Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 15(2): 91–110.
DOI: 10.3934/cam.2023006
Received: 28 September 2022
Revised: 28 March 2023
Accepted: 30 March 2023
Published: 21 April 2023

Research article

A possible interpretation of financial markets affected by dark volatility

Richard Pinčák[1,*], Alexander Pigazzini[2], Saeid Jafari[2,3], Özge Korkmaz[4], Cenap Özel[5] and Erik Bartoš[6]

[1] Institute of Experimental Physics, Slovak Academy of Sciences, Watsonova 47, 043 53 Košice, Slovak Republic
[2] Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark
[3] College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark
[4] Malatya Turgut Özal University, Department of Management Information Systems, Malatya, Turkey
[5] King Abdulaziz University, Department of Mathematics, 21589 Jeddah, Saudi Arabia
[6] Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovak Republic

* **Correspondence:** Email: pincak@saske.sk.

Abstract: The aim of this paper is to use a special type of Einstein warped product manifolds recently introduced, the so-called PNDP-manifolds, for the differential geometric study, by focusing on some aspects related to dark field in financial market such as the concept of dark volatility. This volatility is not fixed in any relevant economic parameter, a sort of negative dimension, a ghost field, that greatly influences the behavior of real market. Since the PNDP-manifold has a "virtual" dimension, we want to use it in order to show how the Global Market is influenced by dark volatility, and in this regard we also provide an example, by considering the classical exponential models as possible solutions to our approach. We show how dark volatility, combined with specific conditions, leads to the collapse of a forward price.

Keywords: dark volatility; negative dimension; virtual dimension; econophysics; financial market; ghost field
Mathematics Subject Classification: 53A31, 91G15

1. Introduction

Einstein warped product manifolds have been investigated both in mathematics and physics. Several papers in literature are dedicated to the study of warped products as solutions of Einstein's field equations, (to name a few, among others, see [1–3]. They are mathematical objects which are also considered in current research. Here, we consider M an Einstein warped product manifold of special kind, the so-called PNDP-manifold, introduced by A. Pigazzini et al. in [4, 5], and we try to use this new approach to "virtual" dimensions and "emerging spaces" (see also [6]), in order to describe the behavior of the financial market conditioned by a ghost field. To present these applications, we take as basic knowledge what is stated in [4–6], recalling the definition of PNDP-manifold.

In recent decades, physicists have increasingly contributed to the modeling of "complex systems" using tools and methodologies developed in statistical mechanics and theoretical physics, introducing what today can be called Econophysics (see [7]), and geometric theories, such as the *Geometric Arbitrage Theory* have also been introduced ([8]). Recently, works have been published (see [9–11]) in which the authors introduce the concepts of Ghost Field, hidden space and extra dimensions in the economic field, also considering spaces with negative dimensions.

In [10] we want to emphasize that the authors used an original idea in the Minkowski spacetime embedded in Kolmogorov space in time series data with the behavior of traders, for GARCH $(1, 1)$ process; to learn more about the GARCH model see also [12]. The result of the work [10] is equivalent to the dark volatility or the hidden risk fear field induced by the interaction of the behavior of the trader in the financial market panic when the market crashes. In particular the authors induce a dark volatility equivalent to dark matter in theoretical physics discovered by Wolfgang Pauli. They consider $g_{ij} \in G$ be a cycle (group G) and $g^{ij} \in G^*$ be a cocycle, a group action acting on the fibre bundle of manifold, $g_{ij} = \sqrt{g_{ij}}\sqrt{g_{ij}}$ and x_t is a time series in the section of the Riemannian manifold, $x_t \in T_{X_0}X = p^{-1}(X)$. Thus there is a differential 2-form of time series arising from the scalar production of the group transformation of the vector bundle, the so-called principal bundle of the time series, $\sigma_t^2 = \langle x_t, x_t \rangle = g_{ij}\|x_t\|^2$ with $g_{ij} = 1$, where σ_t^2 indicates the volatility, and the authors are interested in the case of dark volatility, i. e. $\sigma_t^2 < 0$, which happens when we change the Euclidean metric to the Minkowski metric $g_{ij} = -1$ which allows the negative volatility, $\sigma_t^2 < 0$. Because $\sigma_t^2 = \langle x_t, x_t \rangle = g_{ij}\|x_t\|^2 = -\|x_t\|^2$; the metric used for dark volatility corresponds to the metric used for F in [4].

Our work can also be understood as an extension of the geometric approach described in these papers [13–16]. For instance in [13], objects are represented by open string with 2-endpoints and $D2$-brane, which are continuous enhancement of 1-endpoint open string model. We can, in fact, reinterpret strings and branes as PNDP-strings and PNDP-brane showing the geometric aspects of the "hidden" spaces that our approach introduces, but also as an extension to more "classic" works as [17, 18].

Later, in Section 4, we have demonstrated possible empirical nature of our approach. We have tried to show a particular case, when the forward price represented in its exponential form coincides with the warping function. We have outlined the possible links between the solutions provided by the systems that determine a PNDP-manifold and some classic stochastic models. We have tried to find answer how our concept of dark volatility could be able to stabilize the market crisis.

On the other hand, it would be useful to mention that there is increasing knowledge about the nonstationary, far-from-equilibrium dynamics. The proper understanding such dynamics, e. g., the violation of the ergodic assumption [19], in the systems which econophysics deals, can bring a new perspective on the investigated phenomena [20]. Regarding the performed analysis with volatilities, a number of interesting generalizations of geometric Brownian motion (GBM) models emerged booth in economic and physical literature on stochastic processes. In the future it would be very useful to look at the obtained results from the point of view of the computation with time averaged mean squared displacement for scaled GMB [21, 22] with volatility varying as a power law in time.

The paper is organized as follows: in Sections 2 and 3 we introduce basic properties of PNDP-manifolds with some Ricci-flat examples. In Section 4 we show the relationship between PNDP-manifolds approach and exponential models and provide some empirical analysis on real financial data. In Section 5 we summarize the result of presented work and discuss future ideas.

2. PNDP-manifolds

Before proceeding with the definitions, it is useful to remember the following: if $A \to M$ and $B \to M$ are transversal sub-manifolds of co-dimensions a and b, respectively, then their intersection C, is a new sub-manifold of co-dimensions $a + b$. By removing the requirement for transversality, derived geometry explains how to interpret a non-transversal intersection C as a derived smooth manifold of co-dimension $a + b$. Specifically, $dim(C) = dim(M) - a - b$, and the latter number can be negative. Therefore, we are dealing with "virtual dimensions", which are not linked to the usual geometrical conception of "dimension". In fact, in differential geometry, the transversality condition is a condition that establishes how two submanifolds interact with each other in a larger space. More precisely, two submanifolds are transversal if they intersect in a "non-degenerate" way, i. e., if the sum of their dimensions is equal to the dimension of the largest manifold containing them. The removal of the transversality condition implies that the submanifolds no longer satisfy this condition and therefore their intersection is no longer non-degenerate. This means that their interaction can no longer be described simply as a sum of their dimensions, but requires more advanced techniques than derived geometry, to define what is called "virtual" dimensions.

In this paper, we apply derived geometry and the "virtual" dimension, by utilizing the concept of Kuranishi neighborhood (R^d, E, s), where, by definition, it is the intersection of zero section of the obstruction bundle with itself, therefore, we define the following virtual dimension: $dim(R^d) - rank(E)$, see [23] for more details.

Definition 1: A warped product manifold $(M, \bar{g}) = (B, g) \times_f (F, \ddot{g})$, with metric tensor $\bar{g} = g + f^2\ddot{g}$, is Einstein if only if

$$\bar{Ric} = \lambda\bar{g} \iff \begin{cases} Ric - \frac{d}{f}\nabla^2 f = \lambda g \\ \ddot{Ric} = \mu\ddot{g} \\ f\Delta f + (d-1)|\nabla f|^2 + \lambda f^2 = \mu \end{cases} \tag{2.1}$$

where λ and μ are constants, d is the dimension of F, $\nabla^2 f$, Δf and ∇f are, respectively, the Hessian, the Laplacian and the gradient of f for g, with a smooth positive function $f : (B) \to R^+$.

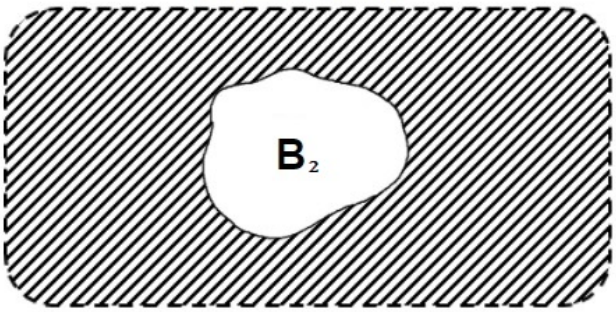

Figure 1. Speculative view of a PNDP-maifold with a positive "virtual" dimension. From the speculative point of view, the following projection, the PNDP-manifold appears as B_2-manifold. Then the rest of the PNDP-manifold remains hidden or invisible.

Definition 2: We call PNDP-manifold a warped product manifold $(M, \bar{g}) = (B, g) \times_f (F, \ddot{g})$ satisfying (1), where the base-manifold (B, g) is a Riemannian (or pseudo-Riemannian) product-manifold $B = B_1 \times B_2$ with $g = \Sigma g_i$, where B_2 is an Einstein manifold (i. e., $Ric_2 = \lambda g_2$ where λ is the same for (1) and g_2 is the metric for B_2), with $dim(B_1) = n_1$, $dim(B_2) = n_2$, so $dim(B) = n = n_1 + n_2$ (Figure 1). The warping function $f : B \to R^+$ is $f(x, y) = f_1(x) + f_2(y)$ (where each is a function on its individual manifold, i. e., $f_1 : B1 \to R^+$ and $f_2 : B_2 \to R^+$) and can also be a constant function. The fiber-manifold $(F, \ddot{g})$ is a derived Riemann-flat manifold with negative "virtual" integer dimensions m, where with *derived smooth manifold* is considered a *smooth Riemannian flat manifolds by adding a vector bundle of obstructions*. In particular for F we consider the space form R^d as a underlying manifold, with orthogonal Cartesian coordinates such that $g_{ij} = -\delta_{ij}$, and as vector bundle of obstructions, $E \to R^d$, a bundle with $rank(E) = 2d$, i. e., twice the dimension of the space form R^d. In this way the dimension of F will always be $m = d - rank(E) = -d$. In fact, in this circumstance, if we consider a Kuranishi neighborhood (R^d, E, S), with manifold R^d, obstruction bundle $E \to R^d$, and section $S : R^d \to E$, then the dimension of the derived smooth manifold F is $dim(R^d) - rank(E)$. Moreover in the case $n - d > 0$ (i.e., M with positive "virtual" dimension) we consider it as PNDP-manifold only the case $n_1 = d = -m$ (so the "virtual" dimension of M, $dim(M)_{\mathcal{V}}$, must coincides with $dim(B_2)$). In the specal case where $n - d > 0$ in which also B_1 is an Einstein-manifold, then we consider it as PNDP-manifold only the case in which $B_1 := B_2$.

Important Note: Since $F := (R^d, E, S)$, and on E (obstruction bundle) the (pseudo-)Riemannian geometry does not work, each (pseudo-)Riemannian geometry operation is performed and defined only on the underlying R^d, but is considered performed and defined also on F (e. g., we will say that the Ricci curvature of F is zero because the Ricci curvature of R^d is zero). The usual (pseudo-)Riemannian geometry works for the underlying smooth manifold (because it is an ordinary manifolds). From now on we will work with the (pseudo-)Riemannian geometry on the derived fibers-manifold F. We will define all (pseudo-)Riemannian geometry operations not directly on F, but on R^d, and considering them made on F, paying attention only to the dimension. Obviously for what has been said, the tangent space and the vector fields are those of R^d. The scalar product with two arbitrary vector fields $\ddot{g}\langle V, W\rangle$ is define on F as: $g_{ij}v^iw^j = -\delta_{ij}v^iw^j = -(v^iw^i)$.

The analysis does not differ from the usual Einstein sequential warped product manifold analysis, $(M_1 \times_h M_2) \times_{\bar{h}} M_3$, (see [24, 25]), where $h = 1$, M_2 is an Einstein-manifold and M_3 is a derived-smooth-manifold with negative "virtual" dimensions. The Riemannian curvature tensor and the Ricci curvature tensor of the product Riemannian manifold can be written respectively as the sum of the Riemannian curvature tensor and the Ricci curvature tensor of each Riemannian manifold (see [26]).

Proposition: If we write the B-product as $B = B_1 \times B_2$, where:

(i) Ric_i is the Ricci tensor of B_i referred to g_i, where $i = 1, 2$,
(ii) $f(x, y) = f_1(x) + f_2(y)$, is the smooth warping function, where $f_i : B_i \to R^+$,
(iii) $Hess(f) = \sum_i \tau_i^* Hess_i(f_i)$ is the Hessian referred on its individual metric, where τ_i^* are the respective pullbacks, (and $\tau_2^* Hess_2(f_2) = 0$ since B_2 is Einstein),
(iv) ∇f is the gradient (then $|\nabla f|^2 = \sum_i |\nabla_i f_i|^2$), and
(v) $\Delta f = \sum_i \Delta_i f_i$ is the Laplacian, (from (iii) then also $\Delta_2 f_2 = 0$),

then the Ricci curvature tensor will be

$$\begin{cases} \bar{Ric}(X_i, X_j) = Ric_1(X_i, X_j) - \frac{d}{f} Hess_1(f_1)(X_i, X_j) \\ \bar{Ric}(Y_i, Y_j) = Ric_2(Y_i, Y_j) \\ \bar{Ric}(U_i, U_j) = \ddot{Ric}(U_i, U_j) - \ddot{g}(U_i, U_j) f^* \\ \bar{Ric}(X_i, Y_j) = 0 \\ \bar{Ric}(X_i, U_j) = 0, \\ \bar{Ric}(Y_i, U_j) = 0, \end{cases} \tag{2.2}$$

where $f^* = \frac{\Delta_1 f_1}{f} + (d-1)\frac{|\nabla f|^2}{f^2}$, and X_i, X_j, Y_i, Y_j, U_i, U_j are vector fields on B_1, B_2 and F, respectively.

A warped product manifold with derived differential fiber-manifold $F := (R^d, E, S)$, and $dim(F)$ a negative integer, is a PNDP-manifold, as defined in Definition 2, if and only if

$$\bar{Ric} = \lambda \bar{g} \iff \begin{cases} Ric_1 - \frac{d}{f} \tau_1^* \nabla_1^2 f_1 = \lambda g_1 \\ \tau_2^* \nabla_2^2 f_2 = 0 \\ Ric_2 = \lambda g_2 \\ \ddot{Ric} = 0 \\ f \Delta_1 f_1 + (d-1)|\nabla f|^2 + \lambda f^2 = 0, \end{cases} \tag{2.3}$$

(since Ric is the Ricci curvature of B, then $Ric = Ric_1 + Ric_2 = \lambda(g_1 + g_2) + \frac{d}{f}\tau_1^* \nabla_1^2 f_1$). Therefore the system (2.3), for $n - d = 0$ and $n - d < 0$, become

$$\bar{R} = \lambda \bar{n} \iff \begin{cases} R_1 f - \Delta_1 f_1 d = n_1 f \lambda \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ f \Delta_1 f_1 + (d-1)|\nabla f|^2 + \lambda f^2 = 0. \end{cases} \tag{2.4}$$

where n_1, n_2, R_1 and R_2 are the dimension and the scalar curvature of B_1 and B_2 respectively. While for $n-d>0$, as by Definition 2, we must set $d=n_1$, so we have

$$\bar{R}=\lambda\bar{n} \Longleftrightarrow \begin{cases} R_1 f - \Delta_1 f_1 n_1 = n_1 f\lambda \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + (n_1-1)|\nabla f|^2 + \lambda f^2 = 0. \end{cases} \tag{2.5}$$

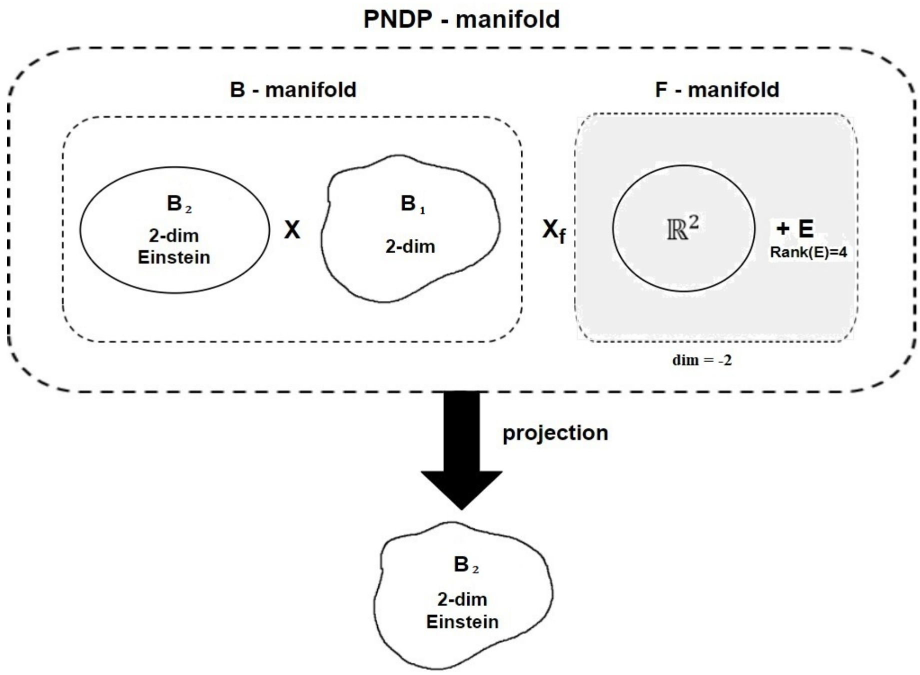

Figure 2. The figure shows a PNDP-maifold with speculative projection into 6-dimensional manifold with an obstruction bundle of $rank = 4$. The "virtual" dimension of a PNDP-manifold (if positive) has to be equal to the dimension of B_2-manifold (see *Definition 2* and projection (2.9)). Also the λ constant of a PNDP-manifold has to be equal to the λ constant of B_2-manifold. So the speculative projection, projects a PNDP manifold on a B_2-manifold "virtually".

Remarks: In the particular case where $d=1$ the systems (2.4) and (2.5) should be modified, in fact for $d=1$, from the system (2.3), we get

$$\bar{Ric}=\lambda\bar{g} \Longleftrightarrow \begin{cases} Ric_1 - \frac{1}{f}\tau_1^*\nabla_1^2 f_1 = \lambda g_1 \\ \tau_2^*\nabla_2^2 f_2 = 0 \\ Ric_2 = \lambda g_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + \lambda f^2 = 0, \end{cases} \tag{2.6}$$

from which the system (2.4) becomes

$$\bar{R} = \lambda\bar{n} \Longleftrightarrow \begin{cases} R_1 f - \Delta_1 f_1 = n_1 f\lambda \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + \lambda f^2 = 0. \end{cases} \tag{2.7}$$

and the system (2.5) becomes

$$\bar{R} = \lambda\bar{n} \Longleftrightarrow \begin{cases} R_1 f - \Delta_1 f_1 = f\lambda \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + \lambda f^2 = 0. \end{cases} \tag{2.8}$$

Finally with regard to the projections/desuspensions we remember that

- if $dim(M) > 0$ (i. e., system solutions (2.5)) we have the projection

$$\pi_{(>0)} : PNDP \to B_2, \tag{2.9}$$

  so M is "virtually" B_2 (see Figure 2),

- if $dim(M) = 0$, (i. e., system solutions (2.4)), we have the projection

$$\pi_{(=0)} : PNDP \to P, \tag{2.10}$$

  where with P we mean a point-like manifold (zero dimension), so M is "virtually" P, and

- if $dim(M) < 0$, (i. e., system solutions (2.4)), we have the projection

$$\pi_{(<0)} : PNDP \to \Sigma^{dim(M)<0}(p), \tag{2.11}$$

  with $\Sigma^{dim(M)<0}(p)$, we mean the ($\|dim(M)\|$)-th desuspension of point. For example, if $dim(M) = -4$ the projection π_{-4} will "virtually" project M into an object which will be given by the fourth desuspension of a point.

Definition 3: We say that dimension of X is -1 if suspension of X, ΣX is diffeomorphic to a point. By induction and suspension operation we can define all negative dimensions. Since if X and Y are diffeomorphic then their suspensions ΣX and ΣY are diffeomorphic. Also a metric on X is compatible with ΣX. It is also well-known that differentials and suspensions are compatible on manifolds. This means that the differential on the suspension space ΣX is induced from the differential on X.

3. PNDP-manifolds in Financial Market

The aim of this work is to interpret the “dark volatility” ghost field, using the fiber-manifold (F) of the PNDP-manifolds. In fact, as per the speculative interpretation of negative dimensions provided earlier, we consider dark volatility as a negative dimension that influences the market (base-manifold (B)), making it emerge as a new manifold.

Let’s consider the global market (B) and the hidden dark volatility (F) that influences it.

In the case of the PNDP-manifolds, it is possible to combine both of them in a single Einstein-manifold (M) in which F will have a negative ”virtual” dimension, $M = B \times_f F$, where $B = B_1 \times B_2$ with Einstein-B_2.

Let us proceed with the following examples to show in advance the types of scenarios that may arise. By utilizing PNDP-manifolds, we can analyze the situation that is related to the Global Market to see how it can possibly react differently if stimulated by a ghost field such as dark volatility, when it is in equilibrium, i.e. when the Ricci curvature is zero, that is Ricci-flat scenario, (arbitrage opportunity disappears and the system of the time series data contains no curvature).

Example 1: Ricci-flat with $\dim(M) < 0$.

Let M be a PNDP-manifolds with $dim(M) < 0$, where $B = B_2 \times B_2$ represents the global market manifold and F the dark volatility.

In this example we show a type of Ricci-flat PNDP-manifold, i. e., with $\lambda = 0$. If we set $B_2 = R^2$, B_1 a 1-dimensional manifold, and $F = (R^4 + E)$, then (2.4) becomes

$$\begin{cases} \Delta_1 f_1 = 0 \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ |\nabla f|^2 = 0. \end{cases} \tag{3.1}$$

The solution is $f_1 = c$, where c is a constant and $\pi_{dim(M)=-1} : (R^2 \times B_1) \times_c (R^4 + E) \to \Sigma^{-1} p$, in which the global market $B = (R^2 \times B_1)$ under the influence of dark volatility F, changes dimension ”virtually”, from 3-dimensional manifold to ”virtual” -1-dimensional manifold, i. e., an object with negative dimension, so even if its Ricci-curvature remains preserved, we cannot consider it as an “equilibrium” situation because it “virtually” degenerates into something like the desuspension of a point. In this case, the scenario points to an extremely negative effect, where the market fails to maintain stability and crashes.

Example 2: Ricci-flat with $\dim(M) = 0$*, point-like.*

For this purpose, we consider fiber-manifold $F = (R^4 + E)$ (with $rank(E) = 8$), 2-dimensional Ricci-flat B_1-manifold and $\lambda = 0$. In this case we still have the situation of (2.4). If for example we choose $B_1 = B_2 = R^2$, then our PNDP-manifold will be $M = (R^2 \times R^2) \times_f (R^4 + E)$, with the following metric: $ds^2 = dr^2 + dx^2 + dy^2 + dz^2 + f^2(du^2 + dv^2 + dw^2)_{(-4)}$, where f is a constant function.

In this case, the Global Market $B = (R^2 \times R^2)$, under the influence of dark volatility F, “virtually” changes dimension from 4-dimensional to “virtual” 0-dimensional manifold, $\pi_{dim(M)=0} : (R^2 \times R^2)_f (R^4 + E) \to$ “virtual” point-like manifold (“virtual” zero-dimension), so we cannot consider it as an "equilibrium" situation because it "virtually" degenerates into a point-like manifold. It is a highly unstable situation, as the market may immediately crash.

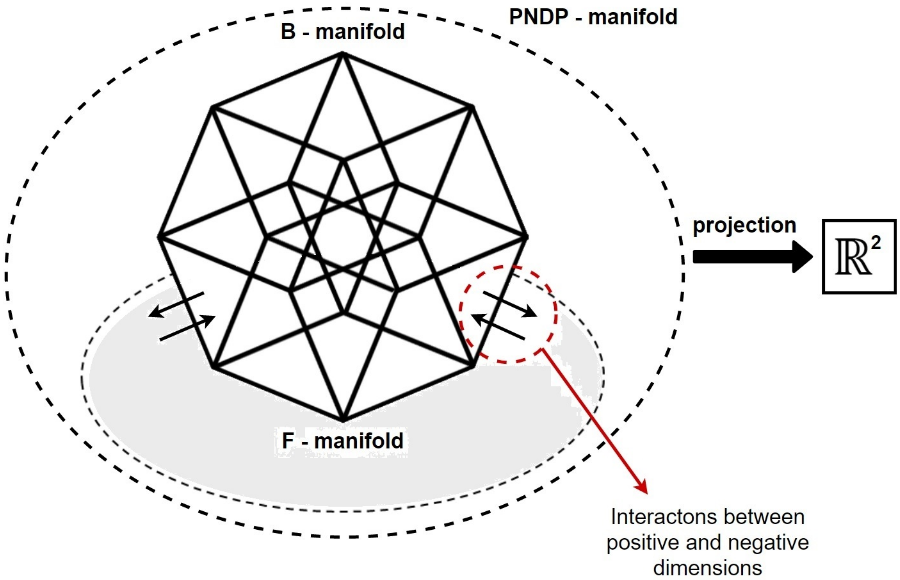

Figure 3. Representation of the Example 3a. The PNDP-manifold is the global market in equilibrium with dark volatility. Here (B-manifold) is represented trivially as a Tesseract of dimension 4. The influence of the dark volatility (F-manifold), acts on the Tesseract by reducing it in dimension (R^2). In this case the market manages to maintain an equilibrium as it maintains a positive dimension and the same Riemanian structure except for the dimension.

Example 3a: Ricci-flat with $\dim(M) > 0$.

Let M be a PNDP-manifold with $dim(M) > 0$, where $B = B_1 \times B_2$ represents the global market manifold and F is the dark volatility. Let $B_1 = B_2$, both Ricci-flat, i. e., $R_1 = R_2 = 0$, and both 2-dimensional, i. e., dimension $n_1 = n_2 = 2$. Then we can consider B_1 and B_2 to be both R^2, with metric: $ds^2 = (dx^2 + dy^2)$. Now let $F = (R^2 + E)$ have "virtual" negative dimension $m = -2$.

In this case the system (2.5) is satisfied for $f = c$, where c is constant (f_1 and f_2 are both constant). Therefore the PNDP-manifold will be $M =: (R^2 \times R^2) \times_c (R^2 + E)$, with metric $ds^2 = dt^2 + dx^2 + dy^2 + dz^2 - c^2(du^2 + dv^2)_{(-2)}$ and "virtual" dimension $dim(M) = 2$ (see Figure 3).

It follows from our projection that $\pi_{dim(M)=2} : (R^2 \times R^2) \times_c (R^2 + E) \to R^2$, which the global market $B = (R^2 \times R^2)$ under the influence of dark volatility F changes from 4-dimensional flat-manifold to its 2-dimensional flat-submanifold, but in this case where the curvature remains preserved and the market maintains a positive dimension ($dim(M) > 0$), we consider that the market also preserves its equilibrium situation.

Example 3b: Ricci-flat with $\dim(M) > 0$.

Let M be a PNDP-manifolds with $dim(M) > 0$, where $B = B_1 \times B_2$ represents the global market manifold and F is the dark volatility. Let $B_1 = B_2$, both Ricci-flat, i. e., $R_1 = R_2 = 0$, and both 2-dimensional, i. e., dimension $n_1 = n_2 = 2$, then we can consider B_1 and B_2 as $S^1 \times S^1$, i. e. B_1 and B_2 are both T^2 torus, and let $F = (R^2 + E)$ with "virtual" negative dimension $m = -2$. Also in this case the system (2.3) is satisfied for $f = c$, where c is constant (f_1 and f_2 are both constant), so the PNDP-manifold will be $M =: (S^1 \times S^1 \times S^1 \times S^1) \times_c (R^2 + E)$, with "virtual" dimension $dim(M) = 2$.

This implies that $\pi_{dim(M)=2} : (S^1 \times S^1 \times S^1 \times S^1) \times_c (R^2 + E) \to (S^1 \times S^1)$, which the global market $B = (S^1 \times S^1 \times S^1 \times S^1)$ under the influence of dark volatility F changes dimension, from 4-torus Ricci-flat to its submanifold, a 2-torus Ricci-flat, but, as in the previous example, it is still in equilibrium.

Time series of financial asset returns often demonstrates volatility clustering (Figure 4). In a time series of stock prices, for instance, it is observed that the variance of returns or log-prices is high for extended periods and then low for extended periods.

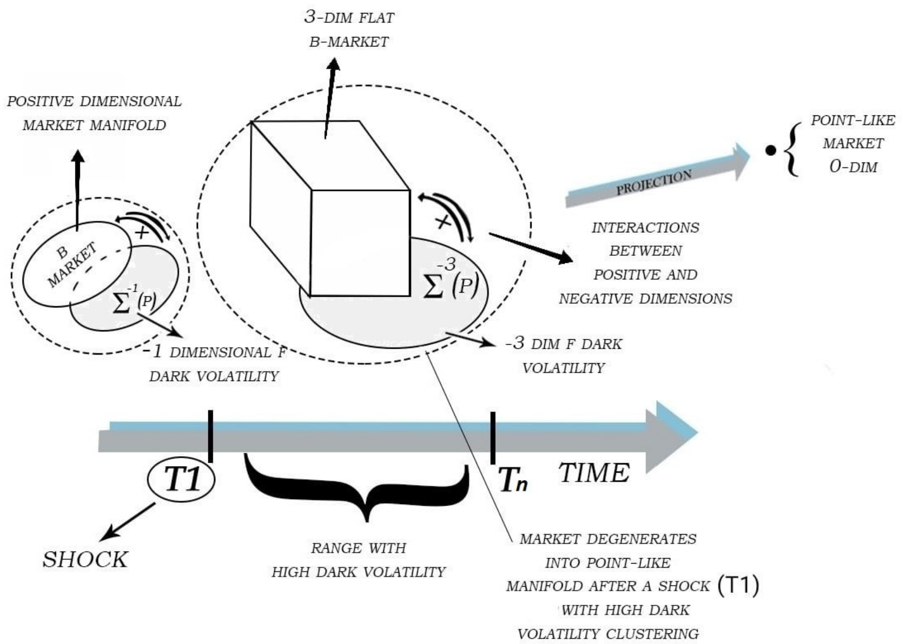

Figure 4. The situation over time regarding the dark volatility clustering phenomenon is shown, according to our approach. We can see a market represented by a positive dimensional manifold (in which the dark volatility is -1 dimensional, i. e., low), over time, at the instant t_1 a shock occurs, a surge in the dark volatility. The clustering phenomenon persists up to an instant t_n, i. e., the volatility will continue to be high from t_1 to t_n, thus allowing the dark volatility to be identified with a new manifold F with an increased negative dimension compared to what it was before t_1. Therefore, through our projection, the example in the figure shows that the market, in the clustering between t_1 and t_n, is identified with a zero dimensional object.

What this means in practice and in the world of investing is that as markets respond to new information with large price movements (volatility), these high-volatility environments tend to endure for a while after the first shock. In other words, when a market suffers a volatile shock , more volatility should be expected.

In our scenario, we assume that the more dim F is negative, the more shocks the market (B) will suffer.

For example, if we consider *Example 2* and *Example 3a*, we see that the same Market B (which is represented by $R^2 \times R^2$, i. e., R^4) suffers a greater shock when dark volatility (F) increases to the point that everything degenerates into a point-like manifold (*Example 2*), compared to when it settles into R^2-manifold (*Example 3a*).

Therefore, we should expect, in our approach, that the clustering of volatility manifests itself, following a shock, as the tendency of a market to a certain change in dimension (sometimes also in curvature) which can be represented by the manifold on which the shock leads it to self projection.

In summary, it is worth noticing that we are dealing with dark volatility, and not volatility, in the sense that B already takes volatility into account, so F (dark volatility) influences B (which already takes into account volatility) by transforming B into another "object" through our projection.

If, for example, we start from market with "virtual" dimension $\dim(M) > 0$ and we have a trend, therefore Dark Volatility Clustering, for example negative, then F will gradually increase its negative dimension, so remaining in $\dim(M) > 0$, i. e., the market is projected in manifolds with positive dimension. However, if the trend turns negative even further, then the market will gradually change into manifolds with ever smaller dimensions and in the end we will reach $\dim(M) = 0$. Here there is a transition point, i. e., beyond this, we have the Crash, identified with $\dim(M) < 0$ (desuspensions of point). If the trend improves then we go back to $\dim(M) > 0$.

Maintaining these trends over a period of time, even a short one, make it possible to determine F, and therefore obtaining a projection on an "object" that will represent the market in that period.

The situation is different if we have a large fluctuation of dark volatility (F), i. e., in every instant the dark volatility continues uninterruptedly to go from very high to very low (Figure 5), which contributes to creating the "Market Panic" (Figure 6). Thus, according to our approach, the big jump in dark volatility corresponds to a change in dimension of F and this means that F continues to change in dimension every instant without a trend takes shape over a period of time, and then the "virtual" dimension of M ($dim(M)$) keeps changing instantaneously by increasing and decreasing, being able to pass from positive to negative and vice versa instantly.

So our projection continues to vary instantly without identifying the market in a specific "object". It is as if the possible states overlap, until a trend takes shape, at which point our projection returns to project the market into one specific "object", since the dark volativity re-identifies with an F, and thus a precise dimension, for a period of time.

4. Relationship between PNDP-manifolds approach and exponential models

In this section we want to relate the PNDP manifolds with the classical exponential models and to do this we want to highlight a particular case in which the warping function f, defined on the base-manifold B of a PNDP-manifold, can coincide with the forward price represented in its exponential form. The objective of this paragraph is in fact to stipulate the possible links between the solutions provided by the systems that determine a PNDP-manifold and some classic stochastic models, to show the possible applicative and empirical nature of this new approach to all effects. Furthermore we demonstrate how it is possible to apply the concept of dark volatility to the well-known classical models.

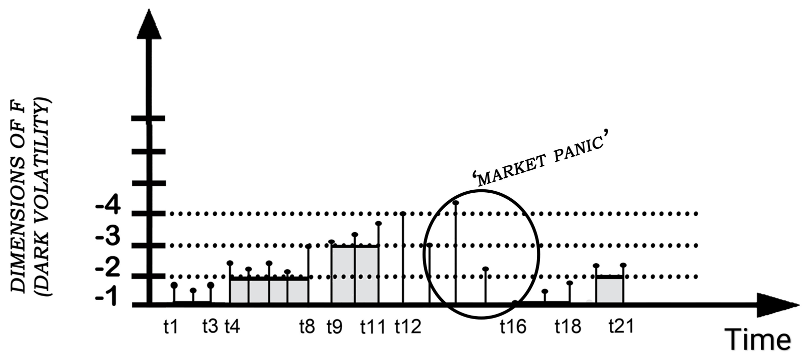

Figure 5. Trend of Dark Volatility. Here we show how hypothetical values of dark volatility affects dimension of F that represents it. From t_1 to t_3, F has dim $= -1$, from t_4 to t_8 its dimension is -2, from t_9 to t_{11} $\dim(F) = -3$. So the dark volatility follows a trend clusterings. From t_{12} to t_{16} there is a fluctuation, every instant $\dim(F)$ changes value: $t_{12} \to \dim(F) = -4$, $t_{13} \to \dim(F) = -3$, $t_{14} \to \dim(F) = -4$, $t_{15} \to \dim(F) = -3$, $t_{16} \to \dim(F) = -1$. In the interval (t_{12}, t_{16}) the market states overlap. From t_{16} it recovers a trend.

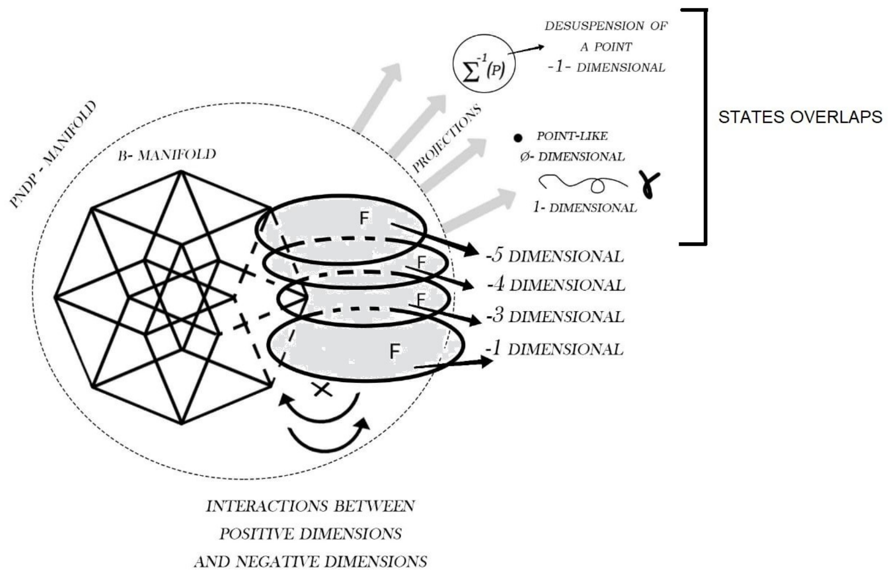

Figure 6. The figure shows a situation of "Market Panic", (another similar situation is also shown in Figure 5 in the interval (t_{12}, t_{16})) in which a global market B undergoes continuous fluctuations in dark volatility F, which continues to change instantly without showing a trend. Due to constant changes in F, the market shows itself as the result of many overlapping projections that we consider as "Market Panic".

4.1. Exponential models

Let's start by recalling that the classic Black '76 model [27] is a generalization of the Black-Scholes [28] model in which a log-normal trend of the underlying is assumed. It also assumes that the forward price of an underlying has the following exponential trend

$$F_t(t) = F_0(t)\mathrm{e}^{-\frac{\sigma^2}{2}t+\sigma W_t}, \tag{4.1}$$

where σ is the constant log-normal volatility, W_t is the Brownian motion and $F_s(t)$ is the forward price of an underlying with expiry in t and valued in $s \leq t$.

The above model can be seen as a particular case of the exponential Levy models. These models provide for the use of Levy processes (see [29]) for the description of the forward price of the underlying. In general, the forward price is represented in its exponential form as

$$F_t(t) = F_0(t)\mathrm{e}^{f_t}, \tag{4.2}$$

where f_t is a Levy process.

It is worth noticing that (from *Definition 4* in [4]) an Einstein warped product manifold is a PNDP-manifold, if and only if, the statements in system (2.3) are satisfied.

If, for example, we consider the $dim(B_1) = n_1 = 1$ (in case with positive dimension), we have $n_1 = d = 1$, then system (2.5) becomes

$$\bar{R} = \lambda\bar{n} \Longleftrightarrow \begin{cases} R_1 f - \Delta_1 f_1 = f\lambda \\ \Delta_2 f_2 = 0 \\ R_2 = \lambda n_2 \\ \ddot{Ric} = 0 \\ f\Delta_1 f_1 + \lambda f^2 = 0. \end{cases} \tag{4.3}$$

At this point if we set $n_2 = 2$, $\lambda = -1$, $R_1 = 0$ and $f_2 = 1$, the system (4.3) becomes

$$\bar{R} = \lambda\bar{n} \Longleftrightarrow \begin{cases} \Delta_1 f_1 = f \\ \Delta_2 f_2 = 0 \\ R_2 = -2 \\ \ddot{Ric} = 0 \\ f^2 = f^2. \end{cases} \tag{4.4}$$

Since the last equation of (4.4) is an identity, the system is reduced to find the solution of a simple linear second order ODE: $\Delta_1 f_1 = f_1 + 1$, by noticing that $f = f_1 + f_2 = f_1 + 1$. The solution will be: $f_1 = C_1 e^t + C_2 e^{-t} - 1$.

As we can see, in this case the market is no longer in equilibrium. In fact we no longer have zero scalar curvature, but we obtain that the warped function $f = f_1 + f_2$ is

$$f = C_1\mathrm{e}^t + C_2\mathrm{e}^{-t}. \tag{4.5}$$

Just to make a parallel with the Black '76 model (4.1), we can set $C_1 = 0$, $C_2 = F_0$ (where F_0 is the forward price, with expiry in t, and constant value in $s \leq t$) and $f_t = -t$. We can also indicate that our solution is considering a Brownian motion in a non-Newtonian fluid. Also our warped function $f : (B_1 \times B_2) \to R^+$ is equivalent to

$$f = F_0 e^{-t}, \tag{4.6}$$

that is, the description of the forward price of the underlying using exponential type model.

This particular case analyzed here suggests that our approach can be traced back to a Levy-type exponential model, only in a market that is not in equilibrium, and highlight that the warping function (solution of the systems that determine the PNDPs and defined on the B-manifold with real positive values), can be linked to exponential models. In particular, in this case, it shows us that the presence of dark volatility is linked to a manifold (F) with only one "virtual" negative dimension and that the forward price of an underlying, again in this particular market situation, will be destined to drop to an asymptotic zero; in other words, it predicts an upper hand of dark volatility and therefore an exponential collapse of the price.

This will lead to a consequent increase in dark volatility corresponding to a market crash, but continuing with the analysis we find that the collapse of the market will continue. In effect, if we consider the increase in dark volatility, subordinated to the collapse of the price, and therefore no longer F with a "virtual" negative dimension, but F with three "virtual" negative dimensions, then the system of Eqs. (4.4) becomes

$$\bar{R} = \lambda \bar{n} \iff \begin{cases} \Delta_1 f_1 = (f_1 + 1)/3 \\ \Delta_2 f_2 = 0 \\ R_2 = -2 \\ \ddot{Ric} = 0 \\ |\nabla_1 f_1|^2 = (f_1 + 1)^2/3. \end{cases} \tag{4.7}$$

We obtain that the first equation has a solution: $f_1 = C_1 e^{t/\sqrt{3}} + C_2 e^{-t/\sqrt{3}} - 1$, while from the fifth equation we get: $f_1 = e^{\frac{\sqrt{3}t+C_3}{3}} - 1$ and $f_1 = e^{\frac{-\sqrt{3}t-C_3}{3}} - 1$. At this point, it is easy to see that, always referring to (4.1) as in the previous case, for $C_1 = C_3 = 0$ and $C_2 = 1$, the solutions of the first equation will coincide with the solution of the fifth, that is, we get

$$f = F_0 e^{-t/\sqrt{3}}, \tag{4.8}$$

which confirms the ongoing crash.

4.2. Empirical analysis for traded options

To demonstrate the behavior of our approach, we performed the empirical analysis of the forward price represented in its exponential form. Further we use the standard formula for the call and put prices [28]

$$\begin{aligned} p_c =& S e^{-qt} N(d_1) - K e^{(-rt)} N(d_2) \\ p_p =& K e^{(-rt)} N(-d_2) - S e^{-qt} N(-d_1) \end{aligned} \tag{4.9}$$

where S is an underlying asset price, K is a strike price, t is time to expiration in years, r is a risk-free interest rate (for simplification q as foreign risk-free interest rate was set to $q = r$) and $N(x)$ is a cumulative distribution function for a Gaussian distribution with standard deviation σ. Forward price was then set as difference of $F = p_c - p_p$.

As the practical implementation of above formulae for the calculations of Black-Scholes option prices we used `py_vollib` python library [30]. For the purpose of analysis we used the online available historical data Yahoo! Finance [31] for selected traded options, namely, Alphabet Inc. (GOOGL), Tesla, Inc. (TSLA), SAP SE (SAP), Diageo plc (DEO), Anheuser-Busch InBev SA/NV (BUD), which provided us with values of parameters S and K. The choosen time period was between 2021/03/24 and 2022/03/24 and the option calls were set for the day 2022/04/22 (time t one month in the future). The risk-free interest rate r was set to 0.0235.

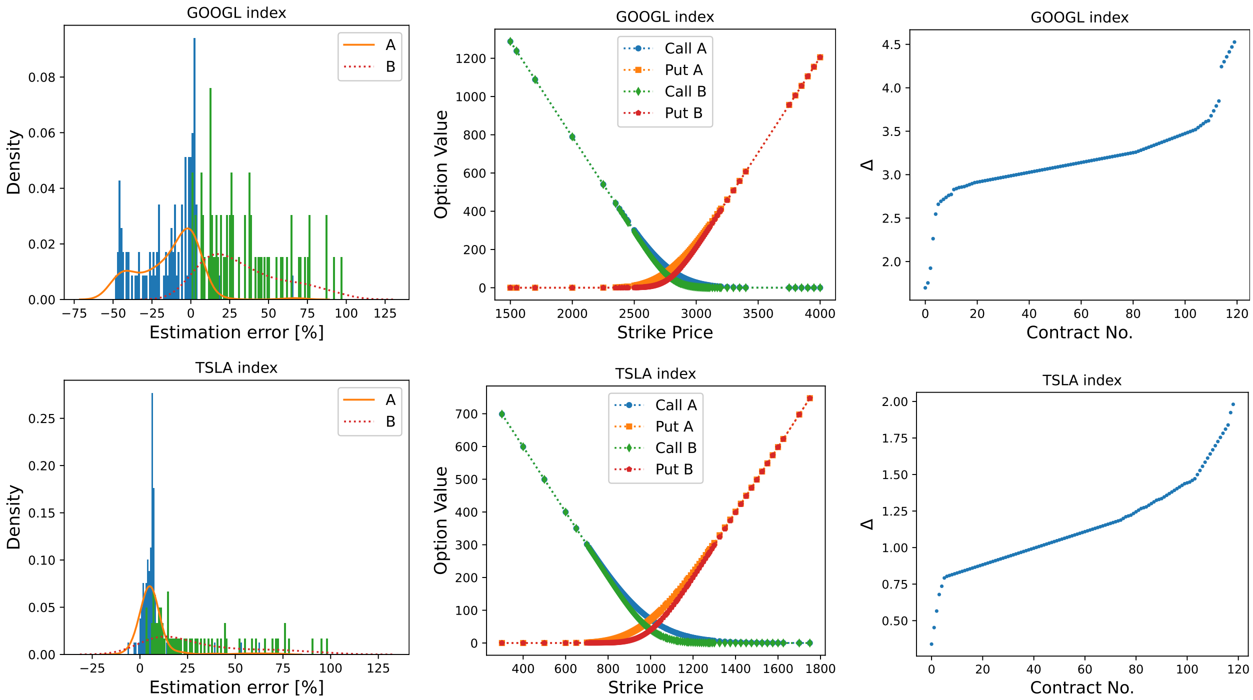

Figure 7. The plot of the analysis results for classical Black-Scholes model [28] (A) and our modified model based on the Eq. (4.8) with incorporated dark volatility (B). The historical data for S and K prices in Eq. (4.9) are from one year time period 2021/03/24 – 2022/03/24. The option calls were set one month in the future for 2022/04/22, i. e., parameter $t = 1/12$, and $r = 0.0235$. (left column) The histograms for the estimation error of selected options, (center column) the calculated values for call and put prices in dependence on initial strika price, (right column) difference of calculated forward price F between both models (A) and (B) for the contracts GOOGL (120) and TSLA (119).

The visualization of comparative difference between the classical Black-Scholes model [28], denoted (A), and our modified model based on the Eq. (4.8) with incorporated dark volatility (B) is presented on Figures 7–8. For both figures, the first column of plots, the histograms for the estimation error, shows how correctly are predicted the market values of the options. In the case of the model (A), for GOOGL options the estimation error is on average underpricing the options, for other options (SAP, DEO, BUD and TSLA) the mean of the estimation error is near the zero. For the model (B), the mean is moving to the positive numbers, the estimation has slightly overpricing effect. The middle column of plots shows the influence of the considering dark volatility on the call and put options values, which is the largest for BUD options. The last column of plots show the difference of forward price values for different strike expectations. The difference has increasing trend for all indexes for both models (A) and (B), but for GOOGL and TSLA contracts is rather stable in the central region.

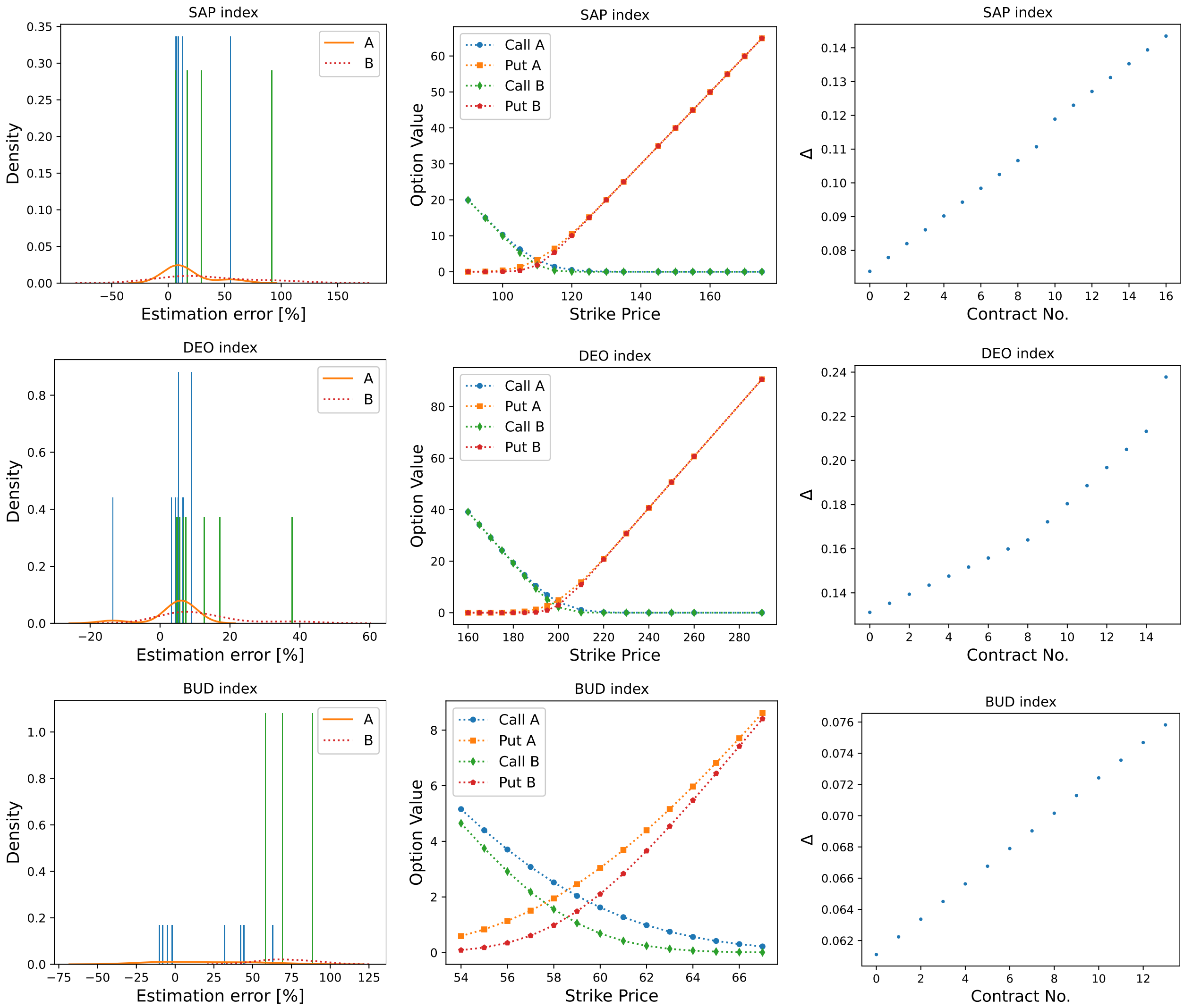

Figure 8. The plot of the analysis results for classical Black-Scholes model [28] (A) and our modified model based on the Eq. (4.8) with incorporated dark volatility (B). The historical data for S and K prices in Eq. (4.9) are from one year time period 2021/03/24 – 2022/03/24. The option calls were set one month in the future for 2022/04/22, i. e., parameter $t = 1/12$, and $r = 0.0235$. (left column) The histograms for the estimation error of selected options, (center column) the calculated values for call and put prices in dependence on initial strika price, (right column) difference of calculated forward price F between both models (A) and (B) for the contracts SAP (17), DEO (16) and BUD (14).

5. Conclusions and remarks

We have shown how the interpretation and the differential geometric approach expressed by the authors in [10], can be re-expressed and reinterpreted in the light of PNDP-manifolds.

Analyzing the above *Examples*, we can say that if the hidden "virtual" dimension of the dark volatility is greater than the positive dimension of the Global Market itself, the result will be an "emergent" Global Market with negative dimensions, something like the desuspension/s of a point, which we could interpret as the boundary of a point and so on. This occurs regardless of which manifold represents the starting Global Market. We consider this situation as a "Market Crash".

When the hidden "virtual" dimension of the dark volatility is equal to the positive dimension of the Global Market itself, will "emerge" a Global Market, which changes into a point-like object, and also in this case, regardless of which manifold represents the starting Global Market. We see the point-like market as a transitional state, in which it can suddenly collapse and crash.

We also want to analyze that when we are in presence of a Global Market expressed by a manifold with zero Ricci curvature, if the hidden "virtual" dimension of the dark volatility is less than the positive dimension of the Global Market itself, the dark volatility can intervene without changing the curvature, but changing only the dimension. We will obtain a manifold with smaller positive dimensions, but in this case, it could preserve its Ricci-flat curvature.

In the case of *Ricci-flat scenario*, i. e., market in equilibrium, or when arbitrage opportunity disappears and physiology of the time series data contains no curvature, the interpretation with PNDP-model with positive "virtual" dimension, shows that the dark volatility may have no influence on it. It will decrease its dimension, but the emerging market may remain in equilibrium. So in the *Examples 3a and 3b* we have shown how it would be possible to construct and interpret a type of differential geometric model of markets, with $dim(M) > 0$, where dark volatility does not affect its curvature.

In Section 4, we have instead shown a possible correlation to classic exponential stochastic models, analyzing a specific case in which for a market, not in equilibrium, the approach suggests a strong influence of dark volatility, predicting an exponential collapse of the price. Thus highlighting the effective negative influence that dark volatility brings also by applying it to the well-known classical models. This could be used to describe the influence of dark volatility on the market crisis that occurred in the United States in 1929, or in Argentina in 2001, but also on the financial crisis of 2007–2009, created by the United States housing bubble. Numerical simulations in this regard will be the subject of our next works.

In fact, here we have only shown a new possible interpretative geometrical approach considering this new manifold, but the main goal of our future work is to investigate this further from the empirical point of view, by analyzing in more detail the Black-Scholes equation. Black-Scholes is a widely used pricing model, also with differential geometric techniques (see [32–34]). We intend to study the Black-Scholes option pricing on Riemannian manifold influenced by dark volatility, this considering Brownian motion on Riemannian manifold as B-market, to determine such pricing on a B-market that changes under the influence of dark volatility F. We will compare the results as F varies, enabling us to see how the change of manifold influences the price trend, and how the dark volatility acts. Since the concept of "virtual" dimensions, from speculative point of view, can create a new concept of "hidden" dimensions, the PNDP-manifolds can be considered in different types of applications, a possible one has already been shown in [6]. Our work can also be understood as an extension of the geometric approach described in these papers [13–16]. For instance, objects are represented by open string with 2-endpoints and $D2$-brane, which are continuous enhancement of 1-endpoint open string model ([13]). We can, in fact, reinterpret strings and branes as PNDP-strings and PNDP-brane showing the geometric aspects of the "hidden" spaces that our approach introduces, but also as an extension to more "classic" works as [17, 18].

In this paper we are using PNDP-manifolds to describe some dark field related aspects in the financial market such as the new concept of dark volatility, in which the latter is considered as the dark matter in the cosmological model. The study considers classical exponential models as possible solutions to our new approach, to show how dark volatility, combined with specific conditions, leads to the collapse of a forward price.

Acknowledgments

The support of the Scientific Grant Agency of the Ministry of Education, Science, Research and Sport of the Slovak Republic and Slovak Academy of Sciences (VEGA) under Grant No. 2/0076/23 is acknowledged.

Conflict of interest

The authors declare there is no conflict of interest.

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