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Research article

Internal friction in the Earth' crust and transverse seismic waves

Boris I. Birger*

Schmidt Institute of Physics of the Earth, Russian Academy of Sciences, Moscow 123242, Russia

* **Correspondence:** Email: birgerbor@mail.ru.

Abstract: A transverse harmonic seismic wave, which propagates through the upper fractured layer of the Earth towards the Earth's surface, is considered. It is shown that the attenuation of a seismic wave in the Earth's upper crust is due to internal friction, i.e., friction between the sides of micro-fractures. Such a damping mechanism does not work in the deeper layers of the Earth where high lithostatic pressure prevents the movement along fractures. The governing equation for a brittle fractured medium is nonlinear. In the study of wave propagation, an approximate method of harmonic linearization is used. It is shown that internal friction in the upper crust leads to the transformation of a transverse harmonic wave into a shock wave.

Keywords: fracture and flow; creep and deformation; friction; non-linear differential equations; seismic attenuation; wave propagation

1. Introduction

It is known that the attenuation of seismic waves is described by the Lomnitz rheological model in which the attenuation is independent of the wave frequency. The Lomnitz model is a linear hereditary rheological model, which is reduced to the Lomnitz creep law in the case of constant stress. The Lomnitz model describes creep, in which only slip of dislocations in crystals occurs. Such a creep micro mechanism dominates at temperatures, low compared with the melting temperature, and for short periods. At higher temperatures and longer periods, creep dominates, the micro mechanism of which is associated with both sliding and crawling of dislocations. High-temperature creep is described by the Andrade linear hereditary model, which is reduced under constant stresses to the Andrade creep law [1,2]. As shown in [3,4], the attenuation of seismic waves is associated with the low-temperature creep of Lomnitz everywhere except the asthenosphere and the boundary layer around the Earth' core where the high-temperature creep of Andrade dominates. However, in the brittle upper layers of the Earth, wave attenuation can be associated not only with creep but also with internal friction, i.e., friction between the sides of micro-fractures [5]. The stress at which sliding along fractures becomes possible depends linearly on lithostatic pressure and is independent of temperature. Lithostatic pressure rapidly grows with depth and, therefore, in sufficiently deep layers of the Earth, sliding along fractures is absent. In the present work, the propagation of seismic waves in the upper layers of the Earth and their attenuation will be investigated using the approximate method of harmonic linearization.

In the first section of the article, the harmonic linearization of the nonlinear equation describing a brittle-elastic medium is carried out and a pseudo-plastic yield strength h is introduced as such a stress upon reaching which the brittleness of the medium is manifested. The second section of the article considers the attenuation of a transverse seismic wave associated with internal friction in a brittle-elastic medium and introduces the dimensionless parameter α, the smallness of which provides weak nonlinearity of the wave attenuation problem and the applicability of the harmonic linearization method. The third section of the article considers the dependence of the parameters h and α on the depth and the conditions for the transformation of a harmonic wave into a shock wave.

2. Harmonic linearization of the governing equation for brittle—elastic medium

Harmonic linearization permits to replace the original nonlinear equation by a linear relation. The essence of this mathematical method, used in the theory of weakly nonlinear oscillations [6], is as follows. Let there be a nonlinear relationship between two variables. We assume that one of the variables depends on time according to a harmonic law. Substituting this dependence into the initial nonlinear relation, we obtain that the second variable is a periodic function of time. Next, we expand this periodic function in a Fourier series and save only the first harmonic in the expansion.

A transverse seismic wave is considered. This wave propagates through the upper fractured layer of the Earth towards the Earth's surface along the vertical z axis. It is assumed that the wave is polarized along the horizontal x axis. For such a wave, only the following components of the displacement, strain tensor, and stress tensor are nonzero

$$u_x \equiv u, \qquad \frac{du_x}{dz} = 2e_{xz} \equiv 2e, \qquad \sigma_{xz} \equiv \sigma. \tag{1}$$

The fractured layer is considered as a brittle-elastic continuous medium in which the mechanism of brittle deformation is movement with friction along the micro-fractures of various orientations.

The brittleness of the medium is described by the equations of ideal plasticity which can be written as [7]

$$\sigma_{ij} = -h\frac{\dot{e}_{ij}}{\dot{e}}, \qquad \sigma = h \tag{2}$$

$$\dot{e}_{ij} = 0, \qquad \sigma < h \tag{3}$$

where h is the yield strength and the second invariants of the deviatoric strain rate and stress tensors are defined as

$$\dot{e}=\sqrt{\dot{e}_{kl}\dot{e}_{kl}}, \quad \sigma = \sqrt{\sigma_{kl}\sigma_{kl}} \tag{4}$$

where summation is assumed over repeated indices.

Since the equations describing the brittleness of material coincide with the equations of ideal plasticity, the brittle medium is called pseudo-plastic. In the case under consideration, when the deviator stresses and strains are given by relations (1), the equations of brittle medium take the form

$$\sigma = -h\frac{\dot{e}^{(br)}}{|\dot{e}^{(br)}|}, \qquad |\sigma| = h \tag{5}$$

$$\dot{e}^{(br)} = 0, \qquad |\sigma| \leq h \tag{6}$$

where $\dot{e}^{(br)}$ is the rate of brittle deformation, h is the stress, determined by the static friction between the sides of the fractures, corresponds to the yield strength in the theory of plasticity and can be called the pseudo-plastic yield strength. As follows from Eq (5), the deviator stresses σ in the brittle medium are caused by friction and change sign when the movement along fractures changes direction. The relation (6) shows that brittle strains occur only when stresses reach the pseudo-plastic yield strength h.

We represent brittle deformation in the form of harmonic oscillation

$$e^{(br)} = \left|e^{(br)}\right|\cos(\omega t) \tag{7}$$

where ω is the oscillation frequency and t is time. Substituting (7) into (5) and performing harmonic linearization, we obtain

$$\frac{\dot{e}^{(br)}}{|\dot{e}^{(br)}|} = sign[-\sin(\omega t)] \approx -\frac{4}{\pi}\sin(\omega t). \tag{8}$$

Passing in the linear relations (7) and (8) to a complex form of writing, we rewrite them in the form

$$e^{(br)} = \left|e^{(br)}\right|\exp(i\omega t), \tag{9}$$

$$\sigma = -i\frac{4}{\pi}h\frac{e^{(br)}}{|e^{(br)}|}. \tag{10}$$

The deformation of the brittle-elastic medium can be represented as the sum of brittle and elastic deformation, characterized by the elastic shear modulus μ,

$$e = e^{(el)} + e^{(br)} \tag{11}$$

where

$$e^{(el)} = \frac{\sigma}{2\mu}. \tag{12}$$

As follows from (10) and (12), elastic and brittle strains are related as

$$e^{(el)} = -i\frac{2h}{\pi\mu}\frac{e^{(br)}}{|e^{(br)}|}. \tag{13}$$

Substituting (13) into (11), we find

$$e = \left(1 - i\frac{2h}{\pi\mu|e^{(br)}|}\right)e^{(br)}. \tag{14}$$

As follows from (14), the brittle deformation is expressed through the total deformation of brittle-elastic medium

$$e^{(br)} = e/\left(1 - i\frac{2h}{\pi\mu|e^{(br)}|}\right), \tag{15}$$

$$|e^{(br)}| = \sqrt{|e|^2 - \left(\frac{2h}{\pi\mu}\right)^2}. \tag{16}$$

Relation (16) is valid under the condition

$$|e| \geq \frac{2h}{\pi\mu}. \tag{17}$$

Substituting (15) and (16) into (10), we obtain the rheological equations that describe, in the framework of the harmonic linearization approximation, the brittle-elastic medium

$$\sigma = 2\mu\left(\frac{2h}{\pi\mu|e|} - i\sqrt{1 - \left(\frac{2h}{\pi\mu|e|}\right)^2}\right)\frac{2h}{\pi\mu|e|}e, \qquad |e| \geq \frac{2h}{\pi\mu}, \tag{18}$$

$$\sigma = 2\mu e, \qquad |e| \leq \frac{2h}{\pi\mu}. \tag{19}$$

Equation (18) shows that if condition (17) is satisfied, we can introduce an effective shear modulus for the brittle-elastic medium

$$\mu_{eff} = \mu\left(\frac{2h}{\pi\mu|e|} - i\sqrt{1 - \left(\frac{2h}{\pi\mu|e|}\right)^2}\right)\frac{2h}{\pi\mu|e|}. \tag{20}$$

Equation (18), obtained for oscillations in the brittle-elastic medium [7], can be easily generalized to the case of oscillations in a creep-brittle-elastic medium, for which the strain can be represented as

$$e = e^{(el)} + e^{(br)} + e^{(cr)} \tag{21}$$

where $e^{(cr)}$ is the creep strain. In the case of harmonic oscillation, the linear integral (hereditary) Boltzmann equation describing the rheology of a creep medium takes the form

$$2e^{(cr)} = \int_0^\infty K(t_1)\sigma(t - t_1)\mathrm{d}t = \sigma\exp(i\omega t)\int_0^\infty K(t_1)\exp(-i\omega t)\mathrm{d}t_1 \tag{22}$$

where *K (t)* is the integral creep kernel. Relation (22) can be rewritten in the form

$$\sigma = 2\frac{1}{K^*(i\omega)}e^{(cr)}, \tag{23}$$

$$K^*(i\omega) = \int_0^\infty K(t_1)\exp\left(-i\omega t\right)\mathrm{d}t_1) \tag{24}$$

where the asterisk denotes the Laplace image of the creep kernel. Using relations (21) and (23), we obtain the effective shear modulus for oscillations in the creep-brittle-elastic medium

$$\mu_{eff} = \mu\left(\frac{2h(1+\mu\mathrm{Re}K^*)}{\pi\mu|e|} - i\sqrt{1-\left(\frac{2h(1+\mu\mathrm{Re}K^*)}{\pi\mu|e|}\right)^2}\right)\frac{2h}{\pi\mu|e|}. \tag{25}$$

Equation (25) is valid if

$$|e| \geq \frac{2h}{\pi\mu}\sqrt{(1+\mu\mathrm{Re}K^*)^2 + (\mu\mathrm{Im}K^*)^2}\ . \tag{26}$$

When

$$|e| \leq \frac{2h}{\pi\mu}\sqrt{(1+\mu\mathrm{Re}K^*)^2 + (\mu\mathrm{Im}K^*)^2}, \tag{27}$$

the effective shear modulus (25) takes the form corresponding to a creep-elastic medium without brittleness

$$\mu_{eff} = \mu\frac{1+\mu\mathrm{Re}K^* + i\mu\mathrm{Im}K^*}{(1+\mu\mathrm{Re}K^*)^2 + (\mu\mathrm{Im}K^*)^2}. \tag{28}$$

3. Seismic wave attenuation

Considering a transverse wave running from below to the Earth's surface, we place the origin of coordinates on this surface and direct the vertical z axis upward. With this choice of the coordinate system, the wave propagates in a brittle layer where $z < 0$. In the case of a traveling wave, for which the displacement and deformation depend on time and spatial coordinate as $\exp(\omega t - kz)$, the rheological relations obtained for harmonic oscillation remain valid. In an elastic medium, the frequency ω is related to the wavenumber k by the equation

$$\rho\omega^2 = \mu k^2. \tag{29}$$

In a brittle-elastic medium, Equation (29) is replaced by the equation, which includes the complex frequency,

$$\rho(\mathrm{Re}\omega + i\mathrm{Im}\omega)^2 = \left(\mathrm{Re}\mu_{eff} + i\mathrm{Im}\mu_{eff}\right)k^2. \tag{30}$$

Having solved Eq (30), we find

$$\mathrm{Re}\omega = k\sqrt{\frac{\mathrm{Re}\mu_{eff} + |\mu_{eff}|}{2\rho}}, \tag{31}$$

$$\mathrm{Im}\omega = k\frac{\mathrm{Im}\mu_{eff}}{\sqrt{\mathrm{Re}\mu_{eff}+|\mu_{eff}|}}. \tag{32}$$

Substituting in (31) and (32) the effective shear modulus (20), we find that the frequency of the wave, traveling in the brittle-elastic medium, is determined by the formulas

$$\mathrm{Re}\omega = k\sqrt{\frac{h}{\pi\rho|e|}\left(1+\frac{2h}{\pi\mu|e|}\right)}, \tag{33}$$

$$\mathrm{Im}\omega = k\sqrt{\frac{2h}{\pi\rho|e|}\left(1-\frac{2h}{\pi\mu|e|}\right)}. \tag{34}$$

Equalities (33) and (34) are valid if $\frac{2h}{\pi\mu|e|} < 1$, i.e., for sufficiently large deformations. When $\frac{2h}{\pi\mu|e|} = 1$, it follows from Eqs (33) and (34)

$$\mathrm{Re}\omega = k\sqrt{\mu/\rho}\,, \qquad \mathrm{Im}\omega = 0. \tag{35}$$

Relations (35) describe a wave in an elastic medium and remain valid when $\frac{2h}{\pi\mu|e|} > 1$.

If we introduce the parameter

$$\alpha = 1 - \frac{2h}{\pi\mu|e|}, \qquad 0 \le \alpha \le 1, \tag{36}$$

the complex frequency is written as

$$\omega = k\sqrt{\mu/\rho}\left(\sqrt{\left(1-\frac{\alpha}{2}\right)(1-\alpha)} + i\sqrt{\alpha(1-\alpha)}\right). \tag{37}$$

The parameter α describes the brittle-elastic medium. If α < 0, the medium behaves as elastic, and it makes no sense to introduce the parameter α. The value α = 0 corresponds to the transition from a pseudo-plastic wave to an elastic wave. Pseudo-plastic is called a wave that propagates in the brittle-elastic medium, and an elastic wave is a wave that propagates in an elastic medium.

Substituting the complex frequency (37) into the expression for the displacements arising during a harmonic wave propagation

$$u = |u|\exp\left(i(\omega t - kz)\right), \tag{38}$$

we obtain an explicit dependence of the horizontal displacement on the vertical coordinate and time

$$u = |u|\exp\left(ikt\sqrt{\frac{\mu}{\rho}\left(1-\frac{\alpha}{2}\right)(1-\alpha)} - ikz\right)\exp\left(-tk\sqrt{\frac{\mu\alpha(1-\alpha)}{\rho}}\right). \tag{39}$$

As follows from (1) and (38), the deformation is related to the displacement by the relation $|e| = k|u|/2$, and the parameter α can be written as

$$\alpha = 1 - \frac{4h}{\pi\mu k|u|}. \tag{40}$$

In the problem considered in this paper, nonlinearity is introduced by the brittleness of the medium. When brittle deformation makes a very small contribution to total deformation, i.e., when

$$\left|e^{(br)}\right| \ll |e|, \tag{41}$$

the brittle-elastic medium degenerates into a linear elastic medium. The parameter α determines the measure of nonlinearity. According to (13), this parameter can be written as

$$\alpha = \frac{\left|e^{(br)}\right|}{|e|} = 1 - \frac{2h}{\pi\mu|e|}. \tag{42}$$

The solution of the problem of oscillations in a weakly nonlinear medium, which degenerates into an elastic medium in the zero-order approximation with respect to the small parameter α, is representable as an infinite series in powers of α [6]. To find the members of this series, we need to substitute the periodic solution (the period $\frac{2\pi}{\omega}$, where ω is the frequency corresponding to the solution for a linear elastic medium) in the nonlinear part of the equation, and then expand the result in a Fourier series. The first term of the series in powers of α, i.e., the first approximation is determined by the first harmonic in the Fourier expansion. It is this harmonic that the harmonic linearization gives. To obtain higher approximations in α, it is necessary to use higher harmonics in the Fourier expansion.

The harmonic linearization method used in the present study is quite correct in the case when the parameter α is small and the problem is weakly nonlinear. For small α, the relations

$$\sqrt{\left(1 - \frac{\alpha}{2}\right)(1 - \alpha)} \approx 1 - \frac{3}{4}\alpha, \qquad \sqrt{\alpha(1 - \alpha)} \approx \sqrt{\alpha} \tag{43}$$

are valid. As the practical usage of harmonic linearization shows [7], this method gives results that correspond to experiments even when the parameter α is not very small, for example, for α ≈ 0.3.

The attenuation of seismic waves is characterized by the function of frequency $Q^{-1}(\omega)$, where Q (ω) is the quality factor of the medium. The quality factor is defined as

$$Q = \pi\tau/T \tag{44}$$

where $\tau = 1/\mathrm{Im}\omega$ is the relaxation time, and $T = 2\pi/\mathrm{Re}\omega$ is the oscillation period. Therefore, the attenuation is associated with the complex oscillation frequency as

$$Q^{-1}(\omega) = 2\frac{\mathrm{Im}\omega}{\mathrm{Re}\omega} = \frac{\mathrm{Im}\mu_{eff}}{\mathrm{Re}\mu_{eff} + \left|\mu_{eff}\right|}. \tag{45}$$

If $\mathrm{Im}\mu_{eff} \ll \mathrm{Re}\mu_{eff}$, it follows from (45)

$$Q^{-1}(\omega) = \frac{\mathrm{Im}\mu_{eff}}{\mathrm{Re}\mu_{eff}}. \tag{46}$$

Equations (45) and (46) are valid for any rheology, which is described using a complex effective shear modulus. For the brittle-elastic medium, it follows from (37) and (45)

$$Q^{-1}(\omega) = 2\sqrt{\alpha/\left(1 - \frac{\alpha}{2}\right)}. \tag{47}$$

For small values of α, formula (47) takes the form

$$Q^{-1}(\omega) \approx 2\sqrt{\alpha}. \tag{48}$$

The creep of the Earth's mantle and the Earth's crust is described by the rheological model of Lomnitz [3,4]. For this model, the integral creep kernel in Eq (22) is defined as

$$K(t) = q/(t+\tau)\ . \tag{49}$$

The Laplace image of the integral kernel (49) is

$$K^*(i\omega) = -q\left(C + \omega\tau + \frac{i\pi}{2}\right) \tag{50}$$

where $C \approx 0.58$ (Euler constant) and $q \approx 2 \cdot 10^{-3}/\mu$. The time τ is very small (small fractions of second) and, therefore, $\omega\tau \ll 1$. For a creep-elastic medium, the creep of which is determined by the Lomnitz rheological model, the effective modulus of elasticity is

$$\mu_{eff} = \mu\frac{1}{1+\mu K^*} = \mu\frac{1+\mu\mathrm{Re}K^* - i\mu\mathrm{Im}K^*}{(1+\mu\mathrm{Re}K^*)^2+(\mu\mathrm{Im}K^*)^2}. \tag{51}$$

In the Lomnitz model, the dimensionless parameter $q\mu \approx 2 \cdot 10^{-3}$ is small. Therefore, $\mu\mathrm{Re}K^* \ll 1$ and Eq (51) takes the form

$$\mu_{eff} \approx \mu\frac{1}{1+\mu\mathrm{Im}K^*} \approx \mu\left(1 + i\frac{\mu q\pi}{2}\right). \tag{52}$$

Substituting (52) into Eq (46), we find

$$Q^{-1}(\omega) \approx \frac{\mathrm{Im}\mu_{eff}}{\mathrm{Re}\mu_{eff}} \approx \frac{\mu q\pi}{2} \approx 10^{-3}\pi. \tag{53}$$

As follows from (53), in the Lomnitz model, wave attenuation does not depend on its frequency.

In Eqs (18), (20), (25), (28), the sign ("plus" or "minus") of the imaginary part is chosen so that the oscillation decays with time. Therefore, in the case of Lomnitz's creep, we must put a minus sign in front of $\mathrm{Im}K^*$ in Eq (28), reducing this equation to Eq (51).

For sufficiently fast processes in the crust and mantle, the effective shear modulus, corresponding to creep, significantly exceeds the elastic shear modulus and, therefore, creep strains are small compared to elastic strains. However, small creep strains should be taken into account when considering the attenuation of seismic waves. In the upper layers of the Earth, brittle deformations significantly exceed creep deformations, and the attenuation of seismic waves (56) due to creep is much weaker than the attenuation (48) caused by brittle deformations. Therefore, these layers can be considered as the brittle-elastic medium.

4. Seismic wave in the brittle-elastic Earth's crust

The yield strength h included in Eq (5) characterizes the friction between the sides of the fractures. The friction force is proportional to the stress directed normal to the fracture and, therefore, depends on the stress field in the vicinity of the fracture and on the direction of the fracture. The coefficient of friction is specified by Byerlee's law [8,9], according to which the coefficient of static friction is estimated as $f_B \approx 0.65$. To determine the value of h for a brittle medium with pre-existing fractures, the minimal (among all possible directions of fractures) value of the friction force is selected [10]. The coefficient $f_B \approx 0.65$ describes the friction in a dry environment. Taking into account humidity, the value of this coefficient decreases approximately 2 times. The lithostatic pressure *p*, which determines the value of the yield strength h, increases with depth as

$$p = \rho g|z| \tag{54}$$

where $\rho \approx 3 \cdot 10^3$ кg/m^3 is the density, $g \approx 10$ m/s^2 is the acceleration of gravity, and | *z* | - depth (*z* axis is directed upward, *z* < 0 under the Earth's surface). Therefore, the pseudo-plastic yield strength depends on the depth as

$$h = \frac{f_B}{2}\rho g|z|, \qquad \frac{f_B}{2}\rho g \approx 10^4 \text{ Па/м}. \tag{55}$$

Solution (38) was obtained under the assumption that the yield strength h is uniform in depth. Taking into account the dependence of h on depth given by relation (55), we represent the desired dependence of horizontal displacement on time and vertical coordinate in the form

$$u = |u| \exp\left(i(\omega(z)t - kz)\right). \tag{56}$$

This form of the desired solution differs from (38) only in that the frequency (but not the wavenumber *k*) is a function of the *z* coordinate. A solution in the form of (56), which describes wave propagation in an inhomogeneous medium, is immediately obtained from Eq (40), in which the parameter α depends on the coordinate *z*

$$\alpha(z) = 1 - \frac{4h(z)}{\pi\mu k|u|} \tag{57}$$

where $\mu \approx 5 \cdot 10^{10}$Pa. Using relations (55), (57), we obtain the depth distribution of the parameter α

$$\alpha(z) = 1 - 2.7 \cdot 10^{-7}\frac{|z|}{k|u|}. \tag{58}$$

According to (58), the parameter α = 0, i.e. the medium behaves as elastic, at a depth

$$|z|_0 = 3.7 \cdot 10^6 k|u|. \tag{59}$$

At shallower depths, brittleness is manifested. A fixed value of the parameter α is reached at a depth

$$|z|_\alpha = 3.7 \cdot 10^6 (1 - \alpha)k|u|. \tag{60}$$

Since the applied method of harmonic linearization works the better, the smaller α, it makes no sense to use this method to consider the passage of a seismic wave through the upper layers of the crust, where α is large. We confine ourselves to the study of the behavior of this wave in such depths where the parameter α is small and, therefore, the effect of brittleness is small.

The wave propagation velocity, as it follows from (39) and (43), is written as

$$c(\alpha) = c(z) = \sqrt{\frac{\mu}{\rho}}\left(1 - \frac{3}{4}\alpha(z)\right) \tag{61}$$

where $\alpha(z)$ is given by relation (58). The velocity $c(z)$ of the wave under consideration is $\sqrt{\frac{\mu}{\rho}} \approx 4\ \mathrm{km/s}$ at a depth where $\alpha = 0$ and decreases with decreasing depth. When a harmonic wave passes through a brittle layer its frequency and velocity decrease.

According to (48) and (58), the dependence of the attenuation, caused by internal friction, on depth is determined by the formula

$$Q^{-1}(\omega) = 2\sqrt{\alpha} = 2\sqrt{1 - 2.7 \cdot 10^{-7}\frac{|z|}{k|u|}}. \tag{62}$$

Hence, the attenuation, associated with internal friction, is zero at a depth given by Eq (59) and grows with decreasing depth. Brittleness causes much stronger wave attenuation than creep. In a brittle layer, the shorter wave, the stronger attenuation. Comparing (62) and (53), it is easy to see that the damping of the wave is determined by the internal friction, and not by the creep of the medium, if $k|u| > 2.7 \cdot 10^{-7}|z|$. When the wave induces the strain $k|u| = 10^{-6}$, the internal friction determines the attenuation at the very small depths $|z| < 3.7\ \mathrm{m}$. However, when the wave is very strong ($k|u| = 3 \cdot 10^{-4}$), the internal friction determines the attenuation at $|z| < 1000$ m.

The pseudo-plastic yield strength h is zero on the Earth's surface and increases with depth. The parameter α is equal to unity on the Earth's surface and decreases with depth, reaching the value $\alpha = 0$ at a certain depth, below which the medium behaves as elastic without showing brittleness. This depth depends on the wave amplitude and wavenumber. The wave propagation velocity given by Eq (61) vanishes on the Earth's surface and increases with depth, reaching the wave propagation velocity in an elastic medium when $\alpha = 0$. The frequency of the wave $\omega(z) = c(z)k$ varies with depth in the same way as velocity. When a seismic wave comes from below into the upper brittle layer, its frequency begins to decrease and the attenuation increases.

Seismological observations [11] show that a transverse seismic wave traveling vertically upward from the hypocenter of a strong earthquake is converted into a shock wave approaching the Earth's surface. The shock wave has a pronounced front and, reaching the Earth's surface, causes significant horizontal acceleration. Passing through the upper layers of the Earth, the wave quickly decays. Therefore, such waves can be observed only during very strong earthquakes that create waves of large amplitude. After passing through the upper layer, the frequency of the seismic wave, coming from the source under the crust, is noticeably reduced. A wave with a frequency of 4 Hz acquires a frequency of the order of 2 Hz which leads to more severe destruction of buildings. Pavlenko [11] explains the occurrence of a shock wave by nonlinear elasticity of the uppermost crustal layer consisting of sedimentary rocks and having a thickness of about 1 km. In the present study, the occurrence of a shock wave is related to a more acceptable rheological model for this uppermost fractured layer.

Pavlenko [11] studies the super strong earthquake in Japan in 2011, where abnormally high seismic accelerations were observed, three times exceeding the acceleration of gravity $\omega^2|u| \approx 3g \approx 30\frac{m}{s^2}$ at frequencies of the order of 4 Hz ($\omega \approx 24s^{-1}$). Consequently, such a wave is characterized by the displacement | u | ≈ 0.05 m and the wavenumber $k = \frac{\omega}{c} \approx 6 \cdot 10^{-3}\mathrm{m}^{-1}$. Hence the deformation is estimated as $k|u| \approx 3 \cdot 10^{-4}$. With such huge deformations (for "ordinary" seismic waves, $k|u|$ is in the range from 10^{-9} to 10^{-7}), rapidly growing cracks appear in the upper layers of the crust, indicating the destruction of mountain rocks.

Taking $k|u| \approx 3 \cdot 10^{-4}$, we find from equality (59) that brittleness arises at a depth of about 1100 m where α = 0, and the value α = 0.3 is attained, as follows from (60), at a depth of about 700 m.

The wave profile is given by displacements at a fixed time. The evolution of the wave profile is associated with the fact that different points of the wave profile move at different speeds. When the rapidly moving points of the wave profile reach the slowly moving points of the profile, the breaking of harmonic wave occurs that leads to the formation of a shock wave [12]. Wave breaking is possible only when the speed decreases in the direction of wave propagation, i.e., when front points of the profile move slower than the rear points. This is the case when a harmonic wave propagates in a brittle-elastic medium.

In a harmonic wave, the displacements depend on the coordinate as $u = |u| \cos kz$. Such displacements correspond to deformation

$$e = -\frac{1}{2} k|u| \sin kz. \quad (63)$$

Profile points, where the displacement is maximal and the deformation is minimal, move with a velocity corresponding to the elastic medium since the stress is small and brittleness does not occur. The remaining points of the profile (see Figure 1) move at a lower velocity given by Eq (61).

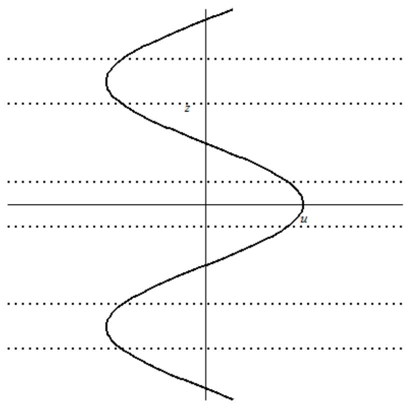

Figure 1. The profile points of the harmonic wave, where the displacement is maximal and the deformation is minimal, move faster than the other profile points.

Deformation (63) is not related to brittleness if the elastic stress corresponding to this deformation does not exceed the pseudo-plastic yield strength

$$\mu k|u| \sin kz < h. \quad (64)$$

Relation (64) is satisfied at the points of the profile where

$$-\arcsin\frac{h}{\mu k|u|} - n\pi < kz < \arcsin\frac{h}{\mu k|u|} - n\pi, \qquad n = 1, 2, 3\ldots \quad (65)$$

Taking into account relation (57), which determines the parameter α, it is convenient to make the replacement in (65)

$$\frac{h}{\mu k|u|} = \frac{\pi}{4}(1-\alpha). \quad (66)$$

At the profile points whose coordinates satisfy condition (65), the propagation velocity corresponds to an elastic wave, and at those profile points, where this condition is not fulfilled, to a pseudo-plastic wave. The time, during which the rapidly moving points of the wave profile reach the slowly moving points of the profile, is determined as

$$\tilde{t} = \frac{T}{4\left(1-\sqrt{\left(1-\frac{\alpha}{2}\right)(1-\alpha)}\right)} \quad (67)$$

where *T* is the wave period. Taking into account the smallness of the parameter α, the relation (67) takes the form

$$\tilde{t} \approx \frac{T}{3\alpha}. \quad (68)$$

With a very small parameter α, the shock wave does not have time to form during the wave period *T*. As shown above for $k|u| \approx 3 \cdot 10^{-4}$, the value α = 0.3 is reached at a depth of about 700 m. At this depth, the time $\tilde{t}$ is approximately equal to the period T, which for the frequency $\omega \approx 24\ s^{-1}$ is estimated as $T = \frac{2\pi}{\omega} \approx 0.25\ s$. For the formation of shock wave, a sufficiently large value of the parameter α is required. In the uppermost layers of the crust, α → 1. However, for such large values of the parameter α (strong nonlinearity), the approximate method of harmonic linearization, strictly speaking, is not applicable.

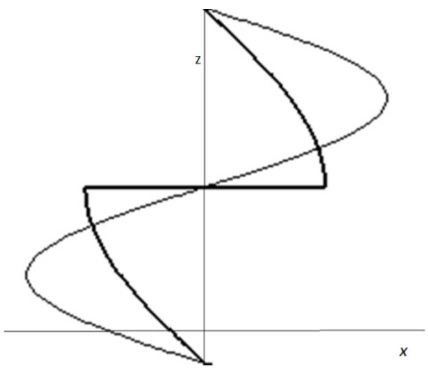

Figure 2. The evolution of the profile of a transverse seismic wave running vertically upward in the Earth’s crust. The thin line shows the profile of the harmonic wave at the initial moment when the wave enters the brittle-elastic layer from below and the thick line shows the profile of the wave after its attenuation in the brittle-elastic layer and the formation of a shock wave.

When a seismic wave enters the brittle-elastic layer from below, its velocity and frequency begin to decrease. When the wave reaches a depth at which the parameter α is sufficiently large (the pseudo-plastic yield strength h is small enough), the initially harmonic elastic wave transforms into a shock wave. This transformation of a harmonic elastic wave into a shock wave (Figure 2 shows the evolution of wave profile) is the result of the action of internal friction.

5. Conclusion

Attenuation of a transverse harmonic seismic wave which propagates through the upper fractured layer towards the Earth's surface is due to internal friction, i.e., friction between the sides of micro-fractures. This damping mechanism does not work in deeper layers of the Earth where high lithostatic pressure prevents movement along fractures. The governing equation for a brittle fractured medium is nonlinear. The use of the approximate method of harmonic linearization gives a simple solution to the problem of the propagation of harmonic wave in a brittle-elastic medium. In geophysics, the quality factor $Q(\omega)$ is widely used. The function $Q^{-1}(\omega)$ characterizes the attenuation of a harmonic wave in a viscoelastic medium. In the article, the quality factor is found for a weakly nonlinear brittle-elastic medium. The solution obtained allows us to assume that internal friction in a brittle-elastic crust distorts the profile of a harmonic wave, leads to its breaking and transformation into a shock wave.

The use of the approximate method of harmonic linearization gives a reliable result only in the case of weak nonlinearity (small value of the parameter α). Therefore, the present work considers wave propagation only in sufficiently deep layers of the Earth's crust where the brittleness, which introduces nonlinearity into the problem, is not too great. Since the estimates of the brittleness of the Earth's layers are not very accurate, it may turn out that although the wave profile in the layers under consideration is distorted (the wave ceases to be harmonic), this distortion is not enough to form a shock wave. The breaking, which leads to the formation of a shock wave, occurs rather in the uppermost brittle layers of the crust but in the case of a very brittle and, therefore, highly nonlinear medium, the method of harmonic linearization is not applicable.

Conflict of interest

The author declares no conflicts of interest in this paper.

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