v)-I(\lambda u,\lambda v),$$

then by a direct computation, we obtain

$$\begin{aligned}h(\lambda)=&\ \lambda^2(p-1)\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2\right)+\frac{2\lambda^{2p}}{p}\|uv\|_p^p\\&+\lambda^2\mu(p-1)\int_\Omega\frac{u^2}{|x|^4}dx+\lambda^2\gamma(p-1)\int_\Omega\frac{v^2}{|x|^4}dx\end{aligned}$$

and

$$\begin{aligned}h'(\lambda)=&\ 2\lambda(p-1)\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2\right)+4\lambda^{2p-1}\|uv\|_p^p\\&+2\lambda\mu(p-1)\int_\Omega\frac{u^2}{|x|^4}dx+2\lambda\gamma(p-1)\int_\Omega\frac{v^2}{|x|^4}dx\\>&\ 0.\end{aligned}$$

So, $h(\lambda)$ is strictly increasing for $\lambda>0$. Hence, by $0<\lambda_*<1$, we can derive $h(1)>h(\lambda_*)$, which translates to

$$2pJ(u,v)-I(u,v)>2pJ(\lambda_* u,\lambda_* v)-I(\lambda_* u,\lambda_* v)\geq 2pd>0.$$

This implies $I(u,v)<2p(J(u,v)-d)$ when $I(u,v)<0$. The proof of Lemma 2.5 is complete. □

Remark 2.1. *For the case where $I(u,v)>0$, similar to Lemma 2.5, if the integral $\int_\Omega|uv|^p\ln|uv|\,dx$ is small enough (i.e., $\int_\Omega|uv|^p\ln|uv|\,dx<0$) in $\Omega_1=\{x\in\Omega|\,|uv|<1\}$, then*

$$\frac{\|uv\|_p^p+(p-1)\int_\Omega|uv|^p\ln|uv|\,dx}{-2(p-1)\|uv\|_p^p}>0.$$

Combining with (2.5) *in Lemma 2.3, there exists a $\lambda_*>1$ such that $I(\lambda_* u,\lambda_* v)=0$.*

In the following, we will introduce several definitions that are essential for the purposes of this paper.

Definition 2.1. *A function $(u(x,t),v(x,t))$ is called a weak solution to problem* (1.1) *if*

$$(u,v)\in L^\infty\left(0,T;H_0^2(\Omega)\right)\times L^\infty\left(0,T;H_0^2(\Omega)\right)$$

with

$$(u_t,v_t)\in L^2\left(0,T;L^2(\Omega)\right)\times L^2\left(0,T;L^2(\Omega)\right)$$

and satisfies

$$(u_t, \phi)_2 + (\Delta u, \Delta\phi)_2 = \left(|v|^p|u|^{p-2}u\ln|uv|\,, \phi\right)_2 - \left(\mu\frac{u}{|x|^4}, \phi\right)_2,$$
$$(v_t, \varphi)_2 + (\Delta v, \Delta\varphi)_2 = \left(|u|^p|v|^{p-2}v\ln|uv|\,, \varphi\right)_2 - \left(\gamma\frac{v}{|x|^4}, \varphi\right)_2,$$

for a.e. $t \in [0,T]$ *and any* $(\phi,\varphi) \in H_0^2(\Omega) \times H_0^2(\Omega)$*, and* $u(x,0) = u_0, v(x,0) = v_0$*. Moreover,*

$$\int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right)d\tau + J(u,v) \le J(u_0,v_0), t \in [0,T].$$

Definition 2.2. *Let* T *be the maximal existence time of a weak solution* $(u(x,t), v(x,t))$ *to the problem* (1.1) *as follows:*
(i) *if* $(u(x,t), v(x,t))$ *exists for all* $0 \le t < +\infty$*, then* $T = +\infty$*, and the weak solution exists globally;*
(ii) *if there is a* $t_0 \in (0,+\infty)$ *such that* $(u(x,t), v(x,t))$ *exists for* $0 \le t < t_0$*, but does not exist at* $t = t_0$*, then* $T = t_0$*, and the weak solution exists locally and blows up in finite time.*

Definition 2.3. *A weak solution* $(u(x,t), v(x,t))$ *to problem* (1.1) *blows up in finite time if the maximal existence time* T *is finite and*

$$\lim_{t\to T^-}\int_0^t \left(\|u\|_2^2 + \|v\|_2^2\right)d\tau = +\infty.$$

Definition 2.4. *A weak solution* $(u(x,t), v(x,t))$ *to the problem* (1.1) *blows up in infinite time if the maximal existence time* $T = +\infty$ *and*

$$\lim_{t\to+\infty}\int_0^t \left(\|u\|_2^2 + \|v\|_2^2\right)d\tau = +\infty.$$

3. Global existence and decay estimates

In this section, we obtain the global existence of a weak solution to the problem (1.1) using the Galerkin method, and then derive a new algebraic decay estimate for this global weak solution using the Gronwall inequality. Furthermore, the behavior of this global weak solution is also presented in the following theorem.

Theorem 3.1. *Let* $(u_0,v_0) \in H_0^2(\Omega) \times H_0^2(\Omega)$*. Assume that* $1 < p < \frac{p^2}{p-1} < \frac{N}{N-4}$*. If* $0 < J(u_0,v_0) < d, I(u_0,v_0) > 0$*, then problem* (1.1) *has a weak solution* $(u(x,t), v(x,t))$ *that exists globally and satisfies the energy inequality*

$$\int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right)d\tau + J(u,v) \le J(u_0,v_0), t \in [0,+\infty). \tag{3.1}$$

Furthermore, a weak solution $(u(x,t), v(x,t))$ *exhibits algebraic decay, namely*

$$\|u\|_2^2 + \|v\|_2^2 \le \left(\|u_0\|_2^2 + \|v_0\|_2^2\right)e^{-2\left(1-\left(\frac{J(u_0,v_0)}{d}\right)^{\frac{2p-2}{2p}}\right)\frac{1}{S_1^2}t}, t \in [0,+\infty), \tag{3.2}$$

where S_1 *is the best constant for the embedding* $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$*. Furthermore, the behavior is given by*

$$\lim_{t\to\infty}\left(\|u(\cdot,t)\|_2^2 + \|v(\cdot,t)\|_2^2\right) = 0.$$

Proof. First, we prove the global existence of a weak solution to problem (1.1) by the Galerkin method. The proof will be divided into 5 steps.

Step 1. Approximation problem

In the Sobolev space $H_0^2(\Omega)$, we choose a basis $\left\{w_j\right\}_{j=1}^{m}$ and define the finite-dimensional space

$$V_m = span\left\{w_1, w_2, \cdots, w_m\right\}.$$

For a positive integer m, we look for the approximate solutions $(u_m(x,t), v_m(x,t))$ to problem (1.1),

$$\begin{cases} u_m(x,t) = \sum\limits_{j=1}^{m} g_{mj}(t)\, w_j(x), \\ v_m(x,t) = \sum\limits_{j=1}^{m} h_{mj}(t)\, w_j(x), \end{cases}$$

satisfying

$$\left(u_{mt}, w_j\right)_2 + \left(\Delta u_m, \Delta w_j\right)_2 = \left(|v_m|^p |u_m|^{p-2} u_m \ln|u_m v_m|, w_j\right)_2 - \left(\mu \frac{u_m}{|x|^4}, w_j\right)_2, \tag{3.3}$$

$$\left(v_{mt}, w_j\right)_2 + \left(\Delta v_m, \Delta w_j\right)_2 = \left(|u_m|^p |v_m|^{p-2} v_m \ln|u_m v_m|, w_j\right)_2 - \left(\gamma \frac{v_m}{|x|^4}, w_j\right)_2, \tag{3.4}$$

and

$$u_m(x,0) = u_{0m} = \sum_{j=1}^{m} g_{mj} w_j(x) \to u_0, m \to +\infty, \tag{3.5}$$

$$v_m(x,0) = v_{0m} = \sum_{j=1}^{m} h_{mj} w_j(x) \to v_0, m \to +\infty, \tag{3.6}$$

where $g_{mj} = g_{mj}(0), h_{mj} = h_{mj}(0)$, $u_{0m}, v_{0m} \in V_m$.

By the Picard iteration method of ordinary differential equations, there exists a positive T such that

$$\left(g_{mj}, h_{mj}\right) \in C^1([0,T]) \times C^1([0,T]),$$

and thus

$$(u_m(x,t), v_m(x,t)) \in C^1\left([0,T], H_0^2(\Omega)\right) \times C^1\left([0,T], H_0^2(\Omega)\right).$$

From this, we obtain a local solution to problem (1.1).

Next, we prove that this solution exists globally.

Step 2. Priori estimates

Multipling (3.3) and (3.4) by $\frac{d}{dt} g_{mj}(t)$ and $\frac{d}{dt} h_{mj}(t)$, respectively, summing for j from 1 to m, and integrating with respect to time variable on $[0,t]$, we arrive at

$$\int_0^t \left(\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2\right) d\tau + J(u_m, v_m) = J(u_{0m}, v_{0m}), t \in [0,T], \tag{3.7}$$

and it follows from (3.5) and (3.6) that $J(u_{0m}, v_{0m}) \to J(u_0, v_0)$. Since $J(u_0, v_0) < d$, we obtain from (3.7) that

$$\int_0^t \left(\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2\right) d\tau + J(u_m, v_m) < d, t \in [0,T], \tag{3.8}$$

for sufficiently large m.

From (3.5), (3.6), and $(u_0, v_0) \in W$, it follows that (u_{0m}, v_{0m}) for sufficiently large m. We can conclude $(u_m, v_m) \in W$ by contradiction, and $t \in [0, T]$. If it does not hold, assume that there is a $t_0 \in (0, T)$ such that $(u_m(t_0), v_m(t_0)) \in \partial W$; $I(u_m(t_0), v_m(t_0)) = 0$ and $(u_m(t_0), v_m(t_0)) \neq (0, 0)$ or $J(u_m(t_0), v_m(t_0)) = d$. Nevertheless, noting that $J(u_m(t_0), v_m(t_0)) = d$ does not occur by (3.8), by virtue of the definition of d, we have $J(u_m(t_0), v_m(t_0)) \geq d$, which is also contradiction with (3.8). So, $(u_m, v_m) \in W$ for sufficiently large m.

From (2.3), we have

$$\begin{aligned} J(u_m, v_m) = &\frac{1}{2p} I(u_m, v_m) + \frac{p-1}{2p}\left(\|\Delta u_m\|_2^2 + \|\Delta v_m\|_2^2\right) + \frac{1}{p^2}\|u_m v_m\|_p^p \\ &+ \frac{\mu(p-1)}{2p}\int_\Omega \frac{{u_m}^2}{|x|^4}dx + \frac{\gamma(p-1)}{2p}\int_\Omega \frac{{v_m}^2}{|x|^4}dx. \end{aligned} \tag{3.9}$$

Since $(u_m, v_m) \in W$ for sufficiently large m, when combined with (3.8) and (3.9), it follows that

$$\begin{aligned} &\int_0^t \left(\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2\right)d\tau + \frac{p-1}{2p}\left(\|\Delta u_m\|_2^2 + \|\Delta v_m\|_2^2\right) + \frac{1}{p^2}\|u_m v_m\|_p^p \\ &+ \frac{\mu(p-1)}{2p}\int_\Omega \frac{{u_m}^2}{|x|^4}dx + \frac{\gamma(p-1)}{2p}\int_\Omega \frac{{v_m}^2}{|x|^4}dx \\ &< d, \end{aligned} \tag{3.10}$$

which implies

$$\frac{p-1}{2p}\left(\|\Delta u_m\|_2^2 + \|\Delta v_m\|_2^2\right) < d, \tag{3.11}$$

$$\frac{\mu(p-1)}{2p}\int_\Omega \frac{{u_m}^2}{|x|^4}dx + \frac{\gamma(p-1)}{2p}\int_\Omega \frac{{v_m}^2}{|x|^4}dx < d, \tag{3.12}$$

$$\int_0^t \left(\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2\right)d\tau < d. \tag{3.13}$$

From (3.13), we know that it implies $T = +\infty$. On the other hand, through a direct calculation, we deduce from Lemma 2.1 that

$$\begin{aligned} &\int_\Omega \left(|v_m|^p |u_m|^{p-2} u_m \ln|u_m v_m|\right)^{\frac{p}{p-1}} dx \\ = &\int_{\Omega_1} \left(|v_m|^p |u_m|^{p-2} u_m \ln|u_m v_m|\right)^{\frac{p}{p-1}} dx + \int_{\Omega_2} \left(|v_m|^p |u_m|^{p-2} u_m \ln|u_m v_m|\right)^{\frac{p}{p-1}} dx \\ \leq &\int_{\Omega_1} \left(\frac{1}{e(p-1)}|v_m|\right)^{\frac{p}{p-1}} dx + \left(\frac{1}{e\sigma}\right)^{\frac{p}{p-1}} \int_{\Omega_2} \left(|u_m|^{p-1+\sigma}|v_m|^{p+\sigma}\right)^{\frac{p}{p-1}} dx \\ \leq &\left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}} \|v_m\|_{\frac{p}{p-1}}^{\frac{p}{p-1}} + \left(\frac{1}{e\sigma}\right)^{\frac{p}{p-1}} \|u_m\|_{\frac{2p(p-1+\sigma)}{p-1}}^{\frac{p(p-1+\sigma)}{p-1}} \|v_m\|_{\frac{2p(p+\sigma)}{p-1}}^{\frac{p(p+\sigma)}{p-1}} \\ \leq &\left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}} \|v_m\|_{\frac{p}{p-1}}^{\frac{p}{p-1}} + \frac{1}{2}\left(\frac{1}{e\sigma}\right)^{\frac{p}{p-1}} \|u_m\|_{\frac{2p(p-1+\sigma)}{p-1}}^{\frac{2p(p-1+\sigma)}{p-1}} + \frac{1}{2}\left(\frac{1}{e\sigma}\right)^{\frac{p}{p-1}} \|v_m\|_{\frac{2p(p+\sigma)}{p-1}}^{\frac{2p(p+\sigma)}{p-1}}, \end{aligned}$$

where $\Omega_1 = \{x \in \Omega| \, |uv| \le 1\}$, $\Omega_2 = \{x \in \Omega| \, |uv| > 1\}$. When $1 < p < \frac{p^2}{p-1} < \frac{N}{N-4}$, by the Rellich-Kondrachov compact embedding theorem, there exists a suitable small positive constant σ such that $H_0^2(\Omega) \hookrightarrow L^{\frac{2p(p+\sigma)}{p-1}}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{\frac{2p(p-1+\sigma)}{p-1}}(\Omega)$. From (3.11), we have

$$
\begin{aligned}
&\int_\Omega \left(|v_m|^p |u_m|^{p-2} u_m \ln |u_m v_m|\right)^{\frac{p}{p-1}} dx \\
\le &\left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}} S_2^{\frac{p}{p-1}} \|\Delta v_m\|_2^{\frac{p}{p-1}} + \frac{S_3^{\frac{2p(p-1+\sigma)}{p-1}}}{2}\left(\frac{1}{e\sigma}\right)^{\frac{p}{p-1}} \|\Delta u_m\|_2^{\frac{2p(p-1+\sigma)}{p-1}} \\
&+ \frac{S_4^{\frac{2p(p+\sigma)}{p-1}}}{2}\left(\frac{1}{e\sigma}\right)^{\frac{p}{p-1}} \|\Delta v_m\|_2^{\frac{2p(p+\sigma)}{p-1}} \\
\le &\, c,
\end{aligned}
\tag{3.14}
$$

where S_2 is the best constant for the embedding $H_0^2(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$, S_3 is the best constant for the embedding $H_0^2(\Omega) \hookrightarrow L^{\frac{2p(p-1+\sigma)}{p-1}}(\Omega)$, and S_4 is the best constant for the embedding $H_0^2(\Omega) \hookrightarrow L^{\frac{2p(p+\sigma)}{p-1}}(\Omega)$; Similar to the proof of (3.14), we have

$$
\int_\Omega \left(|u_m|^p |v_m|^{p-2} v_m \ln |u_m v_m|\right)^{\frac{p}{p-1}} dx < c. \tag{3.15}
$$

Step 3. Pass to the limit

By virtue of Banach-Alaoglu-Bourbaki theorem due to [47], and according to the energy estimates (3.11)-(3.15), we know that there exists a (u, v) and a subsequence of $(\{u_m\}_{m=1}^\infty, \{v_m\}_{m=1}^\infty)$ (still denoted by $(\{u_m\}_{m=1}^\infty, \{v_m\}_{m=1}^\infty)$ for clarity) such that as $m \to \infty$,

$$
\begin{gathered}
u_m \to u \textit{ weakly star in } L^\infty\left(0, +\infty; H_0^2(\Omega)\right), \\
v_m \to v \textit{ weakly star in } L^\infty\left(0, +\infty; H_0^2(\Omega)\right), \\
\frac{u_m}{|x|^2} \to \frac{u}{|x|^2} \textit{ weakly star in } L^\infty\left(0, +\infty; L^2(\Omega)\right), \\
\frac{v_m}{|x|^2} \to \frac{v}{|x|^2} \textit{ weakly star in } L^\infty\left(0, +\infty; L^2(\Omega)\right), \\
\int_\Omega |v_m|^p |u_m|^{p-2} u_m \ln |u_m v_m| \, dx \to \int_\Omega |v|^p |u|^{p-2} u \ln |uv| \, dx \quad \textit{weakly star in } L^\infty\left(0, +\infty; L^{\frac{p}{p-1}}(\Omega)\right), \\
\int_\Omega |u_m|^p |v_m|^{p-2} v_m \ln |u_m v_m| \, dx \to \int_\Omega |u|^p |v|^{p-2} v \ln |uv| \, dx \quad \textit{weakly star in } L^\infty\left(0, +\infty; L^{\frac{p}{p-1}}(\Omega)\right), \\
u_{mt} \to u_t \textit{ weakly in } L^2\left(0, +\infty; L^2(\Omega)\right), \\
v_{mt} \to v_t \textit{ weakly in } L^2\left(0, +\infty; L^2(\Omega)\right).
\end{gathered}
$$

By virtue of Aubin-Lions compactness theorem due to [59], it follows that there exists a subsequence of the given sequence that converges strongly in the desired space,

$$
u_m \to u \textit{ in } C\left([0, +\infty); L^2(\Omega)\right),
$$

$$v_m \to v \text{ in } C\left([0,+\infty);L^2(\Omega)\right).$$

Clearly, this implies that

$$u_m \to u \text{ a.e. in } \Omega \times [0,+\infty),$$

$$v_m \to v \text{ a.e. in } \Omega \times [0,+\infty).$$

Moreover, we can pass to the limit in (3.3) and (3.4) to obtain

$$\left(u_t, w_j\right)_2 + \left(\Delta u, \Delta w_j\right)_2 = \left(|v|^p|u|^{p-2}u\ln|uv|, w_j\right)_2 - \left(\mu\frac{u}{|x|^4}, w_j\right)_2, \tag{3.16}$$

$$\left(v_t, w_j\right)_2 + \left(\Delta v, \Delta w_j\right)_2 = \left(|u|^p|v|^{p-2}v\ln|uv|, w_j\right)_2 - \left(\gamma\frac{v}{|x|^4}, w_j\right)_2. \tag{3.17}$$

Next, we show that the limit function $(u(x,t), v(x,t))$ obtained in (3.16) and (3.17) is a weak solution of problem (1.1). Now, we can fix a positive integer k, such that $m \geq k$ and choose

$$\begin{cases} \phi(x,t) = \sum\limits_{j=1}^{k} g_j(t)\, w_j(x), \\ \varphi(x,t) = \sum\limits_{j=1}^{k} h_j(t)\, w_j(x). \end{cases}$$

Multiplying (3.16) and (3.17) by $g_j(t)$ and $h_j(t)$, respectively, and summing for j from 1 to k, we obtain

$$(u_t, \phi)_2 + (\Delta u, \Delta\phi)_2 = \left(|v|^p|u|^{p-2}u\ln|uv|, \phi\right)_2 - \left(\mu\frac{u}{|x|^4}, \phi\right)_2,$$

$$(v_t, \varphi)_2 + (\Delta v, \Delta\varphi)_2 = \left(|u|^p|v|^{p-2}v\ln|uv|, \varphi\right)_2 - \left(\gamma\frac{v}{|x|^4}, \varphi\right)_2,$$

for a.e. $t \in [0,+\infty)$ and any $(\phi,\varphi) \in H_0^2(\Omega) \times H_0^2(\Omega)$, and $u(x,0) = u_0, v(x,0) = v_0$.

Step 4. Energy inequality

Next, we will prove that a global weak solution $(u(x,t), v(x,t))$ of problem (1.1) satisfies energy inequality (3.1). To achieve this goal, we introduce a nonnegative function $\theta(t) \in C^1([0,+\infty))$. By (3.7), we have

$$\begin{aligned}&\int_0^{+\infty} \theta dt \int_0^t \left(\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2\right) d\tau + \int_0^{+\infty} J(u_m, v_m)\,\theta dt \\ &= \int_0^{+\infty} J(u_{0m}, v_{0m})\,\theta dt.\end{aligned} \tag{3.18}$$

It follows from (3.5) and (3.6) that $J(u_{0m}, v_{0m}) \to J(u_0, v_0)$ as $m \to +\infty$, and therefore the integral $\int_0^{+\infty} J(u_{0m}, v_{0m})\,\theta dt$ (which is the right-hand side of (3.18)) converges to $\int_0^{+\infty} J(u_0, v_0)\,\theta dt$. Since $\int_0^{+\infty} J(u_m, v_m)\,\theta dt$ is lower semi-continuous with respect to the weak topology of $L^2\left(0,+\infty; H_0^2(\Omega)\right) \times L^2\left(0,+\infty; H_0^2(\Omega)\right)$, we have

$$\int_0^{+\infty} J(u,v)\,\theta dt \leq \lim_{m\to+\infty}\inf \int_0^{+\infty} J(u_m, v_m)\,\theta dt. \tag{3.19}$$

Combining (3.18) and (3.19), we have

$$\int_0^{+\infty} \theta dt \int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau + \int_0^{+\infty} J(u,v)\,\theta dt \le \int_0^{+\infty} J(u_0,v_0)\,\theta dt. \tag{3.20}$$

Since the nonnegative function θ is arbitrary, we deduce from (3.20) that energy inequality (3.1) holds.

Step 5. Algebraic decay

Finally, we will prove an algebraic decay of a global weak solution $(u(x,t), v(x,t))$ of problem (1.1). Combining (2.3), (3.1), and $(u(x,t), v(x,t)) \in W$, we have

$$\begin{aligned}&\frac{p-1}{2p}\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) + \frac{1}{p^2}\|uv\|_p^p + \frac{\mu(p-1)}{2p}\int_\Omega \frac{u^2}{|x|^4}dx + \frac{\gamma(p-1)}{2p}\int_\Omega \frac{v^2}{|x|^4}dx \\ &\le J(u,v) \le J(u_0,v_0).\end{aligned} \tag{3.21}$$

From Remark 2.1, since $I(u,v) > 0$, there exists a $\lambda_* > 1$ such that $I(\lambda_* u, \lambda_* v) = 0$, and

$$\begin{aligned}&{\lambda_*}^{2p}\left(\frac{p-1}{2p}\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) + \frac{1}{p^2}\|uv\|_p^p + \frac{\mu(p-1)}{2p}\int_\Omega \frac{u^2}{|x|^4}dx + \frac{\gamma(p-1)}{2p}\int_\Omega \frac{v^2}{|x|^4}dx\right) \\ &\ge J(\lambda_* u, \lambda_* v) \ge d.\end{aligned} \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$\lambda_* \ge \left(\frac{d}{J(u_0,v_0)}\right)^{\frac{1}{2p}} > 1. \tag{3.23}$$

Because of

$$\begin{aligned}I(\lambda_* u, \lambda_* v) &= {\lambda_*}^2\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) - 4{\lambda_*}^{2p}\ln\lambda_*\|uv\|_p^p - 2{\lambda_*}^{2p}\int_\Omega |uv|^p \ln|uv|\,dx \\ &\quad + {\lambda_*}^2\mu\int_\Omega \frac{u^2}{|x|^4}dx + {\lambda_*}^2\gamma\int_\Omega \frac{v^2}{|x|^4}dx \\ &= \left({\lambda_*}^2 - {\lambda_*}^{2p}\right)\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) - 4{\lambda_*}^{2p}\ln\lambda_*\|uv\|_p^p \\ &\quad + \left({\lambda_*}^2 - {\lambda_*}^{2p}\right)\mu\int_\Omega \frac{u^2}{|x|^4}dx + \left({\lambda_*}^2 - {\lambda_*}^{2p}\right)\gamma\int_\Omega \frac{v^2}{|x|^4}dx + {\lambda_*}^{2p}I(u,v) \\ &= 0,\end{aligned} \tag{3.24}$$

it follows from (3.24) that

$$\begin{aligned}4{\lambda_*}^{2p}\ln\lambda_*\|uv\|_p^p &= \left({\lambda_*}^2 - {\lambda_*}^{2p}\right)\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) + \left({\lambda_*}^2 - {\lambda_*}^{2p}\right)\mu\int_\Omega \frac{u^2}{|x|^4}dx \\ &\quad + \left({\lambda_*}^2 - {\lambda_*}^{2p}\right)\gamma\int_\Omega \frac{v^2}{|x|^4}dx + {\lambda_*}^{2p}I(u,v) \\ &> 0,\end{aligned}$$

which implies

$$I(u,v) > \left(1 - \frac{1}{\lambda_*{}^{2p-2}}\right)\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right). \tag{3.25}$$

Combining (3.23) and (3.25), we get

$$I(u,v) > \left(1 - \left(\frac{J(u_0,v_0)}{d}\right)^{\frac{2p-2}{2p}}\right)\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right). \tag{3.26}$$

By virtue of the embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$, we have from (3.26) that

$$I(u,v) > \left(1 - \left(\frac{J(u_0,v_0)}{d}\right)^{\frac{2p-2}{2p}}\right)\frac{1}{S_1^2}\left(\|u\|_2^2 + \|v\|_2^2\right). \tag{3.27}$$

On the other hand, we get from (3.27) that

$$\begin{aligned}\frac{d}{dt}\left(\|u\|_2^2 + \|v\|_2^2\right) &= -2I(u,v)\\ &< -2\left(1 - \left(\frac{J(u_0,v_0)}{d}\right)^{\frac{2p-2}{2p}}\right)\frac{1}{S_1^2}\left(\|u\|_2^2 + \|v\|_2^2\right).\end{aligned} \tag{3.28}$$

From (3.28) and Gronwall′s inequality, we obtain (3.2) in Theorem 3.1, which describes the algebraic decay of a global weak solution of problem (1.1). By the conditions $0 < J(u_0,v_0) < d, 1 < p < \frac{p^2}{p-1} < \frac{N}{N-4}$, and $S_1 > 0$, we have

$$\lim_{t\to\infty} e^{-2\left(1-\left(\frac{J(u_0,v_0)}{d}\right)^{\frac{2p-2}{2p}}\right)\frac{1}{S_1^2}t} = 0.$$

Therefore, we can deduce the behavior from (3.2), specifically

$$\lim_{t\to\infty}\left(\|u(\cdot,t)\|_2^2 + \|v(\cdot,t)\|_2^2\right) = 0.$$

The proof of Theorem 3.1 is complete. □

4. Blow-up and upper bound estimates of blow-up time

In this section, we prove the blow-up of a weak solution of problem (1.1) in finite time by contradiction. Furthermore, a new upper bound estimate for the blow-up time is obtained by solving a minimization problem.

Theorem 4.1. *Let* $(u_0,v_0) \in H_0^2(\Omega)\times H_0^2(\Omega)$*. If* $0 < J(u_0,v_0) < d$*, and* $I(u_0,v_0) < 0$*, problem* (1.1) *has a weak solution* $(u(x,t), v(x,t))$ *that blows up in finite time, namely, there exists a* $T > 0$ *such that*

$$\lim_{t\to T}\int_0^t\left(\|u\|_2^2 + \|v\|_2^2\right)d\tau = +\infty, \tag{4.1}$$

and an upper bound estimate of the blow-up time T *is obtained by*

$$T \le \frac{(2p-1)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)}{2p(p-1)^2(d - J(u_0,v_0))}. \tag{4.2}$$

Proof. We first demonstrate, through a proof by contradiction, that a weak solution $(u(x,t), v(x,t))$ of problem (1.1) experiences blow-up in finite time. Subsequently, we derive an upper bound estimate for the blow-up time T.

Step 1. Blow-up in finite time

Assume that a weak solution $(u(x,t), v(x,t))$ of problem (1.1) exists globally: $T = +\infty$. Setting

$$G(t) = \int_0^t \left(\|u\|_2^2 + \|v\|_2^2\right) d\tau, \tag{4.3}$$

then by (4.3), we have

$$G'(t) = \|u\|_2^2 + \|v\|_2^2, \tag{4.4}$$

$$G''(t) = \frac{d}{dt}\left(\|u\|_2^2 + \|v\|_2^2\right) = 2(u_t, u) + 2(v_t, v) = -2I(u, v). \tag{4.5}$$

By the energy inequality

$$\int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau + J(u, v) \leq J(u_0, v_0), t \in [0, T],$$

and combining it with (2.3), (4.4), and (4.5), we get

$$\begin{aligned} G''(t) &= -2I(u, v) \\ &= -4pJ(u, v) + 2(p-1)\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) + \frac{4}{p}\|uv\|_p^p \\ &\quad + 2\mu(p-1)\int_\Omega \frac{u^2}{|x|^4}dx + 2\gamma(p-1)\int_\Omega \frac{v^2}{|x|^4}dx \\ &\geq 4p\int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right)d\tau - 4pJ(u_0, v_0) + 2(p-1)\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) \\ &\geq 4p\int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right)d\tau - 4pJ(u_0, v_0) + \frac{2(p-1)}{S_1^2}\left(\|u\|_2^2 + \|v\|_2^2\right) \\ &= 4p\int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right)d\tau - 4pJ(u_0, v_0) + \frac{2(p-1)}{S_1^2}G'(t). \end{aligned} \tag{4.6}$$

It is worth noting that

$$\begin{aligned} &\left(\int_0^t ((u_\tau, u) + (v_\tau, v))\, d\tau\right)^2 \\ &= \left(\int_0^t \frac{1}{2}\frac{d}{d\tau}\left(\|u\|_2^2 + \|v\|_2^2\right) d\tau\right)^2 \\ &= \left(\frac{1}{2}\left(\|u\|_2^2 + \|v\|_2^2 - \|u_0\|_2^2 - \|v_0\|_2^2\right)\right)^2 \\ &= \frac{1}{4}\left(\left(\|u\|_2^2 + \|v\|_2^2\right)^2 + \left(\|u_0\|_2^2 + \|v_0\|_2^2\right)^2 - 2\left(\|u\|_2^2 + \|v\|_2^2\right)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)\right) \\ &= \frac{1}{4}\left((G'(t))^2 + \left(\|u_0\|_2^2 + \|v_0\|_2^2\right)^2 - 2G'(t)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)\right), \end{aligned}$$

which implies

$$\left(G'(t)\right)^2 = 4\left(\int_0^t \left((u_\tau, u) + (v_\tau, v)\right) d\tau\right)^2 + 2G'(t)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right) - \left(\|u_0\|_2^2 + \|v_0\|_2^2\right)^2. \tag{4.7}$$

Combining (4.6) and (4.7), we get

$$\begin{aligned} G(t)G''(t) - p(G'(t))^2 \geq\ & 4p\int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau \int_0^t \left(\|u\|_2^2 + \|v\|_2^2\right) d\tau \\ & - 4p\left(\int_0^t \left((u_\tau, u) + (v_\tau, v)\right) d\tau\right)^2 \\ & + \frac{2(p-1)}{S_1^2} G'(t)G(t) - 4pJ(u_0, v_0)G(t) \\ & - 2pG'(t)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right) + p\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)^2. \end{aligned} \tag{4.8}$$

By virtue of Schwarz′s inequality, we have

$$\begin{aligned} & \int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau \int_0^t \left(\|u\|_2^2 + \|v\|_2^2\right) d\tau - \left(\int_0^t \left((u_\tau, u) + (v_\tau, v)\right) d\tau\right)^2 \\ \geq\ & \int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau \int_0^t \left(\|u\|_2^2 + \|v\|_2^2\right) d\tau \\ & - \left(\int_0^t \left(\|u_\tau\|_2 \|u\|_2 + \|v_\tau\|_2 \|v\|_2\right) d\tau\right)^2 \\ \geq\ & \int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau \int_0^t \left(\|u\|_2^2 + \|v\|_2^2\right) d\tau \\ & - \left(\int_0^t \sqrt{\|u_\tau\|_2^2 + \|v_\tau\|_2^2}\sqrt{\|u\|_2^2 + \|v\|_2^2} d\tau\right)^2 \\ \geq\ & 0. \end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8), and combining with the fact that $\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)^2 \geq 0$, it holds that

$$\begin{aligned} & G(t)G''(t) - p(G'(t))^2 \\ \geq\ & \frac{2(p-1)}{S_1^2} G'(t)G(t) - 4pJ(u_0, v_0)G(t) - 2pG'(t)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right) \\ & + p\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)^2 \\ \geq\ & \frac{2(p-1)}{S_1^2} G'(t)G(t) - 4pJ(u_0, v_0)G(t) - 2pG'(t)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right). \end{aligned} \tag{4.10}$$

We can rewrite (4.10) as

$$\begin{aligned} G(t)G''(t) - p(G'(t))^2 \geq\ & \left(\frac{p-1}{S_1^2} G(t) - 2p\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)\right) G'(t) \\ & + \left(\frac{p-1}{S_1^2} G'(t) - 4pJ(u_0, v_0)\right) G(t). \end{aligned} \tag{4.11}$$

From $J(u_0, v_0) < d, I(u_0, v_0) < 0$, it follows that $(u_0, v_0) \in V$. We can prove $(u, v) \in V$, provided $(u_0, v_0) \in V$, by contradiction. Indeed, by contradiction, if it not hold, we assume that (u, v) leaves V at time t_0; there exists a sequence $\{t_n\}$ such that $I(u(t_n), v(t_n)) \leq 0$ when $t_n \to t_0$. By the lower semicontinuity of $H_0^2(\Omega)$, we obtain

$$I(u(t_0), v(t_0)) \leq \lim_{n\to\infty} \inf I(u(t_n), v(t_n)) \leq 0.$$

Since $(u(t_0), v(t_0)) \notin V$, we have $I(u(t_0), v(t_0)) = 0$. By a similar method, we have $J(u(t_0), v(t_0)) = d$. However, if $I(u(t_0), v(t_0)) = 0$, then by the definition of d, we know that

$$d = \inf_{(u,v)\in\mathbb{N}} J(u(t), v(t)) \leq J(u(t_0), v(t_0)),$$

which contradicts with 3.1. And, if $J(u(t_0), v(t_0)) = d$, it also contradicts with 3.1. So, $(u, v) \in V$ provided $(u_0, v_0) \in V$, that the following energy inequality holds:

$$\int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau + J(u, v) \leq J(u_0, v_0), t \in [0, T].$$

Then, from (4.5) and (2.9) in Lemma 2.5, we have

$$G''(t) = -2I(u, v) > 4p(d - J(u, v)) \geq 4p(d - J(u_0, v_0)) > 0. \tag{4.12}$$

It follows from (4.12) that, for any $t \geq 0$,

$$G'(t) \geq 4p(d - J(u_0, v_0))t + G'(0) \geq 4p(d - J(u_0, v_0))t, \tag{4.13}$$

$$G(t) \geq 2p(d - J(u_0, v_0))t^2 + G(0) = 2p(d - J(u_0, v_0))t^2. \tag{4.14}$$

Combining (4.13) and (4.14), for sufficiently large t, there holds

$$\frac{p-1}{S_1^2} G(t) - 2p\left(\|u_0\|_2^2 + \|v_0\|_2^2\right) > 0, \tag{4.15}$$

$$\frac{p-1}{S_1^2} G'(t) - 4pJ(u_0, v_0) > 0. \tag{4.16}$$

It follows from (4.11)-(4.16), for sufficiently large t, that

$$G(t)G''(t) - p(G'(t))^2 > 0.$$

Since

$$\left(\frac{1}{G^{p-1}(t)}\right)'' = \frac{-(p-1)\left(G''(t)G(t) - p(G'(t))^2\right)}{G^{p+1}(t)} < 0,$$

then there exists a finite time $T > 0$ such that

$$\lim_{t\to T} \frac{1}{G^{p-1}(t)} = \lim_{t\to T} \frac{1}{\left(\int_0^t \left(\|u\|_2^2 + \|v\|_2^2\right) d\tau\right)^{p-1}} = 0,$$

namely $\lim_{t\to T}\left(\int_0^t\left(\|u\|_2^2+\|v\|_2^2\right)d\tau\right)^{p-1}=+\infty$, which implies that (4.1) holds. This contradicts the fact that $(u(x,t),v(x,t))$ is a global weak solution of problem (1.1), hence it blows up in finite time.

Step 2. Upper bound estimate of the blow-up time

We next give an upper bound estimate for the blow-up time T.

For any $T^*\in(0,T)$, we can define a positive auxiliary functional

$$M:[0,T^*]\to\mathbb{R},$$

which is defined by

$$M(t)=\int_0^t\left(\|u\|_2^2+\|v\|_2^2\right)d\tau+(T^*-t)\left(\|u_0\|_2^2+\|v_0\|_2^2\right)+\beta(t+\alpha)^2, \tag{4.17}$$

where $t\in[0,T^*]$, $\beta>0$, and $\alpha>0$ are specified later. Through a direct calculation, we have

$$\begin{aligned}M'(t)&=\|u\|_2^2+\|v\|_2^2-\left(\|u_0\|_2^2+\|v_0\|_2^2\right)+2\beta(t+\alpha)\\&=\int_0^t\frac{d}{d\tau}\left(\|u\|_2^2+\|v\|_2^2\right)d\tau+2\beta(t+\alpha)\\&=2\int_0^t((u_\tau,u)+(v_\tau,v))\,d\tau+2\beta(t+\alpha),\end{aligned}\tag{4.18}$$

$$M''(t)=2(u_t,u)+2(v_t,v)+2\beta=-2I(u,v)+2\beta. \tag{4.19}$$

By virtue of (4.19), (3.1), and (2.9), we have

$$\begin{aligned}M''(t)&\ge 4p(d-J(u,v))+2\beta\\&\ge 4p(d-J(u_0,v_0))+4p\int_0^t\left(\|u_\tau\|_2^2+\|v_\tau\|_2^2\right)d\tau+2\beta.\end{aligned}\tag{4.20}$$

It follows from (4.18) by Schwarz′s inequality and Hölder′s inequality that

$$\begin{aligned}(M'(t))^2&=4\left(\int_0^t((u_\tau,u)+(v_\tau,v))\,d\tau+\beta(t+\alpha)\right)^2\\&\le 4\left(\int_0^t\left(\|u_\tau\|_2\|u\|_2+\|v_\tau\|_2\|v\|_2\right)d\tau+\beta(t+\alpha)\right)^2\\&\le 4\left(\int_0^t\sqrt{\|u_\tau\|_2^2+\|v_\tau\|_2^2}\sqrt{\|u\|_2^2+\|v\|_2^2}\,d\tau+\beta(t+\alpha)\right)^2\\&\le 4\left(\left(\int_0^t\left(\|u_\tau\|_2^2+\|v_\tau\|_2^2\right)d\tau\right)^{\frac12}\left(\int_0^t\left(\|u\|_2^2+\|v\|_2^2\right)d\tau\right)^{\frac12}+\beta(t+\alpha)\right)^2\\&\le 4\left(\int_0^t\left(\|u_\tau\|_2^2+\|v_\tau\|_2^2\right)d\tau+\beta\right)\left(\int_0^t\left(\|u\|_2^2+\|v\|_2^2\right)d\tau+\beta(t+\alpha)^2\right)\\&\le 4M(t)\left(\int_0^t\left(\|u_\tau\|_2^2+\|v_\tau\|_2^2\right)d\tau+\beta\right).\end{aligned}\tag{4.21}$$

Combining (4.17), (4.20), and (4.21), there holds

$$M(t)M''(t)-p\left(M'(t)\right)^2 \geq \left(4p\left(d-J\left(u_0,v_0\right)\right)-2\left(2p-1\right)\beta\right)M(t). \tag{4.22}$$

Satisfying $0<\beta\leq \frac{2p(d-J(u_0,v_0))}{2p-1}$, from (4.22), we can conclude that

$$M(t)M''(t)-p\left(M'(t)\right)^2 \geq 0$$

for any $t\in[0,T^*]$. We define $y(t)=\frac{1}{M^{p-1}(t)}$ for any $t\in[0,T^*]$. By virtue of $M(t)>0, M'(t)>0$, we obtain

$$y'(t)=-\frac{(p-1)M'(t)}{M^p(t)}<0,$$

$$y''(t)=-\frac{(p-1)\left(M''(t)M(t)-p\left(M'(t)\right)^2\right)}{M^{p+1}(t)}<0.$$

Through a direct calculation from $y''(t)<0$, we have

$$y(T^*)-y(0)=y'(\xi)T^*<y'(0)T^*, \xi\in(0,T^*),$$

which implies

$$T^*\leq\frac{y(T^*)}{y'(0)}-\frac{y(0)}{y'(0)}<-\frac{y(0)}{y'(0)}=\frac{T^*\left(\|u_0\|_2^2+\|v_0\|_2^2\right)+\beta\alpha^2}{2\beta\alpha(p-1)}, \tag{4.23}$$

where $y(0)>0$, $y(T^*)>0$, and $y'(0)<0$. Hence, from (4.23), we can deduce that

$$T^*\leq\frac{\beta\alpha^2}{2\beta\alpha(p-1)-\left(\|u_0\|_2^2+\|v_0\|_2^2\right)}, \tag{4.24}$$

where $\alpha>\frac{\|u_0\|_2^2+\|v_0\|_2^2}{2\beta(p-1)}$. In order to obtain an upper bound estimate of the blow-up time, we consider a minimizing problem

$$T^*\leq\min_{(\beta,\alpha)\in\Phi} f(\beta,\alpha), \tag{4.25}$$

where

$$f(\beta,\alpha)=\frac{\beta\alpha^2}{2\beta\alpha(p-1)-\left(\|u_0\|_2^2+\|v_0\|_2^2\right)},$$

$$\Phi=\left\{(\beta,\alpha)\,|\,0<\beta\leq\frac{2p\left(d-J\left(u_0,v_0\right)\right)}{2p-1}, \frac{\|u_0\|_2^2+\|v_0\|_2^2}{2\beta(p-1)}<\alpha<+\infty\right\}.$$

Due to the partial derivative

$$f_\beta(\beta,\alpha)=-\frac{\alpha^2\left(\|u_0\|_2^2+\|v_0\|_2^2\right)}{\left(2\beta\alpha(p-1)-\left(\|u_0\|_2^2+\|v_0\|_2^2\right)\right)^2}<0,$$

so $f(\beta,\alpha)$ is decreasing with respect to β, we can obtain

$$\min_{(\beta,\alpha)\in\Phi} f(\beta,\alpha)=f\left(\frac{2p\left(d-J\left(u_0,v_0\right)\right)}{2p-1},\alpha\right)=g(\alpha), \tag{4.26}$$

where

$$g(\alpha)=\frac{2\alpha^2 p\left(d-J\left(u_0,v_0\right)\right)}{4\alpha p\left(p-1\right)\left(d-J\left(u_0,v_0\right)\right)-\left(2p-1\right)\left(\|u_0\|_2^2+\|v_0\|_2^2\right)}. \tag{4.27}$$

Since $g(\alpha)>0$ when $\frac{(2p-1)\left(\|u_0\|_2^2+\|v_0\|_2^2\right)}{4p(p-1)(d-J(u_0,v_0))}<\alpha<+\infty$, then it follows from (4.27) that $g(\alpha)$ achieves its minimum at $\alpha_1=\frac{(2p-1)\left(\|u_0\|_2^2+\|v_0\|_2^2\right)}{2p(p-1)(d-J(u_0,v_0))}$, and

$$g(\alpha_1)=\frac{(2p-1)\left(\|u_0\|_2^2+\|v_0\|_2^2\right)}{2p(p-1)^2\left(d-J\left(u_0,v_0\right)\right)}. \tag{4.28}$$

Combining (4.25), (4.26), and (4.28), we arrive at

$$T^*\leq\frac{(2p-1)\left(\|u_0\|_2^2+\|v_0\|_2^2\right)}{2p(p-1)^2\left(d-J\left(u_0,v_0\right)\right)}, \tag{4.29}$$

hence we can deduce that (4.2) holds from (4.29) by virtue of the arbitrariness of $T^*\to T$. The proof of Theorem 4.1 is complete. □

Author contributions

Tingfu Feng, Yan Dong, Kelei Zhang and Yan Zhu: Methodology; Tingfu Feng and Yan Dong: Writing-original draft; Yan Dong, Kelei Zhang: Writing-review and editing; Yan Zhu: Writing-review.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are grateful to the referees for their valuable comments, which greatly improved the exposition of our paper. This work is sponsored by the National Natural Science Foundation of China (Grant No.12261053), the Special Basic Cooperative Research Programs of Yunnan Provincial Undergraduate Universities Association (Grant No.202301BA070001-002, 202101BA070001-132), and the Scientific Research Fund of Education Department of Yunnan Province (Grant No.2024Y775, 2024Y776, 2025Y1076).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. F. Rellich, *Halbbeschränkte Differentialoperatoren Höherer Ordnung*, Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, 243–250, 1956.

2. F. Rellich, J. Berkowitz, *Perturbation theory of eigenvalue problems*, Gordon and Breach, New York, 1969.
3. E. B. Davies, A. M. Hinz, Explicit constants for Rellich inequalities in $L_p(\Omega)$, *Math. Z.*, **227** (1998), 511–523. https://doi.org/10.1007/PL00004389
4. P. Caldiroli, R. Musina, Rellich inequalities with weights, *Calc. Var. Partial Differential Equations*, **45** (2012), 147–164. https://doi.org/10.1007/s00526-011-0454-3
5. A. Zangwill, Some causes and a consequence of epitaxial roughening, *J. Cryst. Growth*, **163** (1996), 8–21. https://doi.org/10.1016/0022-0248(95)01048-3
6. M. Ortiz, E. A. Repetto, H. Si, A continuum model of kinetic roughening and coarsening in thin films, *J. Mech. Phys. Solids*, **47** (1999), 697–730. https://doi.org/10.1016/S0022-5096(98)00102-1
7. T. P. Schulze, R. V. Kohn, A geometric model for coarsening during spiral-mode growth of thin films, *Phys. D*, **132** (1999), 520–542. https://doi.org/10.1016/S0167-2789(99)00108-6
8. M. Raible, S. Linz, P. Hanggi, Amorphous thin film growth: minimal deposition equation, *Phys. Rev. E*, **62** (2000), 1691–1694. https://doi.org/10.1103/PhysRevE.62.1691
9. M. Raible, S. G. Mayr, S. J. Linz, M. Moske, P. Hnggi, K. Samwer, Amorphous thin film growth:theory compared with experiment, *Europhys. Lett.*, **50** (2000), 61–67. https://doi.org/10.1209/epl/i2000-00235-7
10. M. Raible, S. Linz, P. Hanggi, Amorphous thin film growth: modeling and pattern formation, *Adv. Solid State Phys.*, **41** (2001), 391–403. https://doi.org/10.1007/3-540-44946-9_32
11. O. Stein, M. Winkler, Amorphous molecular beam epitaxy: global solutions and absorbing sets, *European J. Appl. Math.*, **16** (2005), 767–798. https://doi.org/10.1017/S0956792505006315
12. D. Blömker, C. Nolde, J. Robinson, Rigorous numerical verification of uniqueness and smoothness in a surface growth model, *J. Math. Anal. Appl.*, **4** (2015), 311–325. https://doi.org/10.1016/j.jmaa.2015.04.025
13. J. Yang, Energy conservation for weak solutions of a surface growth model, *J. Differ. Equ.*, **283** (2021), 71–84. https://doi.org/10.1016/j.jde.2021.02.040
14. D. Blömker, C. Gugg, On the existence of solutions for amorphous molecular beam epitaxy, *Nonlinear Anal. Real World Appl.*, **3** (2002), 61–73. https://doi.org/10.1016/S1468-1218(01)00013-X
15. D. Blömker, C. Gugg, M. Raible, Thin-film-growth models: Roughness and correlation functions, *Eur. J. Appl. Math.*, **13** (2002), 385–402. https://doi.org/10.1017/S0956792502004886
16. R. V. Kohn, X. Yan, Upper bound on the coarsening rate for an epitaxial growth model, *Comm. Pure Appl. Math.*, **56** (2003), 1549–1564. https://doi.org/10.1002/cpa.10103
17. B. B. King, O. Stein, M. Winkler, A fourth-order parabolic equation modeling epitaxial thin film growth reaction-diffusion systems, *J. Math. Anal. Appl.*, **286** (2003), 459–490. https://doi.org/10.1016/S0022-247X(03)00474-8
18. M. Winkler, Global solutions in higher dimensions to a fourth-order parabolic equation modeling epitaxial thin-film growth, *Z. Angew. Math. Phys.*, **62** (2011), 575–608. https://doi.org/10.1007/s00033-011-0128-1

19. C. Liu, Regularity of solutions for a fourth order parabolic equation, *Bull. Belg. Math. Soc. Simon Stevin*, **13** (2006), 527–535. https://doi.org/10.36045/bbms/1161350694
20. C. Liu, A fourth order parabolic equation with nonlinear principal part, *Nonlinear Anal.*, **68** (2008), 393–401. https://doi.org/10.1016/j.na.2006.11.005
21. X. Li, C. Melcher, Well-posedness and stability for a class of fourth-order nonlinear parabolic equations, *J. Differ. Equ.*, **391** (2024), 25–56. https://doi.org/10.1016/j.jde.2024.01.038
22. J. Zhao, B. Guo, J. Wang, Global existence and blow-up of weak solutions for a fourth-order parabolic equation with gradient nonlinearity, *Z. Angew. Math. Phys.*, **75** (2024), 12. https://doi.org/10.1007/s00033-023-02148-w
23. Y. Feng, B. Hu, X. Xu, Suppression of epitaxial thin film growth by mixing, *J. Differ. Equ.*, **317** (2022), 561–602. https://doi.org/10.1016/j.jde.2022.02.011
24. L. Agelas, Global regularity of solutions of equation modeling epitaxy thin film growth in $\mathbb{R}^d, d = 1, 2$, *J. Evol. Equ.*, **15** (2015), 89–106. https://doi.org/10.1007/s00028-014-0250-6
25. R. Xu, T. Chen, C. Liu, Y. Ding, Global well-posedness and global attractor of fourth order semilinear parabolic equation, *Math. Methods Appl. Sci.*, **38** (2015), 1515–1529. https://doi.org/10.1002/mma.3165
26. G. T. Dee, W. Van Saarloos, Bistable systems with propagating fronts leading to pattern formation, *Phys. Rev. Lett.*, **60** (1988), 2641–2644. https://doi.org/10.1103/PhysRevLett.60.2641
27. Y. Liu, W. Li, A class of fourth-order nonlinear parabolic equations modeling the epitaxial growth of thin films, *Discrete Contin. Dyn. Syst. Ser. S*, **14** (2021), 4367–4381. https://doi.org/10.3934/dcdss.2021112
28. J. Zhou, Global asymptotical behavior and some new blow-up conditions of solutions to a thin-film equation, *J. Math. Anal. Appl.*, **464** (2018), 1290–1312. https://doi.org/10.1016/j.jmaa.2018.04.058
29. Q. Li, W. Gao, Y. Han, Global existence blow up and extinction for a class of thin-film equation, *Nonlinear Anal.*, **147** (2016), 96–109. https://doi.org/10.1016/j.na.2016.08.021
30. J. D. Barrow, P. Parson, Inflationary models with logarithmic potentials, *Phys. Rev. D*, **52** (1995), 5576–5587. https://doi.org/10.1103/PhysRevD.52.5576
31. S. D. Martino, M. Falanga, C. Godano, G. Lauro, Logarithmic Schrodinger-like equation as a model for magma transport, *Europhys, Lett.*, **63** (2003), 472–475. https://doi.org/10.1209/epl/i2003-00547-6
32. D. A. Brown, H. C. Berg, Temporal stimulation of chemotaxis in escherichia coli, *Proc. Natl. Acad. Sci.*, **71** (1974), 1388–1392. https://doi.org/10.1073/pnas.71.4.1388
33. Y. V. Kalinin, L. Jiang, Y. Tu, M. Wu, Logarithmic sensing in Escherichia coli bacterial chemotaxis, *Biophys. J.*, **96** (2009), 2439–2448. https://doi.org/10.1016/j.bpj.2008.10.027
34. Z. Wang, Mathematics of traveling waves in chemotaxis, *Discrete Contin. Dyn. Syst. Ser. B*, **18** (2013), 601–641. https://doi.org/10.3934/dcdsb.2013.18.601
35. F. Gazzola, T. Weth, Finite time blow-up and global solutions for semilinear parabolic equations with initial data at high energy level, *Differ. Integral Equ.*, **18** (2005), 961–990. https://doi.org/10.57262/die/1356060117

36. H. Chen, P. Luo, G. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity, *J. Math. Anal. Appl.*, **422** (2015), 84–98. https://doi.org/10.1016/j.jmaa.2014.08.030

37. Y. Han, Blow-up at infinity of solutions to a semilinear heat equation with logarithmic nonlinearity, *J. Math. Anal. Appl.*, **474** (2019), 513–517. https://doi.org/10.1016/j.jmaa.2019.01.059

38. Y. Han, W. Gao, X. Shi, A class of thin-film equations with logarithmic nonlinearity (in Chinese), *Sci. Sin. Math.*, **49** (2019), 1765–1778. https://doi.org/10.1360/N012018-00024

39. J. Zhou, Global asymptotical behavior of solutions to a class of fourth order parabolic equation modeling epitaxial growth, *Nonlinear Anal. Real World Appl.*, **48** (2019), 54–70. https://doi.org/10.1016/j.nonrwa.2019.01.001

40. M. Liao, Q. Li, A class of fourth-order parabolic equations with logarithmic nonlinearity, *Taiwanese J. Math.*, **24** (2020), 975–1003. https://doi.org/10.11650/tjm/190801

41. J. Zhou, Behavior of solutions to a fourth-order nonlinear parabolic equation with logarithmic nonlinearity, *Appl. Math. Optim.*, **84** (2021), 191–225. https://doi.org/10.1007/s00245-019-09642-6

42. C. Liu, Y. Ma, H. Tang, Lower bound of blow-up time to a fourth order parabolic equation modeling epitaxial thin film growth, *Appl. Math. Lett.*, **111** (2021), 106609. https://doi.org/10.1016/j.aml.2020.106609

43. H. Ding, J. Zhou, Infinite time blow-up of solutions to a fourth-order nonlinear parabolic equation with logarithmic nonlinearity modeling epitaxial growth, *Mediterr. J. Math.*, **18** (2021), 240. https://doi.org/10.1007/s00009-021-01880-9

44. B. Liu, K. Li, F. Li, Asymptotic estimate of weak solutions in a fourth-order parabolic equation with logarithm, *J. Math. Phys.*, **64** (2023), 011513. https://doi.org/10.1063/5.0088490

45. R. Xu, W. Lian, Y. Niu, Global well-posedness of coupled parabolic systems, *Sci. China Math.*, **63** (2020), 321–356. https://doi.org/10.1007/s11425-017-9280-x

46. L. E. Payne, D. H. Sattinger. Saddle points and instability of nonlinear hyperbolic equations, *Israel. J. Math.*, **22** (1975), 273–303. https://doi.org/10.1007/BF02761595

47. D. Sattinger, On global solution of nonlinear hyperbolic equations, *Arch. Rational Mech. Anal.*, **30** (1968), 148–172. https://doi.org/10.1007/BF00250942

48. J. Han, K. Wang, R. Xu, C. Yang, Global quantitative stability of wave equations with strong and weak dampings, *J. Differ. Equations.*, **390** (2024), 228–344. https://doi.org/10.1016/j.jde.2024.01.033

49. Y. Luo, R. Xu, C. Yang, Global well-posedness for a class of semilinear hyperbolic equations with singular potentials on manifolds with conical singularities, *Calc. Var Partial. Dif.*, **61** (2022), 210. https://doi.org/10.1007/s00526-022-02316-2

50. W. Lian, R. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, *Adv. Nonlinear. Anal.*, **9** (2019), 613–632. https://doi.org/10.1515/anona-2020-0016

51. C. Yang, V. D. Radulescu, R. Xu, M. Zhang, Global well-posedness analysis for the nonlinear extensible beam equations in a class of modified Woinowsky-Krieger models, *Adv. Nonlinear. Stud.*, **22** (2022), 436–468. https://doi.org/10.1515/ans-2022-0024

52. X. Wang, R. Xu. Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation, *Adv. Nonlinear. Anal.*, **10** (2020), 261–288. https://doi.org/10.1515/anona-2020-0141
53. W. Lian, J. Wang, R. Xu, Global existence and blow up of solutions for pseudo-parabolic equation with singular potential, *J. Differ. Equations.*, **269** (2020), 4914–4959. https://doi.org/10.1016/j.jde.2020.03.047
54. L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.*, **97** (1976), 1061–1083. https://doi.org/10.2307/2373688
55. E. Lieb, M. Loss, Analysis: Second Edition, *Graduate Studies in Mathematics*, AMS, 2001. https://doi.org/10.1090/gsm/014
56. M. D. Pino, J. Dolbeault, Nonlinear diffusions and optimal constants in Sobolev type inequalities: asymptotic behaviour of equations involving the p-Laplacian, *C. R. Math. Acad. Sci. Paris*, **334** (2002), 365–370. https://doi.org/10.1016/S1631-073X(02)02225-2
57. C. N. Le, X. T. Le, Global solution and blow-up for a class of p-Laplacian evolution equations with logarithmic nonlinearity, *Acta. Appl. Math.*, **151** (2017), 149–169. https://doi.org/10.1007/s10440-017-0106-5
58. P. Drabek, S. I. Pohozaev, Positive solutions for the p-Laplacian: application of the fibering method, *Proc. Roy. Soc. Edinburgh Sect.*, **127** (1997), 703–726. https://doi.org/10.1017/S0308210500023787
59. J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pur. Appl.*, **146** (1986), 65–96. https://doi.org/10.1007/BF01762360

AIMS Press