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Research article

Conformal-type energy estimates on hyperboloids and the wave-Klein-Gordon model of self-gravitating massive fields

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Abstract: In this article we revisit the global existence result of the wave-Klein-Gordon model of the system of the self-gravitating massive field. Our new observation is that, by applying the conformal energy estimates on hyperboloids, we obtain mildly increasing energy estimate up to the top order for the Klein-Gordon component, which clarify the question on the hierarchy of the energy bounds of the Klein-Gordon component in our previous work. Furthermore, a uniform-in-time energy estimate is established for the wave component up to the top order, as well as a scattering result. These improvements indicate that the partial conformal symmetry of the Einstein-massive scalar system will play an important role in the global analysis.

Keywords: conformal energy estimate; partial conformal symmetry; Einstein-massive field system; wave-Klein-Gordon system; nonlinear stability
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1. Introduction

In this article we revisit the global existence of the following wave-Klein-Gordon model of the Einstein-massive scalar field system:

$$\begin{aligned} -\Box u &= P^{\alpha\beta}\partial_\alpha v\partial_\beta v + Rv^2, \\ -\Box v + c^2 v &= H^{\alpha\beta}u\partial_\alpha\partial_\beta v \end{aligned} \tag{1.1}$$

with initial data

$$u|_{\mathcal{H}_2} = u_0, \quad \partial_t u|_{\mathcal{H}_2} = u_1, \quad v|_{\mathcal{H}_2} = v_0, \quad \partial_t v|_{\mathcal{H}_2} = v_1. \tag{1.2}$$

The coefficients $P^{\alpha\beta}, H^{\alpha\beta}, c$ are constants and $c > 0$. In the present article, we restrict our discussion to the compactly supported case, i.e. we suppose that $u_\ell, v_\ell, \ell = 0, 1$ are compactly supported in $\mathcal{H}_2^* = \mathcal{H}_2 \cap \mathcal{K}$ with $\mathcal{H}_2 = \{(t, x)/t = \sqrt{s^2 + |x|^2}\}, \mathcal{K} = \{r < t - 1\}$ and $\mathcal{H}_2^* = \mathcal{H}_2 \cap \mathcal{K}$.

The system (1.1) was introduced in [20, 29] in order to illustrate the main feature of the Einstein-massive scalar field system written in wave coordinates, which is a key step in solving the nonlinear stability problem of the Minkowski space-time with the presence of a real self-gravitating massive scalar field.

Here we give a brief history on the research of nonlinear stability of Minkowski space-time in general relativity. In the *vacuum case*, the first result belongs to Christodoulou and Klainerman [7] who applied a gauge-invariant method via the Bianchi system satisfied by the Riemann curvature. Later on, Lindblad and Rodnianski gave an alternative approach in [23] with formulation in wave coordinates, which was applied by Y. Choquet-Bruhat in [6] for the first local existence result for Einstein equation. See also Bieri-Zipser [3], Bieri [2], Hintz-Vasy [14] etc. In the case of *massless matter fields*, there are works of Zipser [31], Loizelet [25], Taylor [27], Bigorgne et. al. [4], Kauffman-Lindblad [18], Chen [5] etc. for various matter fields.

Compared with the previous two cases, the case with *massive matter fields* possesses a quite different nature. The most important is the *linearzied conformal scaling invariance*, that is, the linearized Einstein equation or Einstein-massless matter field systems enjoy the conformal scaling invariance, while the linearized Einstein-massive matter field systems do not. This symmetry brings lots of properties among which the conformal energy estimate is one of the most important. To be more precise, let us consider the linearization of (1.1). Let

$$u_\lambda(t, x) = u(\lambda t, \lambda x), \quad \lambda \in \mathbb{R}.$$

It is clear that for the free-linear wave equation,

$$\Box u(t, x) = 0 \;\Rightarrow\; \Box u_\lambda(t, x) = \lambda^2 \Box u(\lambda t, \lambda x) = 0,$$

that is, a solution to the free-linear wave equation still solves the same equation after the scaling transform. However, for the free-linear Klein-Gordon equation (which represents the evolution equation of a massive real scalar field, see in detail later on):

$$-\Box v(t, x) + c^2 v(t, x) = 0 \;\Rightarrow\; -\Box v_\lambda(t, x) + \lambda^2 c^2 v_\lambda(t, x) = 0, \tag{1.3}$$

that is, v_λ does not solve the original Klein-Gordon equation in general case.

In [21], the authors relied on the hyperboloidal foliation in the interior of the light-cone $\mathcal{K} := \{r < t - 1\}$ and established the nonlinear stability result of Minkowski space-time with the presence of a real massive scalar field. This was based on a detailed analysis on the model system (1.1) in [20]. See also [29]. This result was later generalized by Ionescu-Pausader in [15, 16] with Fourier-analytic method on non-restricted initial data sets. For other important contributions on various massive matter fields, we refer to [10, 11, 24]. Apart from the above work with Minkowski background which model the astrophysical events and its gravitational wave, we also refer to [1, 12, 28] for the global nonlinear stability results of the Milne space-time with the presence of a massive scalar field in a cosmological context.

Now we give a more detailed explanation on the formulation of (1.1). Recall the Einstein-massive scalar field system written in wave coordinates:

$$
\begin{aligned}
&g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = F_{\alpha\beta}(g,g;\partial g,\partial g) - 16\pi(\partial_\alpha\phi\partial_\beta\phi + (c^2/2)\phi^2 g_{\alpha\beta}),\\
&g^{\mu\nu}\partial_\mu\partial_\nu\phi - c^2\phi = 0,\\
&\Gamma^\gamma_{\alpha\beta}g^{\alpha\beta} = 0,
\end{aligned}
\tag{1.4}
$$

where g is the unknown metric and ϕ the massive scalar field. F is a quartic form, quadratic on g and quadratic on ∂g. When $\phi \equiv 0$, the above system reduces to the vacuum Einstein equation:

$$
\begin{aligned}
&g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = F_{\alpha\beta}(g,g;\partial g,\partial g),\\
&\Gamma^\gamma_{\alpha\beta}g^{\alpha\beta} = 0,
\end{aligned}
\tag{1.5}
$$

which is the Einstein-vacuum equations $R_{\alpha\beta} = 0$ written in the wave coordinates. Since the term $F_{\alpha\beta}(g,g;\partial g,\partial g)$ is already treated with the aid of the wave gauge condition in [23], one focus on the the scalar-metric interaction term $16\pi(\partial_\alpha\phi\partial_\beta\phi + (c^2/2)\phi^2 g_{\alpha\beta})$ and the metric-scalar interaction term $h^{\mu\nu}\partial_\mu\partial_\nu\phi$, where $h^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$. Then we simplify (1.4) by dropping all terms contained in (1.5), and regard $h^{\mu\nu} \simeq -h_{\mu\nu}$ as a scalar. We thus obtain (1.1) who keeps all analytic difficulties arising from the coupling with a massive scalar field. In [20], we have established the global existence result for small initial data with compact support. The proof relies on the hyperbolidal foliation combined with a hierarchy energy estimate on the Klein-Gordon component. More precisely, in [20] we have obtained

$$
\mathbf{E}^N_c(s,v)^{1/2} \lesssim s^{1/2+\delta}, \quad \mathbf{E}^{N-4}_c(s,v)^{1/2} \lesssim s^\delta,
\tag{1.6}
$$

where $\mathbf{E}^p_c(s,v)$ refers to the $p-$order energy of the Klein-Gordon component v defined in (2.27), (2.28), and N describes the regularity of the initial data. The higher-order energies of v have a $s^{1/2+\delta}$ increasing rate (with $\delta \ll 1/2$), while its lower-order energies only have a mild increasing rate, say, s^δ. This technique also leads to a hierarchy of energy bounds for Klein-Gordon component when we regard the complete Einstein-Klein-Gordon system in [21].

To our opinion, this hierarchy is physically counter-intuitive. As we believe that the Klein-Gordon component describes a massive field, thus its propagation speed (the group speed) should be strictly slower than that of massless fields, i.e. the wave component. This demands that when near the light-cone $\{r = t\}$, the Klein-Gordon component should enjoy a strictly faster pointwise decay rate than that of the wave component, because if not so, it happens that an observer detects the fronts of both massive and massless waves from the same source simultaneously, which should not be the case. However if the higher-order energies do have essential increasing rates as, say, $s^{1/2+\delta}$, this indicates that the sufficient-high order derivatives of the Klein-Gordon component may have a decay rate as t^{-1} (see the Klainerman-Sobolev inequality (2.31)), which is exactly the same to that of the wave component. Despite all this, at that moment we did not know whether this hierarchy is a physical phenomenon, or it is only due to our mathematical technical weakness.

The main new contribution in the present article is to answer the above question. We managed to prove that, at least when the initial data enjoy sufficient decay rates at spatial infinity, for example when they are compactly supported, this hierarchy of energy bounds is only due to the technical weakness. In fact we will show that

$$
\mathbf{E}^N_c(s,v)^{1/2} \lesssim s^\delta.
\tag{1.7}
$$

In this new proof, instead of making estimates on the standard energies, we rely on the conformal energy estimate on the wave component. As we will see, this ingredient not only greatly simplifies the original proof, but also brings a uniform bound on the standard energy. Then a scattering property on the wave component is also established up to the top order. On the other hand, the hierarchy on the energy bounds of the Klein-Gordon component is greatly improved. All energy bounds enjoy a mildly increasing rate up to the top order.

From another perspective, applying conformal energy estimate means that we attempt to make use of the conformal invariance of the system (1.1). Although (1.1) does not enjoy conformal scaling invariance (even in the linearized sens, see (1.3)), the application of conformal energy estimate still brings strictly finer estimates, and permits us to obtain better energy bounds. This can be considered as an application of the "partial" conformal scaling invariance of (1.1), which reveals that even a property of symmetry is disturbed, we can still obtain decay from the related (quasi-)conserved quantities. These new observations will have their follow-up influence in the analysis of the full Einstein-massive scalar field system in our coming work.

Now we state the main result of this article.

Theorem 1.1. *Consider the Cauchy problem* (1.1)–(1.2)*. For any positive integer $N \geq 7$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ and*

$$\|(u_0, v_0)\|_{H^{N+1}(\mathbb{R}^3)} + \|(u_1, v_1)\|_{H^N(\mathbb{R}^3)} < \varepsilon, \tag{1.8}$$

the corresponding local solution extends to time infinity. Furthermore, this global solution satisfies the following properties:
1. Uniform energy bound on the wave component up to the top order:

$$\sum_{\alpha=0}^{3} \|\partial_\alpha \partial^I u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon, \quad |I| \leq N. \tag{1.9}$$

2. Linear scattering property for the wave component up to the top order:

$$\lim_{t\to+\infty} \sum_{\alpha=0}^{3} \|\partial_\alpha(\partial^I u(t,\cdot) - \partial^I u^\star(t,\cdot))\|_{L^2(\mathbb{R}^3)} = 0, \quad |I| \leq N, \tag{1.10}$$

where $u^\star$ is a solution to a free-linear wave equation.
3. Non-hierarchy of the Klein-Gordon energy bounds:

$$\mathbf{E}_c^N(s, v)^{1/2} \leq C\varepsilon s^\delta. \tag{1.11}$$

Here $\mathbf{E}_c^N(s, v)$ represents the top order standard hyperboloidal energy defined later on in (2.28)*, and δ a small constant much smaller than* $1/2$.

Remark 1.2. *The restriction on the support of the initial data is not essential. In fact one can generalize the technique applied later on to the non-compactly supported regime via the Euclidean-hyperboloidal foliation (see for example [22]), together with the conformal energy estimate on Euclidean-hyperboloidal hypersurface (see [9, Section 11]).*

The structure of this article is as follows. In Sections 2 and 3, we recall the technical ingredients in the hyperboloidal framework. Sections 4–6 are devoted to the proof of Theorem 1.1, which is composed by three steps. In the first step we rely on the Klainerman-Sobolev inequalities (2.32), (2.33) and the bootstrap assumption (4.2), and obtain a series of L^2 and L^∞ estimations. In the second step we rely on the linear estimates Propositions 3.1, 3.2 and obtain the sharp decay bounds. Finally, we apply energy estimate Proposition 2.1 and obtain (4.3). The last Section is devoted to the proof of the global properties.

2. Basic facts of the hyperboloidal foliation

2.1. *Geometry of the hyperboloidal foliation*

We are working in the (1+3)-dimensional Minkowski space-time with signature $(-,+,+,+)$ and in the Cartesian coordinates we write $(t,x) = (x^0, x^1, x^2, x^3)$. Let $r^2 := \sum_{i=1}^3 (x^i)^2$ and $\partial_0 = \partial_t$. Throughout, Greek indices describe $0,1,2,3$ and Latin indices describe $1,2,3$, and we use the standard convention of implicit summation over repeated indices, as well as raising and lowering indices with respect to the Minkowski metric $\eta_{\alpha\beta}$ and its inverse denoted by $\eta^{\alpha\beta}$.

In this article we focus on the interior of a light-cone

$$\mathcal{K} := \{(t,x)/ r < t-1\}.$$

In the interior of this cone we recall the hyperboloidal foliation

$$\mathcal{K} = \bigcup_{s>1}^{\infty} \mathcal{H}_s^*$$

where

$$\mathcal{H}_s := \{(t,x)/\ t^2 - r^2 = s^2,\ t > 0\}, \quad \mathcal{H}_s^* := \mathcal{H}_s \cap \mathcal{K}.$$

We also denote by $\mathcal{K}_{[s_0,s_1]} := \{(t,x)/\ s_0^2 \le t^2 - r^2 \le s_1^2,\ r < t-1\}$ the subdomain of $\mathcal{K}$ limited by $\mathcal{H}_{s_0}$ and $\mathcal{H}_{s_1}$.

Within the Euclidean metric of $\mathbb{R}^4 \cong \mathbb{R}^{1+3}$, the normal vector and the volume form of $\mathcal{H}_s$ is written as:

$$\vec{n} = \frac{1}{\sqrt{t^2+r^2}}(t, -x^a), \quad d\sigma = \sqrt{1+(r/t)^2}dx,$$

which leads to

$$\vec{n}dx = (1, -x^a/t). \tag{2.1}$$

2.2. *The semi-hyperboloidal frame*

We recall the semi-hyperboloidal frame:

$$\underline{\partial}_0 := \partial_t, \qquad \underline{\partial}_a := (x_a/t)\partial_t + \partial_a. \tag{2.2}$$

Notice that the vector fields $\underline{\partial}_a$ generates the tangent space of the hyperboloid, therefore the normal vector of hyperboloids with respect to the Minkowski metric can be written as $\underline{\partial}_\perp := (t/s)\partial_t + (x^a/s)\partial_a$.

The relation between the semi-hyperboloidal frame and the natural Cartesian frame can be represented as below: $\underline{\partial}_\alpha = \underline{\Phi}_\alpha^\beta \partial_\beta$, where

$$\underline{\Phi}_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

and its inverse

$$\underline{\Psi}_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

Let $T = T^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$ be a two-tensor. We denote by $\underline{T}^{\alpha\beta}$ its components within the semi-hyperboloidal frame, i.e., $T^{\alpha\beta}\partial_\alpha \otimes \partial_\beta = \underline{T}^{\alpha\beta}\underline{\partial}_\alpha \otimes \underline{\partial}_\beta$. Then one has

$$\underline{T}^{\alpha\beta} = \underline{\Psi}_{\alpha'}^\alpha \underline{\Psi}_{\beta'}^\beta T^{\alpha'\beta'}, \qquad T^{\alpha\beta} = \underline{\Phi}_{\alpha'}^\alpha \underline{\Phi}_{\beta'}^\beta \underline{T}^{\alpha'\beta'}.$$

The corresponding semi-hyperboloidal co-frame can be represented as:

$$\theta^0 := dt - (x_a/t)dx^a, \qquad \theta^a := dx^a. \tag{2.5}$$

In the semi-hyperboloidal frame, the Minkowski metric is written as:

$$\underline{\eta}_{\alpha\beta} = \begin{pmatrix} -1 & -x^1/t & -x^1/t & -x^3/t \\ -x^1/t & 1-(x^1/t)^2 & -x^1x^2/t^2 & -x^1x^3/t^2 \\ -x^2/t & -x^2x^1/t^2 & 1-(x^2/t)^2 & -x^2x^3/t^2 \\ -x^3/t & -x^3x^1/t^2 & -x^3x^2/t^2 & 1-(x^3/t)^2 \end{pmatrix}, \tag{2.6}$$

$$\underline{\eta}^{\alpha\beta} = \begin{pmatrix} -(s/t)^2 & -x^1/t & -x^1/t & -x^3/t \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

2.3. The energy estimates on hyperboloids

In this paper, for any functions u defined in $\mathbb{R}^{1+3}$ or it's subset, we define their integral on the hyperboloids as:

$$\|u\|_{L^1_f(\mathcal{H}_s)} := \int_{\mathcal{H}_s} u dx = \int_{\mathbb{R}^3} u(\sqrt{s^2+r^2}, x) dx. \tag{2.8}$$

We recall the following standard energy:

$$\mathbf{E}_c(s,u) = \int_{\mathcal{H}_s} \Big(|\partial_t u|^2 + \sum_a |\partial_a u|^2 + 2(x^a/t)\partial_t u \partial_a u + c^2u^2 \Big) dx, \tag{2.9}$$

and the following standard energy in a curved space-time ($g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}$):

$$\mathbf{E}_{g,c}(s,u) := \mathbf{E}_c(s,u) - \int_{\mathcal{H}_s} \Big(2h^{\alpha\beta}\partial_t u \partial_\beta u X_\alpha - h^{\alpha\beta}\partial_\alpha u \partial_\beta u \Big) dx \tag{2.10}$$

where

$$X_0 = 1, \qquad X_a = -x^a/t.$$

The hyperboloidal conformal energy is defined as

$$\mathbf{E}_{\text{con}}(s,u) := \int_{\mathcal{H}_s} \Big((Ku+2u)^2 + \sum_a (s\bar{\partial}_a u)^2\Big) dx \tag{2.11}$$

where,

$$Ku = (s\partial_s + 2x^a\bar{\partial}_a)u.$$

They satisfied the following energy estimate(cf. [26, Proposition 2.2], see also [30]).

Proposition 2.1. *1. For any function u defined in $\mathcal{K}_{[s_0,s_1]}$ and vanishes near $\partial\mathcal{K}$, for all $s \in [s_0, s_1]$,*

$$\mathbf{E}_c(s,u)^{1/2} \leq \mathbf{E}_c(2,u)^{1/2} + C\int_{s_0}^{s} \|\Box u\|_{L^2_f(\mathcal{H}_{\bar{s}})} d\bar{s}, \tag{2.12}$$

$$\mathbf{E}_{con}(s,u)^{1/2} \leq \mathbf{E}_{con}(s_0,u)^{1/2} + C\int_{s_0}^{s} \bar{s}\|\Box u\|_{L^2_f(\mathcal{H}_{\bar{s}})} d\bar{s}, \tag{2.13}$$

where $\Box u = \eta^{\alpha\beta}\partial_\alpha\partial_\beta u$.
2. Let $g^{\alpha\beta}$ be a C^1 metric and v be a C^2 function. Both are defined in $\mathcal{K}_{[s_0,s_1]}$. Let

$$-g^{\alpha\beta}\partial_\alpha\partial_\beta v + c^2 v = f. \tag{2.14}$$

Suppose $h^{\alpha\beta} := g^{\alpha\beta} - \eta^{\alpha\beta}$ satisfies the following two conditions:

$$\kappa^{-2}\mathbf{E}_{g,c}(s,v) \leq \mathbf{E}_c(s,v) \leq \kappa^2\mathbf{E}_{g,c}(s,v), \tag{2.15a}$$

$$\int_{\mathcal{H}_s} (s/t)\Big(2\partial_\alpha h^{\alpha\beta}\partial_\beta v\partial_t v \ - \partial_t h^{\alpha\beta}\partial_\alpha v\partial_\beta v\Big)dx \leq M(s)\mathbf{E}_c(s,v)^{1/2}, \tag{2.15b}$$

then

$$\mathbf{E}_c(s,v)^{1/2} \leq \kappa^2\mathbf{E}_c(2,v)^{1/2} + \kappa^2\int_2^s \Big(\|f\|_{L^2_f(\mathcal{H}_{\bar{s}})} + M(s)\Big)d\bar{s}. \tag{2.16}$$

For the conformal energy, we have the following estimate:

Lemma 2.2. *Let u be a function defined in $\mathcal{K}_{[s_0,s_1]}$ and vanishes near the conical boundary $\partial\mathcal{K} = \{r = t-1\}$. Then*

$$\|s(s/t)^2\partial_\alpha u\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)u\|_{L^2_f(\mathcal{H}_s)} \leq C\mathbf{E}_{con}(s,u)^{1/2}. \tag{2.17}$$

Proof. We denote by $u_s(x) := u(\sqrt{s^2+|x|^2}, x)$. Remark that $\partial_a u_s = \underline{\partial}_a u$. Then we apply the classical Hardy's inequality:

$$\|(s/t)u\|_{L^2_f(\mathcal{H}_s)} \leq s\|r^{-1}u\|_{L^2_f(\mathcal{H}_s)} \leq Cs\|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)} \leq C\mathbf{E}_{\text{con}}(s,u)^{1/2}.$$

Once $\|(s/t)u\|_{L^2_f(\mathcal{H}_s)}$ is bounded, we see that $\|(s/t)Ku\|_{L^2_f(\mathcal{H}_s)}$ is bounded. Then $s(s/t)^2\partial_t u$ is bounded. Then by recalling $\underline{\partial}_a = (x_a/t)\partial_t + \partial_a$, we obtain the bounds on $\partial_a u$. □

For the convenience of discussion, we introduce the following energy densities:

$$\mathbf{e}_c[u] := \sum_{\alpha=1}^{3}(s/t)|\partial_\alpha u|^2 + \sum_{a=1}^{3}|\underline{\partial}_a u|^2 + c^2u^2, \tag{2.18}$$

$$\mathbf{e}_{\text{con}}[u] := \sum_{\alpha=0}^{3}|s(s/t)^2\partial_\alpha u|^2 + \sum_{a=1}^{3}|s\underline{\partial}_a u|^2 + |(s/t)u|^2. \tag{2.19}$$

From Lemma 2.2, it is clear that

$$\int_{\mathcal{H}_s}\mathbf{e}_{\text{con}}[u]dx \le C\mathbf{E}_{\text{con}}(s,u).$$

2.4. High-order operators and Sobolev decay estimates

For $a = 1, 2, 3$ we recall the Lorentz boosts:

$$L_a := x^a\partial_t + t\partial_a = x_a\partial_0 - x_0\partial_a. \tag{2.20}$$

For a multi-index $I = (i_n, i_{n-1}, \cdots, i_1)$, we note $\partial^I := \partial_{i_n}\partial_{i_{n-1}}\cdots\partial_{i_1}$. Similarly, we have $L^J = L_{i_n}L_{i_{n-1}}, \cdots L_{i_1}$.

Let Z be a high-order derivative composed by ∂_α, L_a. We denote by ord(Z) the order of the operator, and rank(Z) the number of boosts contained in Z. Given two integers $k \le p$, it is convenient to introduce the notations:

$$\begin{aligned}
|u|_{p,k} &:= \max_{\substack{\text{ord}(Z)\le p\\ \text{rank}(Z)\le k}}|Zu|, & |u|_p &:= \max_{0\le k\le p}|u|_{p,k},\\
|\partial u|_{p,k} &:= \max_{\alpha=0,1,2}|\partial_\alpha u|_{p,k}, & |\partial u|_p &:= \max_{0\le k\le p}|\partial u|_{p,k},\\
|\partial^m u|_{p,k} &:= \max_{|I|=m}|\partial^I u|_{p,k}, & |\partial^m u|_p &:= \max_{0\le k\le p}|\partial^I u|_{p,k},\\
|\underline{\partial} u|_{p,k} &:= \max_{a}\{|\underline{\partial}_a u|_{p,k}\}, & |\underline{\partial} u|_p &:= \max_{0\le k\le p}|\underline{\partial} u|_{p,k},\\
|\partial\underline{\partial} u|_{p,k} &:= \max_{a,\alpha}\{|\underline{\partial}_a\partial_\alpha u|_{p,k}, |\partial_\alpha\underline{\partial}_a u|_{p,k}\}, & |\partial\underline{\partial} u|_p &:= \max_{0\le k\le p}|\partial\underline{\partial} u|_{p,k}.
\end{aligned} \tag{2.21}$$

We recall the following estimates established in [26], which can be easily checked by induction:

$$\begin{aligned}
|u|_{p,k} &\le C\sum_{\substack{|I|=p-k\\ |J|\le k}}|\partial^I L^J u|,\\
|\partial u|_{p,k} &\le C\sum_{\substack{|I|+|J|\le p\\ |J|\le k,\alpha}}|\partial_\alpha\partial^I L^J u|,\\
|(s/t)\partial u|_{p,k} &\le C(s/t)\sum_{\substack{|I|+|J|\le p\\ |J|\le k,\alpha}}|\partial_\alpha\partial^I L^J u|,
\end{aligned} \tag{2.22}$$

$$|\underline{\partial} u|_{p,k} \le C\sum_{\substack{|I|\le p-k,a\\ |J|\le k}}|\underline{\partial}_a\partial^I L^J u| + Ct^{-1}\sum_{\substack{|J|\le k,\alpha\\ 0\le|I|\le p-k-1}}|\partial_\alpha\partial^I L^J u| \le Ct^{-1}|u|_{p+1,k+1}, \tag{2.23}$$

$$|\partial\underline{\partial} u|_{p,k} \le Ct^{-1}|\partial u|_{p+1,k+1}. \tag{2.24}$$

On the other hand, one also introduce the high-order energy densities:

$$\mathbf{e}_c^{p,k}[u] := \sum_{\substack{|I|+|J|\le p\\ |J|\le k}} \mathbf{e}_c[\partial^I L^J u], \quad \mathbf{e}_{\text{con}}^{p,k}[u] := \sum_{\substack{|I|+|J|\le p\\ |J|\le k}} \mathbf{e}_{\text{con}}[\partial^I L^J u], \tag{2.25}$$

$$\mathbf{e}_c^{p}[u] := \sum_{k\le p} \mathbf{e}_c^{p,k}[u], \quad \mathbf{e}_{\text{con}}^{p}[u] := \sum_{k\le p} \mathbf{e}_{\text{con}}^{p,k}[u], \tag{2.26}$$

as well as the high-order energies:

$$\mathbf{E}_c^{p,k}(s,u) := \sum_{\substack{|I|+|J|\le p\\ |J|\le k}} \mathbf{E}_c(s,\partial^I L^J u), \quad \mathbf{E}_{\text{con}}^{p,k}(s,u) := \sum_{\substack{|I|+|J|\le p\\ |J|\le k}} \mathbf{E}_{\text{con}}(s,\partial^I L^J u), \tag{2.27}$$

$$\mathbf{E}_c^{p}(s,u) := \sum_{k\le p} \mathbf{E}_c^{p,k}(s,u), \quad \mathbf{E}_{\text{con}}^{p}(s,u) := \sum_{k\le p} \mathbf{E}_{\text{con}}^{p,k}(s,u). \tag{2.28}$$

Combined with (2.18) and (2.19), one obtains

$$|(s/t)\partial u|_{p,k}^2 + |\underline{\partial} u|_{p,k}^2 + c|u|_{p,k}^2 \le C\mathbf{e}_c^{p,k}[u], \tag{2.29}$$

$$|s(s/t)^2\partial u|_{p,k}^2 + |s\underline{\partial} u|_{p,k}^2 + |(s/t)u|_{p,k}^2 \le C\mathbf{e}_{\text{con}}^{p,k}[u]. \tag{2.30}$$

On the other hand, we recall the following Klainerman-Sobolev type estimates established in [13, Chapter VII].

Proposition 2.3. *Let u be a function defined in $\mathcal{K}_{[s_0,s_1]}$ and vanishes near $\partial\mathcal{K} = \{r = t-1\}$. Then*

$$\sup_{\mathcal{H}_s}\{t^{3/2}|u|\} \le C \sum_{|I|+|J|\le 2} \|\partial^I L^J u\|_{L^2_f(\mathcal{H}_s)}. \tag{2.31}$$

Combine this result together with (2.29) and (2.30), we obtain the pointwise estimates on $\mathcal{H}_s$:

$$t^{3/2}(s/t)|\partial u|_{p,k} + t^{3/2}|\underline{\partial} u|_{p,k} + ct^{3/2}|u|_{p,k} \le C\sqrt{\mathbf{E}_c^{p+2,k+2}(s,u)}, \tag{2.32}$$

$$t^{3/2}s(s/t)^2|\partial u|_{p,k} + t^{3/2}s|\underline{\partial} u|_{p,k} + t^{3/2}(s/t)|u|_{p,k} \le C\sqrt{\mathbf{E}_{\text{con}}^{p+2,k+2}(s,u)}. \tag{2.33}$$

3. The L^∞ estimation of wave equation and Klein-Gordon equation

3.1. Estimates on wave equation

Due to the complexity of the system (1.1), the Klainerman-Sobolev inequalities can not supply sufficient decay. We need more precise estimates which regard the linear structure of the wave and/or Klein-Gordon equations. These are estimates which permit us to obtain the linear decay rate when the energies are not uniformly bounded.

Proposition 3.1 (cf. [20])**.** *Let u be a solution to the following Cauchy Problem:*

$$-\Box u = f, \quad u|_{t=2} = 0, \ \partial_t u|_{t=2} = 0, \tag{3.1}$$

where the source f vanishes outside of $\mathcal{K}$, and there exists a global constant C_f depending on f such that:

$$|f| \leq C_f t^{-2-\nu}(t-r)^{-1+\mu}, \quad 0 < \mu, |\nu| \leq 1/2.$$

Then u satisfies the following estimate:

$$|u(t,x)| \lesssim C_f \begin{cases} \frac{1}{\nu\mu}(t-r)^{\mu-\nu}t^{-1}, & 0 < \nu \leq 1/2, \\ \frac{1}{|\nu|\mu}(t-r)^{\mu}t^{-1-\nu}, & -1/2 \leq \nu < 0. \end{cases} \tag{3.2}$$

3.2. Estimations on Klein-Gordon equation

This estimate was established in [19]. The above version is a special case of Proposition 3.2 of [22]. Let v be a sufficiently regular solution to the following Cauchy problem:

$$\begin{aligned} -g^{\alpha\beta}\partial_\alpha\partial_\beta v + c^2 v &= f, \\ v|_{\mathcal{H}_{s_0}} = v_0, \quad \partial_t v|_{\mathcal{H}_{s_0}} &= v_1, \end{aligned} \tag{3.3}$$

where the initial values v_0, v_1 are prescribed on $\mathcal{H}_{s_0}$, and compactly supported in $\mathcal{H}^*_{s_0} = \mathcal{H}_{s_0} \cap \mathcal{K}$. The metric g is sufficiently regular and $g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}$, where $h^{\alpha\beta}$ vanishes near $\partial\mathcal{K}$. For $(t,x) \in \mathcal{K}$, we denote by

$$\bar{H}_{t,x} = (t/s)^2 \underline{h}^{00}|_{(\lambda t/s, \lambda x/s)}.$$

Then we state the following result:

Proposition 3.2. *Suppose that for all $(t,x) \in \mathcal{K}_{[s_0,s_1]}$ and for all $\lambda_0 \leq \lambda \leq s_1$, one has*

$$|\bar{H}_{t,x}| \leq 1/3, \qquad \int_{\lambda_0}^{s_1} |\bar{H}'_{t,x}(\lambda)| d\lambda \leq C \tag{3.4}$$

with C a universal constant. Then for any $\eta \in \mathbb{R}$, the following estimate holds:

$$\begin{aligned} (s/t)^\eta s^{3/2}\left(|v(t,x)| + (s/t)|\partial v(t,x)|\right) \leq_{\eta,s_0} & (s/t)^\eta s^{1/2}|v|_1(t,x) + \sup_{\mathcal{H}_{s_0}}\left(|v| + |\partial v|\right) \\ & + (s/t)^\eta \int_{\lambda_0}^{s} \lambda^{3/2}|f + \mathbb{R}_g[v]|_{(\lambda t/s, \lambda x/s)} d\lambda, \end{aligned} \tag{3.5}$$

in which

$$\lambda_0 = \begin{cases} s_0, & 0 \leq r/t \leq \frac{s_0^2-1}{s_0^2+1}, \\ \sqrt{\frac{t+r}{t-r}}, & \frac{s_0^2-1}{s_0^2+1} \leq r/t < 1 \end{cases} \tag{3.6}$$

and

$$|\mathbb{R}_g[v]| \lesssim s^{-2}|v|_2 + (t/s)^2|\underline{h}^{00}|(s^{-2}|v|_2 + t^{-1}|\partial v|_1) + t^{-1}|h||\partial v|_1. \tag{3.7}$$

Here "$\leq_{\eta,s_0}$" means smaller or equal to up to a constant determined by (η, s_0).

4. Global existence: direct estimates

4.1. The bootstrap argument

From this section we are going to prove Theorem 1.1. Our proof relies on the standard bootstrap argument which is based on the following two observations:

1. The local solution to (1.1) can not approaches its maximal time of existence s^* with bounded energy (of sufficiently high-order). Because if not so, one may apply local existence theory and construct a local solution to (1.1) form $(s^* - \varepsilon)$ with initial data equal to the restriction of the local solution at the time $(s^* - \varepsilon)$. This construction permits us to extend the local solution to $s^* - \varepsilon + \delta$ where δ is determined by the system itself and the high-order energy bounds (which is independent of ε). When $\varepsilon < \delta$, one eventually extends the local solution out of s^* which contradicts the fact that s^* being the maximal time of existence.
2. The high-order energies are continuous with respect to the time variable, whenever the local solution exists. This is also a direct result of local existence theory.

Based on the above observations, and suppose that the local solution (u, v) to (1.1) satisfies a set of high-order energy bounds on an arbitrary time interval $[s_0, s_1]$ (contained in the maximal interval of existence). If we can show that the same set of energies satisfies strictly stronger bounds on the same interval, then one concludes that this local solution extends to time infinity. To see this, suppose that

$$\mathbf{E}^N_{\mathrm{con}}(2, u)^{1/2} \leq C_0\varepsilon, \quad \mathbf{E}^N_c(2, v)^{1/2} \leq C_0\varepsilon. \tag{4.1}$$

Let $[2, s_1]$ be the maximal time interval in which the following energy estimate holds:

$$\begin{aligned} &\mathbf{E}^N_{\mathrm{con}}(s, u)^{1/2} \leq C_1\varepsilon s^{1/2+\delta}, \\ &\mathbf{E}^N_c(s, v)^{1/2} \leq C_1\varepsilon s^{\delta} \end{aligned} \tag{4.2}$$

where $C_1 > C_0$ is sufficiently large, and $\delta > 0$ will be determined later. Then by continuity, when $s = s_1$, at least one of (4.2) becomes equality. However, if we can show that (based on (4.2))

$$\begin{aligned} &\mathbf{E}^N_{\mathrm{con}}(s, u)^{1/2} \leq (1/2)C_1\varepsilon s^{1/2+\delta}, \\ &\mathbf{E}^N_c(s, v)^{1/2} \leq (1/2)C_1\varepsilon s^{\delta}. \end{aligned} \tag{4.3}$$

Then we conclude that (4.2) holds on $[2, s^*)$ where s^* is maximal time of existence. However this is impossible when $s^* < \infty$ due to the first observation. We thus obtain the desired global-in-time existence. Therefore we need to establish the following result:

Proposition 4.1. *Let* $N \geq 7$ *and* $0 < \delta < 1/10$. *Suppose that* (4.2) *holds on* $[2, s_1]$. *Then for* $C_1 > 2C_0$ *and* ε *sufficiently small,* (4.3) *holds on the same time interval.*

The rest of this article is mainly devoted to the proof of the above Proposition. In the following discussion, we apply the expression $A \lesssim B$ for a inequality $A \leq CB$ with C a constant determined by δ, N and the system (1.1).

4.2. Direct L^2 and pointwise estimates

Based on (2.29) and (2.30) together with (4.2), we have the following L^2 estimates:

$$\|(s/t)|u|_N\|_{L^2_f(\mathcal{H}_s)} + \|s|\underline{\partial} u|_N\|_{L^2_f(\mathcal{H}_s)} + \|s(s/t)^2|\partial u|_N\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1\varepsilon s^{1/2+\delta}, \tag{4.4}$$

$$\|c|v|_N\|_{L^2_f(\mathcal{H}_s)} + \|(s/t)|\partial v|_N\|_{L^2_f(\mathcal{H}_s)} + \||\underline{\partial} u|_N\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1 \varepsilon s^\delta. \tag{4.5}$$

Recalling (2.32), (2.33) and (4.2), one has

$$(s/t)|u|_{N-2} + s|\underline{\partial} u|_{N-2} + s(s/t)^2|\partial u|_{N-2} \lesssim C_1 \varepsilon t^{-3/2} s^{1/2+\delta}, \tag{4.6}$$

$$c|v|_{N-2} + (s/t)|\partial v|_{N-2} + |\underline{\partial} v|_{N-2} \lesssim C_1 \varepsilon t^{-3/2} s^\delta. \tag{4.7}$$

4.3. The uniform standard energy estimations on the wave component

We recall Proposition 2.1. For the wave equation, we apply (2.12) with $c = 0$:

$$\mathbf{E}^N(s,u)^{1/2} \le \mathbf{E}^N(2,u)^{1/2} + C\int_2^s \||\Box u|_N\|_{L^2_f(\mathcal{H}_s)} ds. \tag{4.8}$$

Then we remark that, provided that $N \ge 5$,

$$\||\partial_\alpha v \partial_\beta v|_N\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1 \varepsilon s^\delta \|t^{-3/2}|\partial v|_N\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^2 s^{-3/2+2\delta},$$

$$\||v^2|_N\|_{L^2_f(\mathcal{H}_s)} \lesssim C_1 \varepsilon s^\delta \|t^{-3/2}|v|_N\|_{L^2_f(\mathcal{H}_s)} \lesssim (C_1\varepsilon)^2 s^{-3/2+2\delta}.$$

Substitute these bounds into (4.8), we obtain, provided that $C_0 \le C_1/2$ and $C_1\varepsilon$ sufficiently small,

$$\mathbf{E}^N(s,u)^{1/2} \le C_0\varepsilon + C(C_1\varepsilon)^2 \lesssim C_1\varepsilon. \tag{4.9}$$

This uniform energy bound, combined with (2.32), leads us to the following pointwise estimates:

$$(s/t)|\partial u|_{N-2} + |\underline{\partial} u|_{N-2} \lesssim C_1 \varepsilon t^{-3/2}. \tag{4.10}$$

On the other hand, we remark that for $\mathrm{ord}(Z) \le N-3$,

$$\begin{aligned} |\partial_r(Z\underline{\partial}_a u)| =& |\partial_r Z(t^{-1}L_a u)| \lesssim t^{-1} \sum_{\mathrm{ord}(Z')\le N-2} |\partial Z' u| \\ \lesssim& C_1\varepsilon t^{-2}(t-r)^{-1/2}. \end{aligned}$$

Here the first inequality is due to the homogeneity of ∂_α and L_a, and can be checked by induction. Integrate the above bound from $\partial\mathcal{K} = \{r = t-1\}$ to a point (t,x) along the radial direction on a times constant hyperplane, one obtains

$$|\underline{\partial} u|_{N-3} \lesssim C_1\varepsilon t^{-2}(t-r)^{1/2} \lesssim C_1 \varepsilon t^{-5/2} s. \tag{4.11}$$

In the same manner, we can integrate the estimate

$$|\partial_r Zu| \lesssim C_1 \varepsilon t^{-1}(t-r)^{-1/2}$$

for $\mathrm{ord}(Z) \le N-2$ and obtain

$$|u|_{N-2} \lesssim C_1\varepsilon t^{-1}(t-r)^{1/2} \lesssim C_1\varepsilon t^{-3/2} s. \tag{4.12}$$

5. The sharp decay estimates

5.1. Linear estimates on wave and Klein-Gordon equations

For the convenience of discussion, we introduce the following notations:

$$\begin{aligned}\mathbf{A}_k(s) &:= \sup_{\mathcal{K}_{[2,s]}}\{(s/t)^{-2}s^{3/2}(|v|_{N-4,k} + (s/t)|v|_{N-4,k})\},\\ \mathbf{B}_k(s) &:= \sup_{\mathcal{K}_{[2,s]}}\{t|u|_k\}.\end{aligned} \tag{5.1}$$

We will apply Proposition 3.2 on the Klein-Gordon equation. We firstly remark that, following the notations therein and apply (4.10), (4.11) and (4.12):

$$\begin{aligned}|\bar{H}_{t,x}| &\lesssim (s/t)^{-2}|u| \lesssim C_1\varepsilon(t/s)^{1/2}s^{-1/2} \lesssim C_1\varepsilon,\\ |\bar{H}'_{t,x}| &\lesssim (s/t)^{-2}\big((s/t)|\partial u| + (t/s)|\underline{\partial} u|\big) \lesssim C_1\varepsilon(s/t)^{-1/2}s^{-3/2},\end{aligned}$$

which guarantees (3.4). Here we have applied the fact that $t \le s^2$ in $\mathcal{K}$ and the fact that $(t/s) \lesssim \lambda_0$ for all $(t,x) \in \mathcal{K}$.

On the other hand, we remark that for $\mathrm{ord}(Z) \le N-4$, thanks to (4.7) and (4.10), (4.11), (4.12)

$$\mathbb{R}_g[Zv] \lesssim C_1\varepsilon(s/t)^2 s^{-3+\delta}.$$

Finally, we establish the following estimate on commutator.

Lemma 5.1. *Let* $ord(Z) = p \le N-4$ *and* $rank(Z) = k$*, then*

$$\begin{aligned}\big|[Z, H^{\alpha\beta}u\partial_\alpha\partial_\beta]v\big| \lesssim& (s/t)^3 s^{-5/2}\Big(\mathbf{B}_0(s)\mathbf{A}_{k-1}(s) + \sum_{k_1=1}^{k}\mathbf{B}_{k_1}(s)\mathbf{A}_{k-k_1}(s)\Big)\\ &+ (C_1\varepsilon)^2(s/t)^2 s^{-3+\delta}.\end{aligned} \tag{5.2}$$

Especially when $k = 0$*, the first term does not exist.*

Proof. Form (B.2) of [26], Z can be written as a finite linear combination of $\partial^I L^J$ with constant coefficients. Here $|I| = p-k$ and $|J| \le k$. This can be checked by the following commutation relation:

$$[\partial^I, L^J] = \sum_{\substack{|I'|=|I|\\|J'|<|J|}} \Gamma_{I'J'}^{IJ}\partial^{I'}L^{J'} \tag{5.3}$$

where $\Gamma_{I'J'}^{IJ}$ are constants determined by I, J. Then we only need to focus on $[\partial^I L^J, u\partial_\alpha\partial_\beta]v$. For this term, we remark that

$$\begin{aligned}\big|[\partial^I L^J, u\partial_\alpha\partial_\beta]v\big| \lesssim& \sum_{\substack{|I_1|+|I_2|=|I|, |I_1|\ge 1\\|J_1|+|J_2|=|J|}} |\partial^{I_1}L^{J_1}u||\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v|\\ &+ \sum_{\substack{|J_1|+|J_2|=|J|\\|J_1|\ge 1}} |L^{J_1}u||\partial^I L^{J_2}\partial_\alpha\partial_\beta v| + |u|\big|[\partial^I L^J, \partial_\alpha\partial_\beta]v\big|.\end{aligned} \tag{5.4}$$

Here we remark that the last two terms do not exist when $k = 0$. For the first term, we remark that, thanks to (4.10) and (4.7),

$$\begin{aligned}|\partial^{I_1} L^{J_1} u||\partial^{I_2} L^{J_2}\partial_\alpha\partial_\beta v| \lesssim&(C_1\varepsilon)(s/t)^{1/2}s^{-3/2}\,(C_1\varepsilon)t^{-3/2}s^{\delta}\\ \lesssim&(C_1\varepsilon)^2(s/t)^2 s^{-3+\delta}.\end{aligned}$$

For the second term, we remark that in this case $|J_2| = |J| - |J_1|$. Thus

$$|L^{J_1}u||\partial^I L^{J_2}\partial_\alpha\partial_\beta v| \lesssim (s/t)^3 s^{-5/2}\sum_{k_1=1}^{k}\mathbf{B}_{k_1}(s)\mathbf{A}_{k-k_1}(s).$$

For the last term, we remark that

$$\left|[\partial^I L^J, \partial_\alpha\partial_\beta]v\right| \lesssim |\partial v|_{p,k-1},$$

which can be obtained from (5.3). Then

$$|u|\left|[\partial^I L^J, \partial_\alpha\partial_\beta]v\right| \lesssim (s/t)^3 s^{-5/2}\mathbf{B}_0(s)\mathbf{A}_{k-1}(s).$$

□

Now we are ready to establish the estimate on **A** and **B**. We apply Proposition 3.2 on

$$-g^{\alpha\beta}\partial_\alpha\partial_\beta Zv + c^2 Zv = [Z, h^{\alpha\beta}\partial_\alpha\partial_\beta]v$$

where $g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta} = \eta^{\alpha\beta} + H^{\alpha\beta}u$ and $\mathrm{ord}(Z) \le N - 4$. Then by (3.5) and take $\eta = -2$,

$$\begin{aligned}&(s/t)^{-2}s^{3/2}(|Zv|(t,x) + (s/t)|\partial Zv|(t,x))\\ \lesssim&(s/t)^{-2}s^{1/2}|Zv|_1 + \sup_{\mathcal{H}_2}(|Zv| + |\partial Zv|)\\ &+ (s/t)^{-2}\int_{\lambda_0}^{s}\lambda^{3/2}|\mathbb{R}_g[Zv] + [Z, h^{\alpha\beta}\partial_\alpha\partial_\beta]v|_{(\lambda t/s,\lambda x/s)}d\lambda\\ \lesssim&C_1\varepsilon + (s/t)\int_{\lambda_0}^{s}\lambda^{-1}\mathbf{B}_0(\lambda)\mathbf{A}_{k-1}(\lambda)d\lambda + (s/t)\sum_{k_1=1}^{k}\int_{\lambda_0}^{s}\lambda^{-1}\mathbf{B}_{k_1}(\lambda)\mathbf{A}_{k-k_1}(\lambda)d\lambda.\end{aligned}$$

When $k = 0$, the last two terms do not exist. Recall the definition of $|\cdot|_{p,k}$ and $\mathbf{A}_k(s)$, we obtain

$$\mathbf{A}_k(s) \lesssim C_1\varepsilon + \int_2^s \lambda^{-1}\mathbf{B}_0(\lambda)\mathbf{A}_{k-1}(\lambda)d\lambda + \sum_{k=1}^{k}\int_2^s\lambda^{-1}\mathbf{B}_{k_1}(\lambda)\mathbf{A}_{k-k_1}(\lambda)d\lambda \tag{5.5}$$

where the last two terms do not exist when $k = 0$.

Then we turn to the wave equation:

$$-\Box Zu = Z(P^{\alpha\beta}\partial_\alpha v\partial_\beta v + Rv^2) \tag{5.6}$$

for $\mathrm{ord}(Z) \le N - 4$. We remark that $Zu|_{\mathcal{H}_2}$ and $\partial_t Zu|_{\mathcal{H}_2}$ are compactly supported in $\mathcal{H}_2^*$. Then Zu can be decomposed as $Zu = w_i + w_s$ with

$$-\Box w_s = Z(P^{\alpha\beta}\partial_\alpha v\partial_\beta v + Rv^2),\quad w_s|_{\mathcal{H}_2} = \partial_t w_s|_{\mathcal{H}_2} = 0,$$

$$-\Box w_i = 0, \quad w_i|_{\mathcal{H}_2} = Zu|_{\mathcal{H}_2}, \quad \partial_t w_i = \partial_t Zu|_{\mathcal{H}_2}.$$

For w_i we know that it is the solution to the above free linear wave equation with compactly supported initial data. Thus

$$|w_i| \lesssim C_1 \varepsilon t^{-1}. \tag{5.7}$$

For w_s, we apply Proposition 3.1. For this purpose we need to bound the right-hand-side of (5.6). Recall the definition of $\mathbf{A}$, one has, for $k \leq N-4$

$$|\partial_\alpha v \partial_\beta v|_k + |v^2|_k \lesssim t^{-2-1/2}(t-r)^{-1+1/2} \sum_{k_1=0}^{k} \mathbf{A}_{k_1}(s)\mathbf{A}_{k-k_1}(s), \quad \text{in } \mathcal{K}_{[2,s]}. \tag{5.8}$$

Then by Proposition 3.1 with $\mu = \nu = 1/2$, we obtain

$$|w_s| \lesssim t^{-1} \sum_{k_1=0}^{k} \mathbf{A}_{k_1}(s)\mathbf{A}_{k-k_1}(s), \quad \text{in } \mathcal{K}_{[2,s]}.$$

We thus obtain

$$\mathbf{B}_k(s) \lesssim C_1\varepsilon + \sum_{k_1=0}^{k} \mathbf{A}_{k_1}(s)\mathbf{A}_{k-k_1}(s). \tag{5.9}$$

For the case $k = 0$, we recall (5.5) together with (5.9), and obtain

$$\mathbf{A}_0(s) + \mathbf{B}_0(s) \lesssim C_1\varepsilon. \tag{5.10}$$

5.2. *Conclusion by induction*

Substituting (5.10) into (5.5) and (5.9) we obtain the following system of integral inequalities:

$$\begin{aligned}
\mathbf{A}_k(s) \leq & CC_1\varepsilon + CC_1\varepsilon \int_2^s \lambda^{-1}\mathbf{A}_{k-1}(\lambda)d\lambda + CC_1\varepsilon \int_2^s \lambda^{-1}\mathbf{B}_k(\lambda)d\lambda \\
& + C\sum_{k=1}^{k-1} \int_2^s \lambda^{-1}\mathbf{B}_{k_1}(\lambda)\mathbf{A}_{k-k_1}(\lambda)d\lambda, \\
\mathbf{B}_k(s) \leq & CC_1\varepsilon + CC_1\varepsilon\mathbf{A}_k(s) + C\sum_{k_1=1}^{k-1} \mathbf{A}_{k_1}(s)\mathbf{A}_{k-k_1}(s)
\end{aligned} \tag{5.11}$$

where C is a constant determined by N, δ. Then by induction, one obtains

$$\mathbf{A}_k(s) + \mathbf{B}_k(s) \leq CC_1\varepsilon s^{CC_1\varepsilon}, \quad k = 1, 2\cdots, N-4. \tag{5.12}$$

We thus conclude

$$|u|_k \lesssim \begin{cases} C_1\varepsilon t^{-1}, & k = 0, \\ C_1\varepsilon t^{-1} s^{CC_1\varepsilon}, & 1 \leq k \leq N-4, \end{cases} \tag{5.13}$$

$$|v|_{N-4,k} + (s/t)|\partial v|_{N-4,k} \lesssim \begin{cases} C_1\varepsilon (s/t)^2 s^{-3/2}, & k = 0, \\ C_1\varepsilon (s/t)^2 s^{-3/2+CC_1\varepsilon}, & 1 \leq k \leq N-4. \end{cases} \tag{5.14}$$

6. Improved energy estimate and conclusion

In this section we apply the energy estimates Proposition 2.1. For the wave component, we need to establish the following result:

$$\begin{aligned}&\||\partial_\alpha v\partial_\beta v|_{N,k}\|_{L^2_f(\mathcal{H}_s)} + \||v^2|_{N,k}\|_{L^2_f(\mathcal{H}_s)}\\ \lesssim& C_1\varepsilon s^{-3/2}\mathbf{E}_c^{N,k}(s,v)^{1/2} + C_1\varepsilon s^{-3/2+CC_1\varepsilon}\mathbf{E}_c^{N,k-1}(s,v)^{1/2}.\end{aligned} \tag{6.1}$$

This can be checked directly. We only need to remark that

$$|\partial_\alpha v\partial_\beta v|_{N,k} \lesssim |\partial v||\partial v|_{N,k} + \sum_{k_1=1}^{k-1}|\partial v|_{N,k_1}|\partial v|_{N,k-k_1}$$

and then apply (5.14) (under the condition $N \geq 7$). The estimate of v^2 is even easier and we omit the detail. Substitute (6.1) into (2.14), and obtain

$$\begin{aligned}\mathbf{E}_{\mathrm{con}}^{N,k}(s,u)^{1/2} \leq& C_0\varepsilon + CC_1\varepsilon\int_2^s \tau^{-1/2}\mathbf{E}_c^{N,k}(\tau,v)^{1/2}d\tau\\ &+ CC_1\varepsilon\int_2^s\tau^{-1/2+CC_1\varepsilon}\mathbf{E}_c^{N,k-1}(\tau,v)^{1/2}d\tau.\end{aligned} \tag{6.2}$$

For the Klein-Gordon equation, we need to apply Proposition 2.1 (case 2) on

$$-g^{\alpha\beta}Z\partial_\alpha\partial_\beta v + c^2 Zv = [Z, H^{\alpha\beta}u\partial_\alpha\partial_\beta]v. \tag{6.3}$$

In order to check (2.15a), recall (2.10) and (5.13) (case $k = 0$) and remark that $t^{-1} \lesssim (s/t)^2$ in $\mathcal{K}$. Then when $C_1\varepsilon$ sufficiently small, (2.15a) is checked. For (2.15b), we apply directly (4.10) and obtain

$$M(s) \lesssim C_1\varepsilon t^{1/2}s^{-2}\mathbf{E}_c(s,Zv)^{1/2} \lesssim C_1\varepsilon s^{-1}\mathbf{E}_c^{p,k}(s,v)^{1/2}. \tag{6.4}$$

Finally we need to bound the commutator $[Z, H^{\alpha\beta}u\partial_\alpha\partial_\beta]v$. For this purpose, we recall (5.4). The first term in right-hand side of (5.4) is bounded as, thanks to (4.10) and (4.7),

$$|\partial^{I_1}L^{J_1}u||\partial^{I_2}L^{J_2}\partial_\alpha\partial_\beta v| \lesssim \begin{cases}CC_1\varepsilon(s/t)^{1/2}s^{-3/2}|\partial v|_{p,k}, & |I_1|+|J_1| \leq N-1,\\ CC_1\varepsilon t^{-3/2}s^{\delta}|\partial u|_{N-1}, & |I_2|+|J_2|+2 \leq N-2,\end{cases} \tag{6.5}$$

provided that $N \geq 5$. For the second term, we recall (5.13) and (5.14) and suppose that $N \geq 7$,

$$|L^{J_1}u||\partial^I L^{J_2}v| \lesssim \begin{cases}C_1\varepsilon s^{-1+CC_1\varepsilon}(s/t)|\partial v|_{p,k-1}, & |J_1| \leq N-4,\\ C_1\varepsilon s^{-3/2}(s/t)|u|_k, & |J_1| = k,\ |I| \leq N-4,\\ C_1\varepsilon s^{-3/2+CC_1\varepsilon}(s/t)|u|_{k-1}, & |J_1| < k,\ |I|+|J_2| \leq N-4.\end{cases} \tag{6.6}$$

The last term is easier:

$$|u|\left|[\partial^I L^J, \partial_\alpha\partial_\beta]v\right| \lesssim C_1\varepsilon s^{-1}(s/t)|\partial v|_{p,k-1}. \tag{6.7}$$

Now we sum up (6.5) – (6.7), and obtain

$$\begin{aligned}\left\|[Z, H^{\alpha\beta}u\partial_\alpha\partial_\beta]v\right\|_{L^2_f(\mathcal{H}_s)} \lesssim&(C_1\varepsilon)^2 s^{-3/2+2\delta}\\ &+ C_1\varepsilon s^{-1}\mathbf{E}_c^{p,k}(s,v)^{1/2} + (C_1\varepsilon)s^{-3/2}\mathbf{E}_{\rm con}^{p,k}(s,u)^{1/2}\\ &+ C_1\varepsilon s^{-1+CC_1\varepsilon}\mathbf{E}_c^{p,k-1}(s,v)^{1/2}\\ &+ C_1\varepsilon s^{-3/2+CC_1\varepsilon}\mathbf{E}_{\rm con}^{p,k-1}(s,u)^{1/2}.\end{aligned} \tag{6.8}$$

Now we substitute (6.8) and (6.4) into (2.16), and obtain (fix $p = N$)

$$\begin{aligned}\mathbf{E}_c^{N,k}(s,v) \le& C_0\varepsilon + C(C_1\varepsilon)^2 + CC_1\varepsilon\int_2^s \tau^{-1}\mathbf{E}_c^{N,k}(\tau,v)^{1/2}d\tau\\ &+ CC_1\varepsilon\int_2^s \tau^{-3/2}\mathbf{E}_{\rm con}^{N,k}(\tau,u)^{1/2}d\tau\\ &+ CC_1\varepsilon\int_2^s \tau^{-1+CC_1\varepsilon}\mathbf{E}_c^{N,k-1}(\tau,v)^{1/2}d\tau\\ &+ CC_1\varepsilon\int_2^s \tau^{-3/2+CC_1\varepsilon}\mathbf{E}_{\rm con}^{N,k-1}(\tau,u)^{1/2}d\tau.\end{aligned} \tag{6.9}$$

The integral inequalities (6.2) together with (6.9) forms a system. By Cornwall's inequality and induction, we obtain

$$s^{-1/2}\mathbf{E}_{\rm con}^{N,k}(s,u)^{2/3} + \mathbf{E}_c^{N,k}(s,v)^{1/2} \le C_0\varepsilon + C(C_1\varepsilon)^{3/2}s^{C(C_1\varepsilon)^{1/2}} \tag{6.10}$$

provided that $C_1 > C_0$. Now if we take

$$C(C_1\varepsilon)^{1/2} \le \delta, \quad C_1 > 2C_0, \quad \varepsilon < \frac{(C_1-2C_0)^2}{C^2C_1^3}, \tag{6.11}$$

then (6.10) leads to (4.3). Then Proposition 4.1 is established.

7. The uniform energy bounds on hyperplanes and scattering result

In this section we discuss the asymptotic properties of the wave component. We denote by

$$\mathcal{D}_s = \{(t,x)/s \le t \le \sqrt{s^2+r^2}\}.$$

We will make energy estimates of the wave equation in this domain, which permit us to control the standard energies on hyperplanes by the energies on hyperboloids. More precisely,

Proposition 7.1. *Let u be a C^2 function defined in $\mathcal{K}_{[2,s_1]}$. Then for any $5/2 \le s \le s_1$*

$$\|u(s,\cdot)\|_E \le \mathbf{E}(s,u) + \int_{\mathcal{D}_s} |\partial_t u\Box u|dxdt, \tag{7.1}$$

where

$$\|u(t,\cdot)\|_E := \int_{\mathbb{R}^3} |\partial_t u(t,x)|^2 + \sum_{a=1}^3 |\partial_a u(t,x)|^2 dx$$

is the standard energy on a hyperplane.

Proof. We only need to integrate the identity

$$-2\partial_t u\Box u = \partial_t\Big(|\partial_t u|^2 + \sum_{a=1}^{3}|\partial_a u|^2\Big) - 2\sum_{a=1}^{3}\partial_a(\partial_t u\partial_a u)$$

in $\mathcal{D}_s$. □

Then we remark that for $\mathrm{ord}(Z) \le N$,

$$\begin{aligned}\int_{\mathcal{D}_s}|\partial_t Zu\Box Zu|dxdt &\le \int_{\mathcal{K}[\sqrt{2s-1},s]}|\partial_t Zu\Box Zu|dxdt\\ &= \int_{\sqrt{2s-1}}^{s}\int_{\mathcal{H}_\tau}|(s/t)\partial_t Zu|\,|\Box Zu|dxd\tau\\ &\lesssim \int_{\sqrt{2s-1}}^{s}\mathbf{E}^N(\tau,u)^{1/2}\||\Box u|_N\|_{L^2_f(\mathcal{H}_\tau)}d\tau.\end{aligned}$$

Recalling (6.1) and (4.2) (which is now valid on $[2,\infty)$, following the bootstrap argument), one has

$$\int_{\mathcal{D}_s}|\partial_t Zu\Box Zu|dxdt \lesssim (C_1\varepsilon)^2, \tag{7.2}$$

provided that $CC_1\varepsilon \le \delta, \delta \le 1/10$. We then obtain the following uniform energy bound:

$$\|u(t,\cdot)\|_{\mathrm{E}} \lesssim (C_1\varepsilon)^2. \tag{7.3}$$

At the end, we establish the scattering property of the wave component. For this purpose we recall the following result established in [17, Lemma 6.12].

Lemma 7.2. *Let u be a solution to*

$$-\Box u(t,x) = F(t,x), \quad u(2,x) = u_0(x), \quad \partial_t u(2,x) = u_1(x) \tag{7.4}$$

where $F \in C([0,\infty); L^2(\mathbb{R}^3))$, $u_0 \in \dot{H}^1(\mathbb{R}^3), u_1 \in L^2(\mathbb{R}^3)$. If

$$\int_2^\infty \|F(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}d\tau < \infty, \tag{7.5}$$

then there exists $(u_0^\star, u_1^\star)$ with $u_0^\star \in \dot{H}^1(\mathbb{R}^3), u_1^\star \in L^2(\mathbb{R}^3)$ such that

$$\lim_{t\to\infty}\sum_{\alpha=0}^{3}\|\partial_\alpha u(t,\cdot) - \partial_\alpha u^\star(t,\cdot)\|_{L^2(\mathbb{R}^3)} = 0, \tag{7.6}$$

where $u^\star$ is the solution to the Cauchy problem

$$-\Box u^\star = 0, \quad u^\star(2,x) = u_0^\star(x), \quad \partial_t u^\star(2,x) = u_1^\star(x). \tag{7.7}$$

Moreover,

$$\|u_0^\star - u_0\|_{\dot{H}^1(\mathbb{R}^3)} + \|u_1^\star - u_1\|_{L^2(\mathbb{R}^3)} \le C\int_2^\infty \|F(\tau,\cdot)\|_{L^2(\mathbb{R}^3)}d\tau. \tag{7.8}$$

Proposition 7.3. *There exists an initial datum $(u_0^\star, u_1^\star) \in \dot{H}^{N+1}(\mathbb{R}^3) \times H^N(\mathbb{R}^3)$, such that the wave component of the global solution constructed in Theorem 1.1 satisfies:*

$$\lim_{t\to+\infty} \sum_{\alpha=0}^{3} \|\partial_\alpha \partial^I L^J u^\star(t,\cdot) - \partial_\alpha \partial^I L^J u(t,\cdot)\|_{L^2(\mathbb{R}^3)} = 0 \tag{7.9}$$

where $|I| + |J| \leq N$.

Proof. We apply Lemma 7.2 on

$$-\Box \partial^I L^J u = \partial^I L^J (P^{\alpha\beta} \partial_\alpha v \partial_\beta v + R v^2).$$

Recalling (4.7) which is now valid in $\mathcal{K}_{[2,\infty)}$,

$$\| |\partial_\alpha v \partial_\beta v|_N \|_{L^2(\mathbb{R}^3)} \lesssim C_1 \varepsilon \| t^{-3/2} s^\delta |\partial v|_N \|_{L^2(\mathbb{R}^3)} \lesssim C_1 \varepsilon t^{-3/2+2\delta}$$

which is integrable in time. The same estimate holds for v^2. Then by Lemma 7.2 we obtain the desired result. □

8. Conclusion

We first emphasize that the system (1.1) simplified from the Einstein-massive scalar field system with small amplitude regular initial data model the gravitational wave stimulated by an astrophysical event. The background is taken to be the Minkowski space-time. In this context, due to the fact that the Klein-Gordon component always enjoys a mildly increasing energy, the massive wave enjoys a strictly faster decay rate up to the top order, say, $\sim t^{-3/2} s^\delta$ compared with $\sim t^{-1}$ of massless wave (recalling (4.7)). As a consequence, it seems that the extra massive field would not be easily detected directly. If one wants to make tests on the scalar-tensor theory, it is better to concentrate on the wave component, i.e. the metric wave. However, the linear scattering result indicates that in this weak field case, the metric wave looks "very like" (in the sense of L^2 norm) a linear wave by a distant observer, even if a massive scalar field is coupled. If this property still holds for the complete Einstein-massive scalar system (especially with positive ADM mass initial data), it seems to be difficult to distinguish a scalar-tensor gravitational wave from a relativistic one. The same situation holds for many other modified gravity theories such as the $f(R)$ theory and the Brans-Dicke theory. Both of them possess a similar wave-Klein-Gordon structure demonstrated by (1.1).

We should emphasize again that the above discussion is only valid in the astrophysical context with Minkowski space-time background. In the cosmological context where the background space-time is more complicated (for example, the Milne model), it may happen that the massive scalar enjoys a directly detectable feature.

Conflict of interest

The authors declare there is no conflict of interest.

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